Solution to Week 8 Assignment

Q35. By the Euclidean algorithm,

$$\begin{aligned} 87 &= 39.2 + 9 \\ 39 &= 9.4 + 3 \\ 9 &= 3.3 \end{aligned}$$

Hence we deduce that $\gcd(87, 39) = 3$. Working backwards,

$$\begin{aligned} 3 &= 39 - 9.4 \\ &= 39 - (87 - 39.2).4 \\ &= 9.39 - 4.87, \end{aligned}$$

hence $x = 9, y = -4$ is one solution.
Now let (a, b) be another solution, then

$$39a + 87b = 3 = 9.39 - 4.87$$

and so

$$39(9 - a) = 87(b + 4).$$

Dividing by 3 we see that

$$13(9 - a) = 29(b + 4). \tag{1}$$

Since 13 divides the left hand side, it also divides the right hand side. But 13 and 29 are coprime, hence 13 divides $b + 4$. Consquently

$$b = 13k - 4$$

for some $k \in \mathbb{Z}$. Substituting this into (1) gives

$$13(9 - a) = 29.13k$$

and so $9 - a = 29k$, i.e. $a = 9 - 29k$. This shows that the general solution is

$$\{(9 - 29k, 13k - 4) \mid k \in \mathbb{Z}\}.$$

Taking $k = 1$ gives solution $x = -20$, $y = 9$.

Q14.5. Consider two elements $\begin{pmatrix} a_1 & -b_1 \\ b_1 & a_1 \end{pmatrix}$ and $\begin{pmatrix} a_2 & -b_2 \\ b_2 & a_2 \end{pmatrix}$ of C. Then

$$\begin{pmatrix} a_1 & -b_1 \\ b_1 & a_1 \end{pmatrix} + \begin{pmatrix} a_2 & -b_2 \\ b_2 & a_2 \end{pmatrix} = \begin{pmatrix} a_1 + a_2 & -(b_1 + b_2) \\ b_1 + b_2 & a_1 + a_2 \end{pmatrix}$$

which is clearly an element of C. Hence C is closed under addition. Further

$$\begin{pmatrix} a_1 & -b_1 \\ b_1 & a_1 \end{pmatrix} \begin{pmatrix} a_2 & -b_2 \\ b_2 & a_2 \end{pmatrix} = \begin{pmatrix} a_1a_2 - b_1b_2 & -(a_1b_2 + a_2b_1) \\ a_1b_2 + a_2b_1 & a_1a_2 - b_1b_2 \end{pmatrix}$$

which is also an element of C. Hence C is closed under multiplication.

Now if we let $I := \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$, then

$$I^2 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} = -\mathbb{I}.$$

Note that if $\begin{pmatrix} a & -b \\ b & a \end{pmatrix}$ is an element of C, we can write

$$\begin{pmatrix} a & -b \\ b & a \end{pmatrix} = \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} + \begin{pmatrix} 0 & -b \\ b & 0 \end{pmatrix} = a\mathbb{I} + bI,$$

which shows that every element of C can be written as $a\mathbb{I} + bI$ for some $a, b \in \mathbb{R}$. Lastly,

$$\begin{aligned} \begin{pmatrix} 2 & 3 \\ -3 & 2 \end{pmatrix} \begin{pmatrix} -1 & 6 \\ -6 & -1 \end{pmatrix} &= (2\mathbb{I} - 3I)(-\mathbb{I} - 6I) \\ &= -2\mathbb{I} - 12I + 3I + 18I^2 \\ &= -20\mathbb{I} - 9I \\ &= \begin{pmatrix} -20 & 9 \\ -9 & -20 \end{pmatrix}. \end{aligned}$$

Q14.6. In the previous question, a complex number $z = x + iy$ corresponds to the matrix

$$\begin{pmatrix} x & -y \\ y & x \end{pmatrix} := A.$$

Hence the complex conjugate $\overline{z} = x - iy$ corresponds to

$$\begin{pmatrix} x & y \\ -y & x \end{pmatrix}$$

which is the transpose of A. The modulus of z, namely $\sqrt{x^2 + y^2}$, corresponds to the square root of the determinant of A. Finally, the reciprocal of z, namely $\frac{1}{z}$, is equal to $\frac{\overline{z}}{z\overline{z}} = \frac{1}{|z|^2}\overline{z}$. By the previous parts, this corresponds to

$$\frac{1}{\det A} \begin{pmatrix} x & y \\ -y & x \end{pmatrix},$$

which is the inverse of A.

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