THE CLASS OF $\mathbf{LH}\mathfrak{F}$ GROUPS

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The precise definition of the class $\mathbf{LH}\mathfrak{F}$ is stated on page 2 of this note. The class contains many groups commonly found in nature:

- all soluble-by-finite groups, and more generally all elementary amenable groups;
- all linear groups (i.e. all subgroups of $GL_n(K)$ where K is a field), and more generally all groups of automorphisms of a noetherian module over a commutative ring;
- all groups of finite cohomological dimension, and more generally all groups which admit a finite dimensional classifying space for proper actions: in particular, all groups of finite virtual cohomological dimension.
- For sufficiently large e it is known that the free Burnside groups of exponent e admit actions on contractible 2-dimensional complexes with cyclic stabilizers and these groups therefore belong to $\mathbf{LH}\mathfrak{F}$.

$\mathbf{LH}\mathfrak{F}$ satisfies a number of closure operations: it is

- subgroup closed: every subgroup of an $\mathbf{LH}\mathfrak{F}$-group belongs to $\mathbf{LH}\mathfrak{F}$;
- extension closed: if $N \lhd G$ and both N and G/N belong to $\mathbf{LH}\mathfrak{F}$ then G also belongs to $\mathbf{LH}\mathfrak{F}$; and
- closed under directed unions: a group belongs to $\mathbf{LH}\mathfrak{F}$ if and only if all its finitely generated subgroups belong to $\mathbf{LH}\mathfrak{F}$.
- Moreover, the fundamental group of a graph of $\mathbf{LH}\mathfrak{F}$-groups belongs to $\mathbf{LH}\mathfrak{F}$, in particular $\mathbf{LH}\mathfrak{F}$ is closed under forming free products with amalgamation and HNN-extensions.

The class of $\mathbf{LH}\mathfrak{F}$ groups is distinct from the class of all groups:

- Thompson's group F with presentation
$$\langle x_0, x_1, x_2, \cdots : x_i^{-1} x_j x_i = x_{j+1}\ (i < j)\rangle$$
does not belong to $\mathbf{LH}\mathfrak{F}$ because it is torsion-free of infinite cohomological dimension and of type FP_∞.

There are a number of groups arising in modern mathematics for which membership of $\mathbf{LH}\mathfrak{F}$ is undecided. These include many branch groups: see Wilson's survey [11]. In general constructions of residually finite groups using wreath products or actions on rooted trees lead to important examples.

There are two key theorems concerning $\mathbf{LH}\mathfrak{F}$-groups.

Theorem 1 (Kropholler–Mislin [9]).
Every $\mathbf{LH}\mathfrak{F}$-group of type FP_∞ has a finite dimensional classifying space for proper actions. This result has a number of important special cases:

- *Elementary amenable groups of type* FP_∞ *are virtually of type* FP.
- *Linear groups of type* FP_∞ *are virtually of type* FP.
- *Torsion-free $\mathbf{LH}\mathfrak{F}$-groups of type* FP_∞ *have type* FP.

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Theorem 2 (Benson [1, 2]).
If G is an $\mathbf{LH}\mathfrak{F}$*-group and M is a $\mathbb{Z}G$-module of type* FP_∞ *then there is a natural number n such that in any partial projective resolution*

$$P_n \to \cdots \to P_1 \to P_0 \to M \to 0$$

by finitely generated projective modules P_i, then the kernel $M' := \mathrm{Ker}(P_n \to P_{n-1})$ *is a direct summand of a $\mathbb{Z}G$-module M'' which has a finite filtration*

$$0 = L_0 \leq L_1 \leq \cdots \leq L_t = M''$$

such that for each i with $0 < i \leq t$, the section L_i/L_{i-1} is isomorphic to an induced module $\mathrm{Ind}_H^G V_i$ *for some finite elementary abelian subgroup H of G and some finitely generated $\mathbb{Z}$-torsion-free $\mathbb{Z}H$-module V_i. In particular*

- *if G is a torsion-free* $\mathbf{LH}\mathfrak{F}$*-group and M is a $\mathbb{Z}G$-module of type* FP_∞ *then M has finite projective dimension.*

Both of these theorems depend on a theory of modules and groups of type FP_∞ developed by Cornick and Kropholler in a series of papers [3, 4, 5] using techniques of complete cohomology. Benson actually proves more general results which apply to group algebras kG where k is a commutative Noetherian ring of finite global dimension. The most general form of Benson's theorem applies to strongly group graded rings, [8].

Definition (Kropholler [6, 7]). The definition of $\mathbf{LH}\mathfrak{F}$ makes use of the spirit of Philip Hall's calculus for classes of groups and closure operations. We always use the term *class of groups* to mean a class of groups closed under isomorphism. Let $\mathfrak{X}$ be a class of groups. We now define a new class $\mathbf{H}_1\mathfrak{X}$ as follows:

- G belongs to $\mathbf{H}_1\mathfrak{X}$ if and only if there is a finite dimensional contractible G-complex X with cell stabilizers in $\mathfrak{X}$. Here the notion of G-complex is as defined by tom Dieck [10]: it is a CW-complex on which G acts by self-homeomorphisms in such a way that the set-wise stabilizer of each cell coincides with its point-wise stabilizer.

We can now define a hierarchy of classes $\mathbf{H}_\alpha\mathfrak{X}$ for each ordinal α by transfinite recursion:

- if $\alpha = 0$ then $\mathbf{H}_\alpha\mathfrak{X} = \mathfrak{X}$;
- if α is a successor ordinal then $\mathbf{H}_\alpha\mathfrak{X} = \mathbf{H}_1(\mathbf{H}_{\alpha-1}\mathfrak{X})$;
- if α is a limit ordinal then $\mathbf{H}_\alpha\mathfrak{X} = \bigcup_{\beta<\alpha} \mathbf{H}_\beta\mathfrak{X}$.

We now define an operator $\mathbf{H}$ on classes of groups by

- G belongs to $\mathbf{H}\mathfrak{X}$ if and only if G belongs to $\mathbf{H}_\alpha\mathfrak{X}$ for some ordinal α.

There is also the classically defined operator $\mathbf{L}$ on classes of groups which is defined as follows:

- $\mathbf{L}\mathfrak{X}$ is the class of groups G such that every finite subset F of G is contained in a subgroup of G which belongs to $\mathfrak{X}$.

Now let $\mathfrak{F}$ denote the class of all finite groups. The class $\mathbf{LH}\mathfrak{F}$ is defined.

In a nutshell, $\mathbf{LH}\mathfrak{F}$ is the smallest $\mathbf{H}$-closed $\mathbf{L}$-closed class containing the class of finite groups.

1. Open Questions

- There is no known proof that $\mathbf{LH}\mathfrak{F}$ is distinct from $\mathbf{H}\mathfrak{F}$.
- It is unknown if there exists an ordinal α such that

$$\mathbf{H}\mathfrak{F} = \mathbf{H}_\alpha\mathfrak{F}.$$

  Such an ordinal must be ≥ 3. It is known that the subgroup of $GL_2(\mathbb{Q}(x))$ comprising matrices of form

$$\begin{pmatrix} 1 & * \\ 0 & * \end{pmatrix}$$

  does not belong to $\mathbf{H}_2\mathfrak{F}$ but does belong to $\mathbf{H}_3\mathfrak{F}$. So the class $\mathbf{H}_3\mathfrak{F}$ is genuinely bigger than the class $\mathbf{H}_2\mathfrak{F}$. It is also known that the free abelian group of rank $\aleph_\omega$ belongs to $\mathbf{H}_3\mathfrak{F}$ but does not belong to $\mathbf{H}_2\mathfrak{F}$.
- It is an open problem whether $\mathbf{H}\mathfrak{F} = \mathbf{H}_3\mathfrak{F}$.
- It is unknown which branch groups belong to $\mathbf{LH}\mathfrak{F}$.
- It is known that abelian group of cardinality $\leq \aleph_\omega$ belong to $\mathbf{H}\mathfrak{F}$. It is unknown if abelian groups of larger cardinality belong to $\mathbf{H}\mathfrak{F}$ although they obviously belong to $\mathbf{LH}\mathfrak{F}$.

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