ACCELERATED ALGEBRA AND CALCULUS

Assignment 3 – Solution

Exercise 11.2

The complex number $z = re^{i\theta}$ is a fourth root of -1 if

$$z^4 = r^4 e^{4i\theta} = -1.$$

Writing -1 in polar form as $e^{i\pi}$, this means

$$r^4 e^{4i\theta} = e^{i\pi},$$

so $r = 1$. Now since $e^{2\pi i} = 1$, we have

$$e^{4i\theta} = e^{i\pi} = e^{3i\pi} = e^{5i\pi} = e^{7i\pi}$$

hence the four complex roots are

$$\begin{aligned} e^{i\frac{\pi}{4}} &= \frac{1}{\sqrt{2}} + i\frac{1}{\sqrt{2}}, \\ e^{i\frac{3\pi}{4}} &= -\frac{1}{\sqrt{2}} + i\frac{1}{\sqrt{2}}, \\ e^{i\frac{5\pi}{4}} &= -\frac{1}{\sqrt{2}} - i\frac{1}{\sqrt{2}}, \\ e^{i\frac{7\pi}{4}} &= \frac{1}{\sqrt{2}} - i\frac{1}{\sqrt{2}}. \end{aligned}$$

Note that the first and last are complex conjugates, as are the middle two.

Since these are roots of $x^4 + 1 = 0$, we have that $(x - z)$ is a factor of $x^4 + 1$, with z replaced by any of the above roots. So

$$\begin{aligned} x^4 + 1 &= \left(x - \left(\tfrac{1}{\sqrt{2}} + i\tfrac{1}{\sqrt{2}}\right)\right)\left(x - \left(\tfrac{1}{\sqrt{2}} - i\tfrac{1}{\sqrt{2}}\right)\right)\left(x - \left(-\tfrac{1}{\sqrt{2}} + i\tfrac{1}{\sqrt{2}}\right)\right)\left(x - \left(-\tfrac{1}{\sqrt{2}} - i\tfrac{1}{\sqrt{2}}\right)\right) \\ &= (x - \sqrt{2}x + 1)(x + \sqrt{2}x + 1). \end{aligned}$$

Note: in general, if z and $\bar{z}$ are conjugates we have

$$(x - z)(x - \bar{z}) = x^2 - (z + \bar{z})x + z\bar{z}.$$

As a quick exercise, you can show that $z + \bar{z}$ and $z\bar{z}$ are both real numbers.

Problem 12.5

Claim:

$P \in \mathbb{R}[x], P(a) = 0, P'(a) = 0 \implies P(x) = (x-a)^2 Q(x)$ for some $Q \in \mathbb{R}[x]$.

Proof. Since $P(a) = 0$ we know

$$P(x) = (x-a)R(x)$$

for some $R \in \mathbb{R}[x]$ (by Theorem 10.1.10).

Then, using the product rule to differentiate,

$$P'(x) = R(x) + (x-a)R'(x).$$

Since $P'(a) = 0$, this implies $R(a) + (a-a)R'(a) = 0$, i.e. $R(a) = 0$. So

$$R(x) = (x-a)Q(x)$$

for some $Q \in \mathbb{R}[x]$ (again by Theorem 10.1.10).

Putting this together, we have

$$\begin{aligned} P(x) &= (x-a)R(x) \\ &= (x-a)(x-a)Q(x) \\ &= (x-a)^2 Q(x), \end{aligned}$$

where $Q \in \mathbb{R}[x]$. $\square$