Solution to Week 6 Assignment

Let $k \in \mathbb{R}$ and consider $f(x) := (1+x)^k$. Then

$$\begin{aligned}
f'(x) &= k(1+x)^{k-1} && \text{so} \quad f'(0) = k. \\
f^{(2)}(x) &= k(k-1)(1+x)^{k-2} && \text{so} \quad f^{(2)}(0) = k(k-1). \\
f^{(3)}(x) &= k(k-1)(k-2)(1+x)^{k-3} && \text{so} \quad f^{(3)}(0) = k(k-1)(k-2). \\
& && \vdots
\end{aligned}$$

Case 1: $k \in \mathbb{N}$. Then $f^{(i)}(0) = k(k-1)\dots(k-i+1)$ for all $0 \le i \le k$ and $f^{(i)}(0) = 0$ for all $i > k$. Hence in this case the Maclaurin series is

$$\sum_{i=0}^{\infty} \frac{f^{(i)}(0)}{i!} x^i = 1 + \sum_{i=1}^{k} \frac{k(k-1)\dots(k-i+1)}{i!} x^i.$$

In particular this is a polynomial (i.e. the sum is finite). We claim that this converges for all x. Indeed, considering the partial sums

$$s_j := \sum_{i=0}^{j} \frac{f^{(i)}(0)}{i!} x^i$$

we see that $s_k = s_{k+1} = s_{k+2} = \dots$ and so (for all x) the limit of $(s_n)_{n\in\mathbb{N}}$ is just s_k. Since $(s_n)_{n\in\mathbb{N}}$ has limit s_k for all x, by definition this means that the Maclaurin series converges to s_k for all x. Since $k \in \mathbb{N}$ we know (from the binomial theorem) that $(1+x)^k = s_k$, hence the Maclaurin series converges to $(1+x)^k$ for all x.

Case 2: $k \notin \mathbb{N}$. Then $f^{(i)}(0) = k(k-1)\dots(k-i+1)$ for all $i \ge 0$. Hence in this case the Maclaurin series is

$$\sum_{i=0}^{\infty} \frac{f^{(i)}(0)}{i!} x^i = 1 + \sum_{i=1}^{\infty} \frac{k(k-1)\dots(k-i+1)}{i!} x^i.$$

We claim first that this converges if $|x| < 1$. Let $a_i := \frac{f^{(i)}(0)}{i!} x^i$, then

$$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{\frac{k(k-1)\dots(k-n)}{(n+1)!} x^{n+1}}{\frac{k(k-1)\dots(k-n+1)}{n!} x^n}\right| = \frac{|k-n|}{n+1}|x| = \frac{\left|1-\frac{k}{n}\right|}{1+\frac{1}{n}}|x| \to |x| \text{ as } n \to \infty.$$

Hence by the ratio test the Maclaurin series converges for all $|x| < 1$.

(From here is quite difficult, and was not part of the assessment) We claim that for all $|x| < 1$, the Maclaurin series converges to $f(x) = (1+x)^k$. Set

$$g(x) := 1 + \sum_{i=1}^{\infty} \frac{k(k-1)\dots(k-i+1)}{i!} x^i$$

By the above we know that $g(x)$ converges for all $x \in (-1,1)$, hence in this range we can differentiate term by term to obtain

$$g'(x) = \sum_{i=1}^{\infty} \frac{k(k-1)\dots(k-i+1)}{(i-1)!} x^{i-1} = \sum_{i=0}^{\infty} \frac{k(k-1)\dots(k-i)}{i!} x^i$$

for all $x \in (-1, 1)$. Adding term by term (which we can do since $g(x)$ converges on the interval $(-1, 1)$) we see that

$$(1 + x)g'(x) = kg(x) \tag{1}$$

for all $x \in (-1, 1)$. Now let $h(x) := \frac{g(x)}{(1+x)^k}$, then by the product rule

$$h'(x) = \frac{(1 + x)^{k-1} \left((1 + x)g'(x) - kg(x)\right)}{(1 + x)^{2k}} \overset{(1)}{=} 0.$$

This shows that $h(x) = c$ for some constant c, but since we know that $h(0) = 1$, it follows that $h(x) = 1$. From this, we deduce that $g(x) = (1 + x)^k$ for all $x \in (-1, 1)$, as required.

m.wemyss@ed.ac.uk