Solution to Week 5 Assignment

5.4. Let $a_n := \frac{1}{n^3}$ for all $n \in \mathbb{N}$. We claim that the limit of the sequence $(a_n)_{n\in\mathbb{N}}$ is zero. Let $\varepsilon > 0$, and choose $N \in \mathbb{N}$ such that $N > \frac{1}{\varepsilon}$. Then for all $n \geq N$

$$|a_n - 0| = \tfrac{1}{n^3} \leq \tfrac{1}{n} \leq \tfrac{1}{N} < \varepsilon,$$

which proves that the limit is 0.

5.6. Let $r \in \mathbb{R}$ and define $a_n := r^n$ for all $n \in \mathbb{N}$. We claim that $(a_n)_{n\in\mathbb{N}}$ has a limit when $1 < r \leq 1$. To prove this, we split into cases:

(a) When $r = 1$, we claim that the limit is 1. Let $\varepsilon > 0$, and choose $N = 1$. Then for all $n \geq N$

$$|a_n - 1| = |1^n - 1| = 0 < \varepsilon,$$

which proves that the limit is 1.

(b) When $r = 0$, we claim that the limit is 0. Let $\varepsilon > 0$, and choose $N = 1$. Then for all $n \geq N$

$$|a_n - 0| = |0^n - 0| = 0 < \varepsilon,$$

which proves that the limit is 0.

(c) When $-1 < r < 1$ with $r \neq 0$, we claim that the limit is 0. Let $\varepsilon > 0$, and choose $N > \frac{\ln \varepsilon}{\ln |r|}$. Note that

$$|a_n - 0| < \varepsilon \iff |r|^n < \varepsilon \iff n \ln |r| < \ln \varepsilon \iff n > \tfrac{\ln \varepsilon}{\ln |r|}. \qquad (1)$$

Hence, by (1), for all $n \geq N$ we have

$$|a_n - 0| < \varepsilon,$$

which proves that the limit is 0.

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