Excitation Wave Breaking in Excitable Media with Linear Shear Flow

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If an excitable medium is moving with relative shear, the waves of excitation may be broken by the motion. We consider such breaks for the case of a constant linear shear flow. The mechanisms and conditions for the breaking of solitary waves and wave trains are essentially different: the solitary waves require the velocity gradient to exceed a certain threshold, while the breaking of repetitive wave trains happens for arbitrarily small velocity gradients. [S0031-9007(98)07183-X]

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Introduction.—Many nonlinear dissipative distributed systems display excitable behavior; the return from a perturbed state to a single stable equilibrium state may be achieved by two qualitatively different routes, one of which entails a large excursion from equilibrium, the so called "excitation." Examples of excitability abound in chemical systems, notably in the celebrated Belousov-Zhabotinsky reaction, but also in the context of spontaneous ignition of stored solids or of leaking reactant fluids [1]. Many fundamental physiological processes are driven by excitable behavior, and excitability is important in biological morphology and some examples of population dynamics [2].

In appropriate circumstances, a distributed excitable system can support a traveling wave of excitation followed by a return to the equilibrium state, and the characteristic spiral, target pattern, or scroll wave forms have attracted enormous attention. In virtually all of this work, the supporting medium has been assumed to be at rest or in uniform motion. However, the effects of medium movement on the excitation wave dynamics have been studied for slight deformations [3], and effects of nonuniform advection onto chemical reactions but without excitable properties were studied in [4,5]. Little or no attention has been given to the situation where the excitable medium with recovery undergoes relative straining motion, as in a shear flow or even a nonuniform elastic deformation—although experiments with the Belousov-Zhabotinsky reaction have demonstrated that sufficiently strong convective motion of the chemically reacting medium can break the excitation waves [6].

In this Letter we identify and analyze the influence on excitation waves of one of the simplest examples of relative motion of the medium, a constant linear shear.

Plane waves in linear shear flows.—Mathematical models of excitable media take the form of reaction-diffusion systems of equations, and the generic reaction-diffusion system in a shear flow in the (x, y) plane can be written in the form

$$\frac{\partial u}{\partial t} = f(u) + \alpha y \frac{\partial u}{\partial x} + D\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right), \qquad (1)$$

where u is the column vector of reacting species, f represents the nonlinear reaction rates, D is the diffusion matrix, and α is the gradient of the advection velocity, $\vec{c} \equiv (\alpha y, 0)$. We assume that, for $\alpha = 0$, the $(1 + 1)$-dimensional version of this system has solutions in the form of periodic waves (solitary waves, in the limit of infinite period), with the speed and shape determined by the period (or being unique for a solitary wave), which is typical for excitation waves.

The analysis of the propagation of excitation in this system is readily performed for plane waves. It is easily seen that self-similar solutions are possible only for the trivial case of the wave propagating exactly across the flow. The generic substitution defining plane waves is

$$u(x, y, t) = v(\eta, t), \qquad \eta = xC(t) + yS(t), \qquad (2)$$

where the functions $C(t)$, $S(t)$ determine the direction of propagation of the waves, $\tan\theta(t) = S(t)/C(t)$, and are defined up to a multiplicative constant. We choose that constant so that at $t = 0$, $C(0) = \cos\theta(0)$ and $S(0) = \sin\theta(0)$. To satisfy the system (1), these functions must obey differential equations

$$\dot{C} = 0, \qquad \dot{S} = \alpha C, \qquad (3)$$

and the wave profile must obey the $(1 + 1)$-dimensional PDE system

$$\frac{\partial v}{\partial t} = f(v) + K(t)D\,\frac{\partial^2 v}{\partial \eta^2}, \qquad (4)$$

where the effective diffusion matrix $K(t)D$ is determined by a scaling factor of $K(t)$,

$$K(t) = C(t)^2 + S(t)^2 = 1 + 2\alpha t \cos\theta_0 \sin\theta_0 + \alpha^2 t^2 \cos^2\theta_0, \qquad (5)$$

$\theta_0 = \theta(0)$, and $\theta(t)$ is the angle between the normal to the wave front and the x axis (or between the wave front and the y axis). In physical space, the dependence (5) corresponds to a change of the distance between isophase lines according to the equation (see Fig. 1)

$$\lambda(t) = \lambda(0)\,(1 + 2\alpha t \cos\theta_0 \sin\theta_0 + \alpha^2 t^2 \cos^2\theta_0)^{-1/2}. \tag{6}$$

The propagation angle at time t is

$$\theta(t) = \arctan(S/C) = \arctan[\tan(\theta_0) + \alpha t]. \tag{7}$$

Thus, for plane waves, the problem reduces to that for propagation of excitation waves in a 1-dimensional cable with diffusion depending explicitly on time. Now we are going to study the conditions under which this dependence can cause a propagation block.

Solitary wave blocking.—We assume the following properties of the unperturbed version of this equation ($K \equiv 1$): (i) There exists a stable solution in the form of a solitary wave, $v(\eta, t) = V(\eta - c_v t)$, with width $L = L_{\text{norm}}$, defined, for example, as the separation between points in which some component of v has a chosen value. (ii) This solitary wave can develop from initial conditions in the form of this same wave laterally squeezed, i.e., $v(\eta, 0) = V(k\eta)$, $k > 1$, with width, $L_{\text{init}} = L_{\text{norm}}/k$, shorter than normal but longer than some minimal width, L_{min} ($L_{\text{min}} < L_{\text{init}} < L_{\text{norm}}$), but if $L_{\text{init}} < L_{\text{min}}$ the wave decays. (iii) The typical time for the development (establishment) of the wave profile is τ_{prof}. If, as is often the case, the excitation waves are supported by processes of different time scales, then τ_{prof} will be the slowest of them (i.e., the time constant of the limiting stage). The meaning of this parameter will be seen more clearly from its use below. If $K \neq 1$ but is any positive constant, then assumption (i) implies that (4) has a stationary solitary wave solution $v(\eta, t) = V(K^{-1/2}\eta - c_v t)$, with width $L_{\text{stat}}(K) = L_{\text{norm}} K^{1/2}$. If $K(t)$ is not constant but changes slowly, we may expect that the solution will have the same form as this wave, slowly adjusting its width accordingly. If $K(t)$ changes too rapidly, only then may the wave solution collapse.

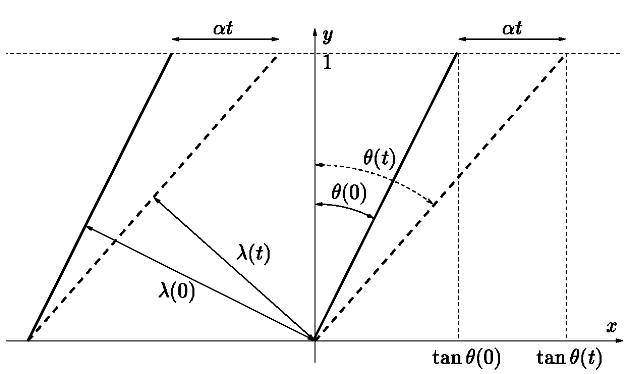

FIG. 1. Schematic diagram of the plane wave deformation by the shear flow. Bold solid lines are equiphase lines at time moment 0, and bold dashed lines are the same lines at time moment t.

With the assumptions listed above, this can be formalized in a phenomenological model. Let us assume, for simplicity, that the dynamics of the wave width is linear with constant relaxation time τ_{prof}. As the instantaneous equilibrium state should be changing in accordance with $K(t)$, this gives

$$\dot{L} = \tau_{\text{prof}}^{-1}[K^{1/2}(t) L_{\text{norm}} - L], \qquad L(0) = L_{\text{norm}},$$
$$L(t) \geq K^{1/2} L_{\text{min}} \quad \forall\, t. \tag{8}$$

This equation is in terms of the spatial variable η used in (4), which physically corresponds to the width measured in the direction of the flow. The rescaled width $\bar{L} = K^{-1/2} L$, corresponding to the real width measured perpendicularly to the wave front, then obeys

$$\dot{\bar{L}} = \tau_{\text{prof}}^{-1}(L_{\text{norm}} - \bar{L}) - \frac{\dot{K}}{2K}\bar{L}, \qquad \bar{L} \geq L_{\text{min}}. \tag{9}$$

The starting and the final asymptotic value of $\bar{L}$ is L_{norm}. In between, it decreases below L_{norm} but always remains positive. The minimal value of $\bar{L}$ is achieved at $\dot{\bar{L}} = 0$ which gives

$$1 + \tau_{\text{prof}} \frac{\dot{K}}{2K} = \frac{L_{\text{norm}}}{L_{\text{min}}}. \tag{10}$$

The system of equations (8) and (10) determines, in principle, the critical shear α_* and the corresponding time t_* of the break; however, the exact solution is rather tedious.

To obtain a simple analytical estimate, let us consider the case of practical interest, when the ratio of the activation time scale to the inhibition time scale is

$$\tau_{\text{act}}/\tau_{\text{inh}} = \epsilon \ll 1. \tag{11}$$

Propagation of excitation waves in this limit is described by the Fife approximation [7]. In particular, the limiting stage of wave formation is the establishment of its width, which happens on the time scale of $\tau_{\text{prof}} \propto \tau_{\text{inh}}$, and the minimal possible width from which the wave can recover is of the order of the activation front width, $L_{\text{min}} \propto c_v \tau_{\text{act}}$, whereas, in contrast, the normal wave width is, obviously, $L_{\text{norm}} \propto c_v \tau_{\text{inh}}$.

In this limit we should expect that the breakup occurs only if the effective diffusivity changes very quickly compared with $\tau_{\text{prof}} = \tau_{\text{inh}}$. Assuming, in a self-consistent way, that $t_* \propto \tau_{\text{act}}$ and neglecting the change of $L(t)$ during this time interval, $L(t) \approx L_{\text{norm}}$, to leading order we obtain a system of two equations:

$$L_{\text{norm}}/L_{\text{min}} = \epsilon^{-1} = K^{1/2}, \tag{12}$$

$$L_{\text{norm}}/L_{\text{min}} = \epsilon^{-1} = \left(1 + \frac{\tau_{\text{prof}}\dot{K}}{2K}\right). \tag{13}$$

Using (5) in (12), $\tan\theta(t_*) \approx \alpha_* t_* \approx \epsilon^{-1}$, then (13) implies $\tau_{\text{prof}}/t_* \approx \epsilon^{-1}$, and so the solution is

$$t_* \approx \epsilon\tau_{\text{prof}} = \tau_{\text{act}}\,,$$
$$\theta(t_*) \approx \pi/2 - \epsilon\,, \tag{14}$$
$$\alpha_* \approx \tau_{\text{prof}}^{-1}\epsilon^{-2} = \frac{\tau_{\text{inh}}}{\tau_{\text{act}}^2}\,,$$

in agreement with the original assumptions [8].

Thus, in this case, there is a *critical* shear above which the wave is quenched, and the value of this shear depends on the difference between the time scales of activator and inhibitor, which is a measure of the excitability of the system; the more excitable the system, the bigger the shear needed to destroy the wave.

Periodic wave conduction block.—For periodic wave trains the situation is completely different, as the spatial period measured in terms of the variable η (the period in the direction of the flow) is fixed. In physical space, if measured in the instantaneous direction of propagation, the spatial period changes according to (6). But in excitable media, there is always a minimal wavelength, λ_*, at which a periodic wave train can propagate, and so $\lambda(t)$ decreasing below this value is a sufficient condition for the propagation block. From (6) we obtain the following estimate of the time and the angle for which the break will occur for any value of the shear however small,

$$\theta_*^0 = \arctan(1/k), \qquad t_* = \alpha^{-1}(k - 1/k),$$
$$\theta(t_*) = \arctan(k), \tag{15}$$

where

$$k = \lambda(0)/\lambda_*\,. \tag{16}$$

Numerical illustrations.—So far we have considered only plane waves. To verify the estimates and analyze the effect of the shear flow on more complicated autowave patterns, we have performed numerical simulations for the FitzHugh-Nagumo system, with the flow incorporated, in the following form:

$$\frac{\partial E}{\partial t} = c_1 E(E - a)(1 - E) - g + \alpha y\,\frac{\partial E}{\partial x} + D\nabla^2 E\,,$$
$$\frac{\partial g}{\partial t} = \epsilon(c_2 E - g) + \alpha y\,\frac{\partial g}{\partial x} + \delta D\nabla^2 g\,. \tag{17}$$

We shall refer to the space and time units of this equation as s.u. and t.u., respectively. The parameter values were chosen: $c_1 = 10$, $a = 0.02$, $\epsilon = 0.1$, $c_2 = 5$, $\delta = 1$, and $D = 1$. We solved this system with explicit Euler scheme (forward time, centered space), with space step $h_s = 0.5$ s.u., in a rectangular medium $(x, y) \in [0, L] \times [-M/2, M/2]$ with periodic boundary conditions at $x = 0, L$ and nonflux boundary conditions at $y = \pm M/2$. The sizes of the medium, L and M, the velocity gradient, α, and the time step, h_t, were varied in different experiments. The minimum wavelength of a periodic train in quiescent

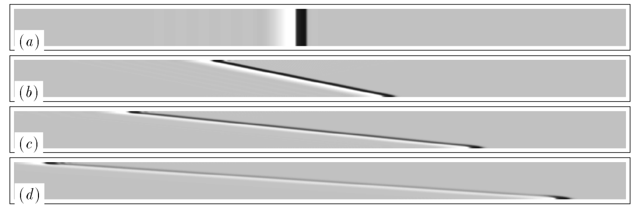

FIG. 2. Conduction block of a solitary plane wave in the linear shear flow. Velocity gradient $\alpha = 10$ t.u.$^{-1}$, medium size 500 s.u. $\times$ 30 s.u., time step $h_t = 5 \times 10^{-5}$ t.u. Shown are snapshots of the E field (darker shade corresponds to higher value) with interval 0.5 t.u. Panel (d) corresponds to the moment just before the disappearance of the wave: no excitation (black) and only the recovery tail (lighter shade) left in the medium; some excitation survived only near the boundaries.

($\alpha = 0$) medium is 19.0, and the wavelength of the spiral wave is 41.0.

The evolution of a solitary plane wave is shown in Fig. 2. The horizontally propagating wave was initiated in the quiescent medium and then the flow was switched on, which corresponds to the conditions of the analytical estimation. The activator and inhibitor are E and g and their characteristic times are, respectively, $\tau_{\text{act}} \propto 1$ t.u. and $\tau_{\text{inh}} \propto 1/\epsilon = 10$ t.u. According to (14), this means that $\alpha_* \propto 10$ t.u.$^{-1}$ for our choice of parameters. The threshold value of the velocity gradient necessary for breaking a single plane wave was found numerically to be $\alpha_* = 6$ t.u.$^{-1}$, the time $t_* = 1.5$ t.u., and the wave orientation at the moment of the break $\tan\theta(t_*) = 15$, which, to this order of magnitude, is consistent with the analytical estimates of (14), $\alpha_* = 10$ t.u.$^{-1}$, $t_* = 1$ t.u., and $\tan\theta(t_*) = 10$.

We have studied what happens to autowave structures at velocity gradients much less than this threshold. According to (15) and (16), we have for this medium $k \approx 2.16$, and at $\alpha = 0.06$ t.u.$^{-1}$ the wave break of the periodic train occurs at $t_* \approx 28$ t.u. at the angle of $\theta(t_*) \approx 1.14$ rad. These predictions, obtained for plane waves, agree, in order of magnitude, with simulations of the evolution of more complicated autowave patterns, the spiral wave and the target pattern.

In the first example (see Fig. 3) we initiated a spiral wave in a quiescent medium ($\alpha = 0$) and then switched on the shear flow. The spiral wave breaks, and the

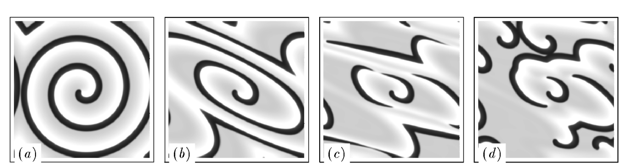

FIG. 3. Breakup of a spiral wave. Velocity gradient $\alpha = 0.06$ t.u.$^{-1}$, medium size 200 s.u. $\times$ 200 s.u., time step $h_t = 5 \times 10^{-3}$ t.u. Shown are snapshots of the E field with interval 25 t.u.

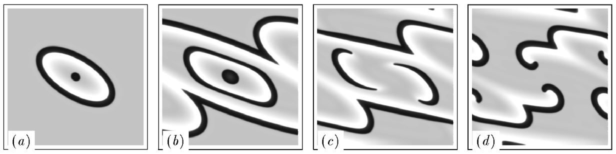

FIG. 4. Breakup of a target pattern. Velocity gradient $\alpha = 0.06$ t.u.$^{-1}$, medium size 200 s.u. $\times$ 200 s.u., time step $h_t = 5 \cdot 10^{-3}$ t.u. Shown are snapshots of the E field with interval 25 t.u.

time ($t_* \approx 48$) and orientation of the wave [$\theta(t_*) = 1.3$ rad] are in good correspondence with the theory. The correspondence in this case would be better but for the phenomenon of "plasticity" of the excitation wave: as the process is nonstationary, the visual breakup happens later than the time at which conditions of stationary propagation are violated. This explanation was confirmed by numerical experiments when the flow was stopped before the wave broke, but after it should have happened accordingly to the analytical estimate; the wave subsequently broke, despite the absence of the flow.

In the other example (see Fig. 4) we initiated a series of topologically circular waves by periodical stimulation of a point in the medium with the period equal to the period of the spiral wave. It can be seen that while the first wave propagates without problems, the propagation of the second is suppressed and the third wave is blocked. This block occurs not for the whole wave but only at some points, leading to wave breaks which curl up into spirals. The first breaks occur to the third wave, at time $t \approx 63$ t.u., that is 19.4 t.u. after its initiation, and at a propagation angle of about $\theta \approx 1.3$ rad, i.e., the same as in the previous case.

These two examples show that, at least for the particular model chosen, the conditions of the wave break are almost the same whatever the origin and shape of the pattern is, and the order of magnitude of the velocity gradient necessary for the wave break agrees with the estimate (15) obtained for plane periodic waves based on the quasistationary arguments.

As we see in Fig. 3, a spiral wave at time t_* succumbs to wave breaks which then develop into new spiral waves. Thus, in a linear shear flow a "chain reaction" of spiral wave births and deaths leads to a "frazzle gas" of excitation wavelets. The mechanism for the generation of this frazzle gas is different from that described in [9]. As this mechanism requires only a finite deformation of the medium, we may expect that it can play its role not only in constant flows, but in a more wide variety of situations. If oceanic plankton dynamics are considered as an excitable medium [10], then currents can influence their spatial dynamics.

An ODE system of analogous form to the one considered in this Letter can also describe the action of an electric field on a chemical excitable medium, where the role of the advection velocity is played by the electric field multiplied by the mobility of the reagent. If all the reagents have the same mobility, the corresponding PDE system is formally equivalent to that for reaction diffusion with convective flow [12].

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