Solution to Week 9 Assignment

Q2(a). To compute A^{-1}, we use Gaussian ellimination on $(A \mid \mathbb{I})$:

$$\left(\begin{array}{rrr|rrr} 1 & -2 & 1 & 1 & 0 & 0 \\ 2 & 1 & -2 & 0 & 1 & 0 \\ -2 & 1 & 1 & 0 & 0 & 1 \end{array}\right) \underset{\sim}{\scriptstyle R_2 \mapsto R_2 + R_3} \left(\begin{array}{rrr|rrr} 1 & -2 & 1 & 1 & 0 & 0 \\ 0 & 2 & -1 & 0 & 1 & 1 \\ -2 & 1 & 1 & 0 & 0 & 1 \end{array}\right)$$

$$\underset{\sim}{\scriptstyle R_3 \mapsto R_3 + 2R_1} \left(\begin{array}{rrr|rrr} 1 & -2 & 1 & 1 & 0 & 0 \\ 0 & 2 & -1 & 0 & 1 & 1 \\ 0 & -3 & 3 & 2 & 0 & 1 \end{array}\right)$$

$$\underset{\sim}{\scriptstyle R_1 \mapsto R_1 + R_2} \left(\begin{array}{rrr|rrr} 1 & 0 & 0 & 1 & 1 & 1 \\ 0 & 2 & -1 & 0 & 1 & 1 \\ 0 & -3 & 3 & 2 & 0 & 1 \end{array}\right)$$

$$\underset{\sim}{\scriptstyle R_3 \mapsto \frac{1}{3} R_3} \left(\begin{array}{rrr|rrr} 1 & 0 & 0 & 1 & 1 & 1 \\ 0 & 2 & -1 & 0 & 1 & 1 \\ 0 & -1 & 1 & \frac{2}{3} & 0 & \frac{1}{3} \end{array}\right)$$

$$\underset{\sim}{\scriptstyle R_2 \mapsto R_2 + R_3} \left(\begin{array}{rrr|rrr} 1 & 0 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & \frac{2}{3} & 1 & \frac{4}{3} \\ 0 & -1 & 1 & \frac{2}{3} & 0 & \frac{1}{3} \end{array}\right)$$

$$\underset{\sim}{\scriptstyle R_3 \mapsto R_3 + R_2} \left(\begin{array}{rrr|rrr} 1 & 0 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & \frac{2}{3} & 1 & \frac{4}{3} \\ 0 & 0 & 1 & \frac{4}{3} & 1 & \frac{5}{3} \end{array}\right)$$

Hence we deduce that

$$A^{-1} = \begin{pmatrix} 1 & 1 & 1 \\ \frac{2}{3} & 1 & \frac{4}{3} \\ \frac{4}{3} & 1 & \frac{5}{3} \end{pmatrix}.$$

Check:

$$\begin{pmatrix} 1 & -2 & 1 \\ 2 & 1 & -2 \\ -2 & 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 \\ \frac{2}{3} & 1 & \frac{4}{3} \\ \frac{4}{3} & 1 & \frac{5}{3} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 \\ \frac{2}{3} & 1 & \frac{4}{3} \\ \frac{4}{3} & 1 & \frac{5}{3} \end{pmatrix} \begin{pmatrix} 1 & -2 & 1 \\ 2 & 1 & -2 \\ -2 & 1 & 1 \end{pmatrix}$$

Q18.2. First, we investigate even n. When $n = 2$ we have

$$\begin{vmatrix} 0 & a_1 \\ a_2 & 0 \end{vmatrix} = - \begin{vmatrix} a_2 & 0 \\ 0 & a_1 \end{vmatrix} = (-1)a_1 a_2$$

since there is precisely one row swap. When $n = 4$ we see

$$\begin{vmatrix} 0 & 0 & 0 & a_1 \\ 0 & 0 & a_2 & 0 \\ 0 & a_3 & 0 & 0 \\ a_4 & 0 & 0 & 0 \end{vmatrix} = - \begin{vmatrix} a_4 & 0 & 0 & 0 \\ 0 & 0 & a_2 & 0 \\ 0 & a_3 & 0 & 0 \\ 0 & 0 & 0 & a_1 \end{vmatrix} = (-1)^2 \begin{vmatrix} a_4 & 0 & 0 & 0 \\ 0 & a_3 & 0 & 0 \\ 0 & 0 & a_2 & 0 \\ 0 & 0 & 0 & a_1 \end{vmatrix} = (-1)^2 a_1 a_2 a_3 a_4$$

Continuing in this way, if n is even, we see that we require $\frac{n}{2}$ row swaps to bring the matrix into diagonal form. Hence, when n is even, the determinant is $(-1)^{\frac{n}{2}} a_1 a_2 \ldots a_n$.

We now investigate the case when n is odd. When $n = 3$ we have

$$\begin{vmatrix} 0 & 0 & a_1 \\ 0 & a_2 & 0 \\ a_3 & 0 & 0 \end{vmatrix} = - \begin{vmatrix} a_3 & 0 & 0 \\ 0 & a_2 & 0 \\ 0 & 0 & a_1 \end{vmatrix} = (-1)a_1a_2a_3$$

When $n = 5$ we have

$$\begin{vmatrix} 0 & 0 & 0 & 0 & a_1 \\ 0 & 0 & 0 & a_2 & 0 \\ 0 & 0 & a_3 & 0 & 0 \\ 0 & a_4 & 0 & 0 & 0 \\ a_5 & 0 & 0 & 0 & 0 \end{vmatrix} = - \begin{vmatrix} a_5 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & a_2 & 0 \\ 0 & 0 & a_3 & 0 & 0 \\ 0 & a_4 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & a_1 \end{vmatrix} = (-1)^2 \begin{vmatrix} a_5 & 0 & 0 & 0 & 0 \\ 0 & a_4 & 0 & 0 & 0 \\ 0 & 0 & a_3 & 0 & 0 \\ 0 & 0 & 0 & a_2 & 0 \\ 0 & 0 & 0 & 0 & a_1 \end{vmatrix}$$

Continuing in this way, if n is odd, we see that we require $\frac{n-1}{2}$ row swaps to bring the matrix into diagonal form. Hence, when n is odd, the determinant is $(-1)^{\frac{n-1}{2}} a_1a_2 \dots a_n$.

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