Inertial Wave Convection in Rotating Spherical Fluid Shells

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Abstract

The report represents a study of the linear properties of thermal convection in rotating spheres and shells. An “inertial wave” analytical approximation based on the similarities of the convection flows and the solutions of the Poincaré equation is developed for the region of small Prandtl numbers. Theoretical expression for the Rayleigh number of convection with insulating thermal boundary conditions is obtained and tested against full numerical solutions. A second part of the report describes the results of a numerical study of the convection regimes and transitions found in the region of small Prandtl numbers. The transition between retrograde and prograde drifting modes is examined in detail. A new multi-columnar solution is found to bound the region of wall-attached convection at very high rotation rates. Various predictions of the developed analytical theory are also tested and verified numerically. Finally convection patterns in the case of insulating thermal boundary conditions are observed and compared with the better studied case of perfectly conducting boundaries.

1. Introduction

In this report we study the linear problem of the onset of thermal convection in rotating spheres and spherical shells.

The interest in this problem is motivated mainly by geophysical and astrophysical applications and dates back at least half a century, during which period a vast amount of literature has been accumulated. The most comprehensive formulation of the problem and some early results were presented by Chandrasekhar (1961). The fundamental theoretical work was carried out by Roberts (1968) and Busse (1970a). It was predicted that convection would be in the form of slowly drifting columnar rolls with small azimuthal scale but the precise structure of the flow was not determined by the theories. Experimental studies (Carrigan and Busse (1983)) confirmed the qualitative features predicted by the theories for the range of Prandtl numbers of their working fluids. Later numerical studies (Zhang and Busse (1987), Ardes et al. (1997)) found that the form of convection pattern is strongly dependent on almost all parameters that enter the formulation of the problem and they identified various regimes of convection at onset. A number of excellent reviews on the subject exist and the reader is referred to Busse (2002) as one of the most recent ones.

However certain mathematical and numerical difficulties prevent the complete solution of the problem. The preferred mode of convection is usually non-axisymmetric and strongly time dependent even at the onset. Another analytical difficulty arises from the geometry of the system and more precisely from the fact that the role of the Coriolis force varies with the angle between gravity and the vector of angular velocity. On the numerical side the
investigation of the linear problem is hindered by the large number of parameters, including
the Rayleigh and Prandtl numbers, the rotation rate parameter as well as the radius ratio
of the spherical shell. In addition to that various choices of the boundary conditions, the
heating model and the variation of gravity can be made at formulation. Having in mind all
these difficulties, it is not surprising that many open questions still exist and that there are
many possibilities for further investigations of the linear problem.

An efficient way of overcoming the lack of a complete solution has been to obtain ana-
ytical approximations in special cases. In the case of a thin shell the effects of rotation
can be treated as a perturbation (Busse(1970b)). An analytical method for the description
of low Prandtl number convection is based on the idea that the thermal convection can
be treated as a perturbation of inertial oscillations, which on the other hand emerge as
solutions of the Poincaré equation in rotating spheres (Zhang(1994)). Another method, ap-
licable to the same parameter region, is the equatorial approximation described by Ardes
et al.(1997). These analytical approximations need to be validated and tested against full
numerical solutions. The numerics in addition reveal many new properties and phenomena
not predicted by the theories.

Such an approach to the problem of convection in rotating spheres and shells has been
adopted in the papers of Zhang and Busse(1987) and Ardes et al.(1997). Apart from
testing various theoretical predictions, these papers report the most detailed numerical
investigations of the parameter space so far. The preferred types of convection flow at
relatively low and moderate Prandtl numbers are determined. Wall-attached regime is
observed at lower values and columnar type flows at higher values of the Prandtl number.
When the rotation parameter is varied, eigenmode competition is observed and transitions
between several new modes and patterns are identified, including modes traveling in the
retrograde or prograde direction. Many properties of these phenomena are investigated
below in detail.

These two papers have provided useful ideas and starting points for the studies un-
dertaken in the present report. Here we choose to focus our attention on the properties
of inertial wave convection, which is observed in the region of small Prandtl numbers and
intermediate to high values of the rotation parameter. This choice is motivated by several
facts. Firstly, a convenient analytical approach is possible in this parameter region. The
thermal convection can be considered to be a perturbation of inertial oscillations of the
Poincaré equation. Based on this idea we follow the method described by Zhang(1994) but
show a different approach to obtain the results published by him. Furthermore we extend
the analysis and solve the heat equation for a new case of perfectly insulating thermal
boundary conditions. As a result complete analytical convection solutions are obtained
and a theoretical expression for the critical Rayleigh number in the limit of small Prandtl
numbers and high rotation rates is derived. The validity of the results is tested against
numerical solutions of the full set of linear equations. Secondly, all previous studies agree
that at low Prandtl numbers convection is much richer in dynamical behavior than at high
Prandtl numbers of the order one or higher and that there can be regions where the pre-
ferred mode is still unknown. At the same time neither very high rotation rates nor very
small Prandtl number cases have been reached so far. Several already known phenomena
need to be studied in more detail as well. A particular goal is to outline the border between
regions of retrograde and prograde drifting modes in the $P - \tau - \eta$ space. Furthermore,
several analytical facts based on the inertial wave approximation developed in the first part of the report have to be tested. More precisely we wish to verify the prediction that in the regime of wall-attached convection the ratio between the frequency and the rotation parameter must remain constant for a broad range of rotation rates and at the same time have no radial dependence. Another prediction of the theory is the fact that the critical Rayleigh number of a particular mode does not depend on the rotation rate for small values of the Prandtl number and high rotation rates. Finally, a study of the dependences of the critical Rayleigh number and frequency on the radius ratio as well as a test of the validity of the analytical approximations for various thickness of the spherical shell has not yet been reported in previous studies.

A third part of the report, which deserves a study of its own, addresses the problem of the onset of thermal convection in rotating spherical shells, but in contrast with the second part in the case of perfectly insulating thermal boundary conditions. Since the case of insulating thermal boundary conditions has not been previously reported in the literature, we begin its exploration with a number of comparisons between the new case and the much better studied case of perfectly conducting thermal boundaries. We try to outline any differences or similarities in the Rayleigh number and frequency relationships as well as to observe whether regimes of patterns similar to the flows in the conducting case exist.

The report starts with a short description of the geometrical configuration and the formulation of the problem in section 2. Section 3 describes the inertial wave approximation and extends the results of Zhang (1994) to the case of insulating thermal boundary conditions. The numerical methods used for solving the governing equations are introduced in section 4. Section 5 presents the main results emerging from the numerical study of low Prandtl number convection. In section 7 some preliminary results from the numerical study of convection with thermally insulating boundaries are reported and concluding remarks are given in section 8.

2. Mathematical Formulation of the Problem

We consider the problem of convection in rotating spherical shells in its classical formulation with a uniform distribution of heat sources and a gravity force that increases with distance from the center of the sphere. Accordingly the distributions of temperature and gravity in the spherically symmetric basic state are given by

\[ T = T_0 - \beta \tilde{r}^2/2, \quad g = -\gamma \tilde{r} \]

where \( \tilde{r} \) is the position vector with respect to the center of the sphere and \( \tilde{r} \) is its absolute value. The sphere is rotating with angular velocity \( \Omega \) about a fixed axis given by the unit vector \( \mathbf{k} \). It will be convenient to introduce dimensionless variables. As length scale we use the difference \( d \) between inner and outer radius of the spherical fluid shell. As time scale we use \( d^2/\nu \) where \( \nu \) is the kinematic viscosity of the fluid and as temperature scale we take \( \beta d^2\nu/\kappa \) where \( \kappa \) denotes the thermal diffusivity of the fluid. The dimensionless equations of motion for the velocity vector \( \mathbf{u} \) and the heat equation for the deviation \( \Theta \) of the temperature from the static distribution are given by

\[ \frac{\partial}{\partial t} \mathbf{u} + \mathbf{u} \cdot \nabla \mathbf{u} + \tau \mathbf{k} \times \mathbf{u} = -\nabla \pi + r R \Theta + \nabla^2 \mathbf{u} \]

\[ (2.2a) \]
\[ \nabla \cdot \mathbf{u} = 0 \quad (2.2b) \]

\[ \left( \frac{\partial}{\partial t} \Theta + \mathbf{u} \cdot \nabla \Theta \right) P = r \cdot \mathbf{u} + \nabla^2 \Theta \quad (2.2c) \]

where \( r \) is the dimensionless position vector and the Rayleigh number \( R \), the Taylor number \( \tau^2 \), and the Prandtl number \( P \) are defined by

\[ R = \frac{\alpha \gamma \beta d^6}{\nu \kappa}, \quad \tau = \frac{2 \Omega d^2}{\nu}, \quad P = \frac{\nu}{\kappa} \quad (2.3) \]

The thermal expansivity has been denoted by \( \alpha \) and the Boussinesq approximation has been assumed.

Stress-free velocity boundary conditions and two different types of temperature boundary conditions are imposed on the governing equations (2.2)

\[ \frac{\partial}{\partial r} \left( \frac{u_\phi}{r} \right) = \frac{\partial}{\partial r} \left( \frac{u_\theta}{r} \right) = 0 \]

Type A: \( \Theta = 0 \)

Type B: \( \frac{\partial \Theta}{\partial r} = 0 \)

at \( r = r_i \equiv \frac{\eta}{1 - \eta} \) and \( r = r_0 \equiv \frac{1}{1 - \eta} \), \quad (2.4)

where \( \eta \) denotes the radius ratio of the spherical shell.

This is a general formulation which is not restricted to the linear problem. In the present report we neglect the nonlinear terms in equations (2.2) and use them as a starting point for both the analytical and the numerical studies reported below.

### 3. The Inertial Wave Approximation

In this section we present an analytical approximation for the description of the linear properties of thermal convection in the region of small Prandtl numbers and intermediate to high values of the rotation parameter. We follow the perturbation analysis as described by Zhang(1994) and consider the thermal convection as a small perturbation of the inertial oscillations of the Poincaré equation, the solutions of which are known, but utilize a different approach in order to improve his results and extend the analysis to a new case of convection with insulating thermal boundary conditions.

The justification for such a perturbation analysis is based on the exceptionally good agreement between the numerical values of the frequencies of the preferred modes of convection and the frequencies of the Poincaré inertial modes which will be demonstrated in the following section 5 of the report.

#### The Perturbation Analysis

We consider the linear problem in rotating fluid spheres and omit all nonlinear terms. Since we are interested in the small Prandtl number limit we neglect the time derivative in the
heat equation (2.2c). After substituting the time derivative of the velocity field by its eigenvalue \( \frac{\partial}{\partial t} \rightarrow i\omega \) the basic equations (2.2) reduce to,

\[
i\omega U + k \times U = -\nabla \pi + \frac{R}{\tau} r\Theta + \frac{1}{\tau} \nabla^2 U
\]  

\[
\nabla \cdot U = 0
\]  

\[
\nabla^2 \Theta = -r \cdot U.
\]  

In order to benefit from the results presented in Zhang(1994), for technical convenience, and because the approximation is valid for fluid spheres rather than shells, we need to rescale our basic equations. The connection with the dimensionless parameters (2.3) defined in the preceding section and used in the numerical analysis in the rest of the report is,

\[
R = R_{eq.(2.3)}(1 - \eta)^6, \quad \tau = \tau_{eq.(2.3)}(1 - \eta)^2, \quad \omega = -\omega_{eq.(2.3)}/\tau
\]  

Stress-free velocity boundary conditions and conducting or insulating temperature boundary conditions, as given by equation (2.4) are assumed. We expand the dependent variables,

\[
U = u + u_1, \quad \pi = \pi_0 + \pi_1, \quad \Theta = \Theta, \quad \omega = \omega_0 + \omega_1,
\]  

where \( u_1, \pi_1, \Theta \) and \( \omega_1 \) are small perturbations from the solutions of the Poincaré equation for the limit \( 1/\tau \rightarrow 0 \), which are denoted by \( u, \pi \) and \( \omega_0 \). The above expansion is valid for

\[
\frac{1}{m^\frac{5}{2}} < 1.
\]  

Substituting the expansions into (3.5)–(3.7), gives the zeroth order of the perturbation problem,

\[
i\omega_0 u + k \times u = -\nabla \pi_0
\]  

\[
\nabla \cdot u = 0.
\]  

This system is equivalent the the Poincaré equation as shown in Greenspan(1968). The complete analytical solution of the Poincaré equation has been recently obtained by Zhang et al.(2001), but here we make use only of a particular class of solutions as will be described shortly. The next order of the perturbation analysis in the limit \( \eta = 0 \) gives rise to,

\[
i\omega_0 u_1 + k \times u_1 = -\nabla \pi_1 + \frac{R}{\tau} r\Theta + \frac{1}{\tau} \nabla^2 (u + u_b) - i\omega_1 u
\]  

\[
\nabla \cdot u_1 = 0
\]  

\[
\nabla^2 \Theta = -r \cdot u.
\]
where \( u = u_i + u_b \), \( u_i \) is the perturbation of the interior flow and \( u_b \) is the boundary flow associated with the Ekman boundary layer, which is non-zero only in the vicinity of the outer spherical surface. While \( u_i \) is small compared to \( u_b \), \( u_b \) has to be large enough so that \( u_b + u \) satisfies the stress-free boundary condition. It can be shown that the Ekman layer plays an essential role because the integral \( H_b = \int_V u^* \cdot \nabla^2(u + u_b) dV \) appearing later on in the expression for the Rayleigh number is non-zero only if \( u_b \) is non-zero (Zhang(1994)). If we multiply eq. (3.12) by the complex conjugate of \( u \), \( u^* \), which also satisfies \( \nabla \cdot u^* = 0 \) and the boundary condition \( u^*_r = 0 \) and integrate over the volume of the sphere, we obtain

\[
\int_V u^* \cdot (i\omega_0 u_1 + 2k \times u_1 + \nabla \pi_1) dV = \int_V u_1 \cdot (i\omega_0 u^* - 2k \times u^*) dV. \tag{3.15}
\]

We use,

\[
\int_V u^* \cdot \nabla \pi_1 dV = \int_s \pi_1 u^*_r dS = 0 \tag{3.16}
\]

and the fact that \( u^* \) satisfies,

\[
i\omega_0 u^* + k \times u^* = \nabla \pi_0^*
\]

and observe that the integral given by (3.15) vanishes. Then the solvability conditions, where the real part gives the critical Rayleigh number and the imaginary part gives the correction of the frequency are,

\[
Re \left[ R \int_V u^* \cdot r \Theta dV \right] = -Re \left[ \frac{1}{\tau} \int_V u^* \cdot \nabla^2(u + u_b) dV \right] \tag{3.17}
\]

\[
Im \left[ R \int_V u^* \cdot r \Theta dV \right] = -Im \left[ \frac{1}{\tau} \int_V u^* \cdot \nabla^2(u + u_b) dV \right] + \omega_1 \int_V u^* \cdot u dV.
\]

Hence the critical Rayleigh number of a particular mode is given in the first order by,

\[
R = \frac{-\int_V u^* \cdot \nabla^2(u + u_b) dV}{\int_V u^* \cdot r \Theta dV} = -\frac{H_b}{H_0}. \tag{3.18}
\]

Now in order to evaluate (3.19) we need to obtain a solution \( \Theta(r) \) of the heat equation (3.14).

**Solutions of the Zeroth Order Problem and the Heat Equation**

A general solution of the Poincaré equation is available (Zhang et al. (2001)), but for the purposes of the present report only a particular class of solutions, as suggested by the numerical analysis of the problem is relevant. In this subsection we demonstrate a convenient way to construct such solutions and using them solve the heat equation for both types of thermal boundary conditions.
We first consider the zeroth order problem in the perturbation analysis (3.11). In order to eliminate the continuity equation we introduce the general representation for the solenoidal vector field \( \mathbf{u} \),

\[
\mathbf{u} = \nabla \times (\nabla \times \mathbf{r} v) + \nabla \times \mathbf{r} w
\]  

(3.19)

where the scalar variables \( v \) and \( w \) are uniquely defined if the condition is imposed that their averages over surfaces \( \mathbf{r} = \text{const.} \) vanish. By acting with the operators \( \mathbf{r} \cdot \nabla \times \nabla \times \) and \( \mathbf{r} \cdot \nabla \times \) on the first equation of the system we obtain two equations for the poloidal and toroidal scalars \( v \) and \( w \),

\[
(-i\omega L_2 + im)\nabla^2 v = -Q w
\]  

(3.20)

\[
(-i\omega L_2 + im)w = Qv,
\]  

(3.21)

where \( L_2 \) is the negative Laplacian on the unit sphere and \( Q \) is defined by

\[
Q \equiv \mathbf{k} \cdot \nabla - \frac{1}{2} (L_2 \mathbf{k} \cdot \nabla + \mathbf{k} \cdot \nabla L_2).
\]  

(3.22)

The boundary conditions are also transformed to,

\[
v = \frac{\partial^2}{\partial r^2} v = \frac{\partial}{\partial r} \left( \frac{w}{r} \right) = 0, \text{ at } r = 1.
\]  

(3.23)

The expressions,

\[
v = (r^m - r^{m+2})P_m^m \exp(i\omega t + im\phi)
\]  

(3.24)

\[w = r^{m+1} \frac{1}{\omega(m+1)(m+2) - m(2m+1)} P_{m+1}^m \exp(i\omega t + im\phi),\]

provide an exact solution of the zeroth order problem for

\[
\omega_{\pm} = \frac{1}{m+2} \left( 1 \pm \sqrt{\frac{m^2 + 4m + 3}{2m + 3}} \right)
\]  

(3.25)

as can be easily verified by substituting (3.24) into (3.21) and using formula (7.41) of Appendix A.

Now the heat equation,

\[
\nabla^2 \Theta = -\mathbf{r} \cdot \mathbf{u} = -L_2 v = -(r^m - r^{m+2})m(m+1)P_m^m \exp(i\omega t + im\phi),
\]  

(3.26)

suggests that \( \Theta(r) \) contains a single spherical harmonic of the same order and degree as the poloidal scalar, and its radial dependence can be obtained as a solution of the radial part of (3.26),

\[-m(m+1)(r^m - r^{m+2}) = \nabla^2 f(r) = \left( \frac{1}{r} \frac{\partial^2}{\partial r^2} - \frac{m(m+1)}{r^2} \right) f(r).\]
Assuming a radial dependence of the type,

\[ f(r) = K_1 r^{c_1} + K_2 r^{c_2}, \]

plugging it into the radial equation and determining the constants \( K_1, K_2, c_1, c_2 \), we obtain the solution,

\[
\Theta(r) = -m(m+1)P_m^m \exp(i\omega t + im\phi) \times \\
\times \left[ \frac{r^{m+2}}{(m+3)(m+2)-(m+1)m} - \frac{r^{m+4}}{(m+5)(m+4)-(m+1)m} \right],
\]

where a harmonic term \( ar^m \) has been added to the radial dependence. After using the insulating and the conducting thermal boundary conditions (2.4) of Types A and B at \( r = 1 \) and determine the constant \( a \) in both cases we finally obtain,

\[
\Theta(r) = -(m+1)P_m^m \exp(i\omega t + im\phi) \times \\
\times \left[ \frac{mr^{m+2} - (m+2)r^m}{(m+3)(m+2)-(m+1)m} - \frac{mr^{m+4} - (m+4)r^m}{(m+5)(m+4)-(m+1)m} \right]
\]

for the insulating case and

\[
\Theta(r) = -m(m+1)P_m^m \exp(i\omega t + im\phi) \times \\
\times \left[ \frac{r^{m+2} - r^m}{(m+3)(m+2)-(m+1)m} - \frac{r^{m+4} - r^m}{(m+5)(m+4)-(m+1)m} \right]
\]

for the conducting one.

**Evaluation of the Integrals**

After having obtained the solutions of the zeroth order problem and the heat equation the integrals \( H_b \) and \( H_\theta \), which appear in the solvability condition (3.18), need to be evaluated.

Making use of the formula for the components of a solenoidal vector in terms of poloidal and toroidal scalars,

\[
\mathbf{u} = \left( \frac{L_2}{r} \frac{1}{r} \partial^2 r^v + \frac{1}{\sin \theta} \frac{\partial}{\partial \phi} w, \frac{1}{r} \frac{\partial}{\partial \phi} r^v - \frac{\partial}{\partial \theta} w \right),
\]

we proceed to the evaluation of the first integral.

\[
H_b = - \int \mathbf{u}^* \cdot \nabla^2 (\mathbf{u} + \mathbf{u}_b) \, dV = - \int \mathbf{u}^* \cdot \nabla^2 \mathbf{u}_b \, dV = \int \mathbf{u}^* \cdot \left( \mathbf{r} \cdot \nabla \frac{\mathbf{u}_b}{r} \right) \, dS
\]

\[
= \int \left( u^*_\phi \frac{\partial}{\partial r} u^*_{\phi} + u^*_\theta \frac{\partial}{\partial r} u^*_{\theta} \right) \, dS
\]

\[
= 2\pi \int_0^\pi \left\{ \left( \frac{1}{r} \frac{\partial^2}{\partial \phi r} r^v + \frac{1}{\sin \theta} \frac{\partial}{\partial \theta} w^* \right) \left( \frac{\partial^2}{\partial \phi r} r^v + \frac{1}{\sin \theta} \frac{\partial^2}{\partial \theta r} w \right) \right\} \sin \theta \, d\theta
\]

\[
+ 2\pi \int_0^\pi \left\{ \left( \frac{1}{\sin \theta} \frac{\partial}{\partial \phi r} r^v - \frac{\partial}{\partial \theta r} r^v \right) \left( \frac{1}{r} \frac{\partial^2}{\partial \theta r} r^v - \frac{\partial}{\partial \theta} w^* \right) \right\} \sin \theta \, d\theta
\]

\[
= 2\pi m(m+1)[2(4m+2) \int_0^1 |P_m^m|^2 d(\cos \theta) + \\
+ (m+2) \frac{(m+1)^2 - 1}{\omega(m+1)(m+2) - m(2m+1)} \int_0^1 |P_m^m|^2 d(\cos \theta)].
\]
The second integral may also be readily calculated. For the insulating thermal boundary case we have,

\[
H_\theta = \int_v u^* \cdot r \Theta dV = \int_v L_2 v^* \Theta dV
\]

\[
= 2 \pi m (m + 1)^2 \int_0^1 |P_m^m|^2 |P_m^m| d(\cos \theta) \ast
\]

\[\ast \int_0^1 (r^m - r^{m+2}) \left[ \frac{mr^m + 1 - (m + 4)r^m}{(m + 3)(m + 4) - (m + 1)m} - \frac{mr^{m+2} - (m + 2)r^m}{(m + 3)(m + 2) - (m + 1)m} \right] r^2 dr \]

\[
= \frac{8 \pi m (m + 1)^2 (14m^2 + 59m + 63)}{(2m + 9)(2m + 7)(2m + 5)^2(2m + 3)^2} \int_0^1 |P_m^m|^2 d(\cos \theta),
\]

and in an analogous manner for the perfectly conducting case,

\[
H_\theta = 2 \pi m (m + 1)^2 \int_0^1 |P_m^m|^2 |P_m^m| d(\cos \theta) \ast
\]

\[\ast \int_0^1 (r^m - r^{m+2}) \left[ \frac{r^{m+2} - r^m}{(m + 3)(m + 2) - (m + 1)m} - \frac{r^{m+4} - r^m}{(m + 5)(m + 4) - (m + 1)m} \right] r^2 dr \]

\[
= \frac{2 \pi m^2 (m + 1)^2 (40m + 108)}{(2m + 9)(2m + 7)(2m + 5)^2(2m + 3)^2} \int_0^1 |P_m^m|^2 d(\cos \theta).
\]

**The Complete Analytical Solution**

After evaluating the integrals \(H_b\) and \(H_\theta\) we can readily write the expression for the critical Rayleigh number for a given \(m\),

\[
R = m^2 (m + 2)^3 \left\{ \left[ \pm (m + 1) \sqrt{ \frac{m^2 + 4m + 3}{2m + 3} + 1 } \right]^2 (2m + 3)^{-1} + (2m + 1) \right\} \ast \left( \frac{1}{2m + 9}(2m + 7)[(2m + 5)(2m + 3)]^2 \right) \left[ \frac{(m + 1)(14m^2 + 59m + 63)^{-1}}{(m + 1)m(10m + 27)} \right]^{-1}
\]

insulating case

conducting case

where the following property of the associated Legendre polynomials is used,

\[
\frac{\int_0^1 |P_{m+1}^m|^2 d(\cos \theta)}{\int_0^1 |P_m^m|^2 d(\cos \theta)} = \frac{(2m + 1)^2}{2m + 3}.
\]

As a result of the perturbation analysis we obtained a complete solution for the convection in the limit of small Prandtl numbers and high rotation rates, which in the first order is represented by

\[
(R, m, u, \omega_0, \Theta),
\]

where all state variables depend only on \(m\). In order to obtain the critical Rayleigh number at onset, one has to loop through all values of \(m\) and select the smallest possible \(R\). Before proceeding to the numerical analysis several properties of the obtained solution must be
outlined. Firstly, we notice that the frequency \( \omega \) of the zeroth order solution has two values - a positive value corresponding to a retrograde drifting convection pattern and a negative one corresponding to a prograde drifting mode. These two modes are competing at onset and will be demonstrated in the numerical analysis in the following sections. Secondly, we observe that for a given \( m \) the critical Rayleigh number (3.33) does not depend on the radial component as well as on the rotation rate in this approximation. These are particularly interesting features that we will address in the numerical simulations reported below.

**Numerical Test of the Analytical Solution**

Using the numerical method described in the following section we check the obtained analytical result (3.33) against numerical values. Since the approximation is derived for \( \tau \rightarrow \infty \), \( P \rightarrow 0 \) and \( \eta = 0 \) we use values of these parameters as close to the analytical assumptions as numerically reasonable. The comparison is represented in Fig. 1 with the numerical results for several different values of \( m = 2, 4, 8, 12 \) for parameters \( P = 10^{-6}, \eta = 0.1 \) and varying \( \tau \). Naturally a strict independence of \( R \) on \( \tau \) as predicted by (3.33) can not be expected but an approximate independence is fulfilled over a large span of values of \( \tau \). At very low and high values of \( \tau \) the convection enters other regimes of flow as will be discussed in the following sections.

![Figure 1: Comparison of numerical results (solid lines) with the analytical expression (dashed lines) for \( P = 10^{-6}, \eta = 0.1 \) and \( m = 2, 4, 8, 12 \) in direction of the arrow.](image)

4. **Numerical Methods**

Following the earlier analyses by Zhang and Busse(1987) and Ardes et al.(1997), we use a Galerkin method for the numerical solution of equations (2.2).

In a manner similar to the described in the previous section we transform the full
nonlinear vector equation (2.2a) to a couple of scalar equations for \( v \) and \( w \),

\[
[(\nabla^2 - \frac{\partial}{\partial t}) L_2 + \tau \mathbf{k} \times \mathbf{r} \cdot \nabla] \nabla^2 v + \tau Qw - RL_2 \Theta = -\mathbf{r} \cdot \nabla \times (\nabla \times (\mathbf{u} \cdot \nabla \mathbf{u}))
\]

(4.35a)

\[
[(\nabla^2 - \frac{\partial}{\partial t}) L_2 + \tau \mathbf{k} \times \mathbf{r} \cdot \nabla] w - \tau Qv = \mathbf{r} \cdot \nabla \times (\mathbf{u} \cdot \nabla \mathbf{u}).
\]

(4.35b)

The boundary conditions are given by (2.4) and modified as in (3.24) at \( r = r_i \equiv \eta_1 - \eta \) and \( r = r_0 \equiv \frac{1}{1 - \eta} \).

In order to solve equations (4.35) and (2.2c) by the Galerkin method, it is convenient to expand the dependent variables into complete systems of functions satisfying the boundary conditions,

\[
v = \sum_{\nu, l, n} a_{\nu ln} \exp \{i \nu (m \varphi - \omega t)\} P_{l}^{\nu|m}(\cos \theta) \sin n\pi (r - r_i)
\]

(4.36a)

\[
w = \sum_{\nu, \hat{l}, n} c_{\nu ln} \exp \{i \nu (m \varphi - \omega t)\} P_{\hat{l}}^{\nu|m}(\cos \theta) \cos (n - 1)\pi (r - r_i)
\]

(4.36b)

Type A:

\[
\Theta = \sum_{\nu, l, n} b_{\nu ln} \exp \{i \nu (m \varphi - \omega t)\} P_{l}^{\nu|m}(\cos \theta) \sin n\pi (r - r_i)
\]

Type B:

\[
\Theta = \sum_{\nu, \hat{l}, n} b_{\nu ln} \exp \{i \nu (m \varphi - \omega t)\} P_{\hat{l}}^{\nu|m}(\cos \theta) \cos (n - 1)\pi (r - r_i). \quad (4.36c)
\]

Note the difference in the sign of \( \omega \) in (4.36) and (3.25). This representation has been chosen in such a way that solutions in the form of drifting waves which are \( m \)-periodic in the azimuthal direction are described by constant coefficients \( a_{\nu ln} \) etc. More complex solutions can be described by time dependent coefficients \( a_{\nu ln} \) etc. In both cases the conditions

\[
a_{\nu ln} = a_{-\nu ln}^+ , \quad b_{\nu ln} = b_{-\nu ln}^+ , \quad c_{\nu ln} = c_{-\nu ln}^+
\]

(4.37)

must be satisfied for real expressions (4.36) where the superscript + indicates the complex conjugate. Of particular interest are solutions that are symmetric with respect to the equatorial plane in which case the subscript \( l \) runs through \( | \nu | m + 2j \) for \( j = 0, 1, 2 \ldots \) while the subscript \( \hat{l} \) runs through \( | \nu | m + 2j + 1 \) for the same \( j \). The associated Legendre polynomials will be assumed in such a form that the average of \( [P_{l}^{\nu|m}]^2 \) over the unit sphere is unity.

After the equations for the coefficients \( a_{\nu ln} \) etc. have been obtained through a projection of the basic equations (4.35) and (2.2c) onto the space of the expansion functions used in (4.36) the system of equations for the coefficients must be truncated. We shall use the
truncation condition that all coefficients and corresponding equations are neglected whose subscripts satisfy

\[ 2n + l - |\nu| m + 2 |\nu| > 3 + 2N_T \]  \hspace{1cm} (4.38)

The same condition applies for \( \hat{l} \) instead of \( l \). The condition is the same as used by Zhang and Busse (1987) and provides a triangular truncation such that as many functions in the radial as in the latitudinal direction are used for the representation of the solution.

For the linear problem of the onset of convection the right hand side of eqs. (4.35) and the term \( \mathbf{u} \cdot \nabla \Theta \) in eq. (2.2c) can be neglected and only terms with \( \nu = 1 \) need to be kept in the representation (4.36). The linear homogeneous system of complex algebraic equations for the coefficients \( a_{1l} \) represents an eigenvalue problem for \( R \) and \( \omega \) as the real and imaginary parts of the eigenvalue.

For the most cases presented in this report a truncation parameter of \( N_T = 18 \), or a total of 513 coefficients have been used. In a few cases in the high rotation regime the resolution was increased to \( N_T = 30 \), which already poses significant requirements on the computer capacity and computational times.

5. Onset of Thermal Convection at Low Prandtl Numbers and High Rotation Rates

In this section we present the basic results gained from the numerical investigation of the onset of convection at low Prandtl numbers in the case of conducting thermal boundary conditions of the type A and address the questions posed in the introductory section of the report.

The parameter exploration included a large number of points in the ranges of \( 10^{-5} \leq P \leq 10 \), \( 0 < \tau \) and \( 0.1 \leq \eta \leq 0.8 \). The computations with \( P \geq 1 \) do not actually belong in the low Prandtl number region and are only included for the sake of finding the border between the retrograde and prograde drifting modes, since these modes are found to exist even at Prandtl numbers of order unity and higher. Of course, the scenario of the evolution of the onset of convection with increasing rotation parameter \( \tau \), as will be described below, is valid for a smaller region, which we believe to be approximately \( 10^{-5} \leq P \leq 10^{-2} \).

Ideally the proper way to investigate a given parameter region is to keep the values of all parameters of the problem fixed and vary in a continuous way only one of them. When the dependence on this parameter is well understood the process is repeated for all remaining parameters. In practice, of course, this is impossible. In an experimental situation for example, the variation of the Prandtl number is limited to the set of available working fluids. In a numerical study one has much weaker restrictions, but even then parameters such as the Prandtl number \( P \) and the radius ratio \( \eta \) are comparatively difficult to vary because of their relatively small domain and big changes of the properties of the flow with small changes of the values of these parameters. On the other hand the rotation rate can be varied comparatively easily in a wide range and as a result much smoother dependences are observed. Furthermore, any non-monotonic behavior guarantees a significant and well-defined transition between different states.
Following this approach, numerous cases in the low Prandtl number region were investigated. Fig. 2 represents a typical example of the results obtained in this region and provides an excellent overview and introduction to most of the various regimes of convection flow that can be expected here. Although all other parameters, $P = 0.0001$, $\eta = 0.3$, $m = 8$, are fixed, this case is a typical example and is situated in the middle of the parameter region of interest. No substantial qualitative differences have been observed at other parameter values. The $\omega/\tau$ curve is particularly instructive and several different states of the preferred mode can be immediately noticed.

At very low rotation parameters of order unity the convection cells form near the outer
Figure 4: In the middle part: the Rayleigh numbers $R$ of the competing prograde (dotted line, empty circles) and retrograde mode (dashed line, filled circles) as well as its actual critical value (thick solid line) and the corresponding frequencies as a function of $\tau$ in the case $P = 0.001$, $\eta = 0.2$, $m = 6$. On the left and right: Contours of constant radial velocity $u_r$ (down) and toroidal scalar $w$ on the spherical surface $r = 0.9$ at both sides of the transition for $\tau = 58000$ and 64000.

surface in the equatorial region of the spherical shell as can be seen in Fig. 3, which illustrates the various states of the flow in an equatorial projection. At this values of the rotation parameter $\tau$ the critical Rayleigh number is still rather small. The preferred mode has a negative frequency which indicates a retrograde drift with time. This drift cannot, of course, be seen in the snapshots of Fig. 3. The solution has a relatively small toroidal component, which vanishes in the limit $\tau \rightarrow 0$.

A new mode which does not exist at $\tau = 0$ enters the picture and approaches the initial mode indicated above. As a result of the switch-over phenomenon described by Zhang and Busse (1987), the $R(\tau)$ and $\omega(\tau)$ curves of the two competing modes exhibit smooth bends, but for some more time the initial mode is still preferred. During this gradual transition the pattern of convection also changes gradually. The convection rolls are no longer straight and strictly radially oriented as they were near $\tau = 0$, but change shape and become inclined as illustrated by the second plot of Fig. 3. The frequency still keeps its negative sign and the pattern exhibits a retrograde drift.

Past a particular value of the rotation parameter $\tau$, the value of the critical Rayleigh number $R$ of the competing mode becomes so much lower than the value of $R$ of the initially preferred mode that an abrupt jump occurs. This is especially obvious from the discontinuity of the $\omega(\tau)$ curve in Fig. 2. During this transition the frequency even changes sign. The change of sign indicates a change in the direction of the drift of the pattern. In the new state the preferred mode drifts in the prograde direction. Because the effects of the switch-over competition still act on the $\omega(\tau)$ curve of the new mode, it is also bend, which as before results in a pattern of spiraling rolls, as can be noticed in the third plot of Fig. 3.

In a completely analogous way a second transition happens and the preferred mode
changes back to retrograde drift. This next transition occurs at about \( \tau = 8000 \) for the parameters of the case represented on Fig. 2 and brings the flow in the region described well by the inertial wave approximation.

We would like to draw the attention to the last two transitions seen in Fig. 2. After the second transition and between values of about \( 8000 < \tau < 3.5 \cdot 10^5 \), in the particular case of Fig. 2, the convection is strictly in the, so called, wall-attached regime which is well-described by the inertial wave as well as the equatorial approximation of Ardes et al. (1997). The theoretical values given by (3.25) have been plotted with dotted lines in Fig. 2. As can be seen they agree perfectly with the numerical values. It is worth noting that these values do not depend on \( \tau \). What is more they do not depend on the aspect ratio as well and this will be a subject of further discussion in the present report. Since the expression (3.25) gives the frequencies of the inertial oscillations, found as solutions of the Poincaré equation in rotating spheres, this perfect agreement with the numerically computed frequency of the preferred mode of the convective flow is the basis for the argument that convection is small perturbation of inertial oscillations in this particular parameter regime. In the middle of the region the convection undergoes a new transition from a retrograde to prograde drifting modes, which is illustrated for somewhat different parameters in Fig. 4. These modes represent the generic case of wall-attached type of convection and the most notable difference between them is the direction of their drift. The two solutions differ mainly by a phase shift in the poloidal part of the velocity field and the corresponding amplitudes are quite close to each other. As can be seen in Fig. 4 at lower
Figure 6: The streamfunctions $r \sin \theta \partial_{\theta} v$ in the equatorial plane illustrating the transition to mult-columnar state in the case $P = 10^{-4}$, $\eta = 0.6$, $m = 6$ and values of $\tau = 10^7$ for $2 \cdot 10^7$.

Figure 7: The boundary separating the retrograde (below the surface) and prograde mode (above the surface) for a fixed value of the wavenumber $m = 6$.

value of the rotation parameter $\tau$ the retrograde mode has a lower critical Rayleigh number and therefore is preferred to the prograde mode. As $\tau$ increases the Rayleigh number $R$ of the retrograde mode grows faster than that of the competing prograde mode and at some critical value of $\tau$ eventually becomes larger. At this point the transition from retrograde to prograde mode occurs. The frequency exhibits a discontinuity, changes sign and the whole pattern starts drifting in the opposite direction. Apart from this obvious process no other physical reason for the transition is identified. Mathematically the two frequencies emerge as the two roots of a quadratic dispersion relation and the inertial wave approximation does not provide an explanation for the transition, since according to (3.33), the retrograde mode has a smaller Rayleigh number and thus is always preferred in contrast to the numerical results. Thus one may suggest that an explanation for the transition must be sought in higher orders of the perturbation analysis.

The last transition observed in Figs. 2 and 3 and shown in more detail in Fig. 5 is arguably the most interesting one of all. It was completely unexpected and represents a
novel pattern observed for the first time at onset of convection. Examining the plots of the Rayleigh numbers and the frequencies of the competing modes, the same mechanism of transition as the one just described in connection with the retrograde - prograde switch is identified. But in all other aspects this transition is rather different from the previous one. Although the frequency again exhibits a finite discontinuity, it does not change sign this time and the pattern continues to drift in prograde direction. A major change in the structure of the convection solution can be observed. Besides the wall-attached mode which persists, several other concentric layers of convection columns appear. The whole pattern resembles a chess play-board in the sense that the streamlines of any given column are directed in the opposite direction with respect to the streamlines of all neighboring columns. This multi-column solution seems to emerge as a result of the competition between the wall-attached type convection and the effects of the extremely high rotation which asymptotically favors columnar structures at a distance of about half the radius of the outer spherical shell (Busse(1970)). While in other parameter regions this competition leads to a multi-hump solutions and finally to spiraling columnar convection, here the tendency for wall-attachment is compatible in strength to the effects of high rotation and leads to a breaking of the equatorially wall-attached cell into multiple columns that obey more closely the Taylor constraint of rotation. In contrast to the multi-hump and spiraling columnar solutions the multi-columns are strictly oriented in radial direction. As might be seen from the plots of $\frac{\omega}{\tau}$, this transition does not occur in a gradual manner but rather sharply at a well-determined value of $\tau$. The transition to multi-columnar convection is found to bound from above the regime of purely wall-attached convection at high rotation rates. Although the precise border between the two regimes is not investigated in the present report, the switch to multi-columnar convection has been found in all examined cases, regardless of the radius ratio $\eta$ and for all Prandtl numbers $P \leq 10^{-2}$. However at large $\eta$ the number of concentric layers diminishes because of the smaller size of the shell gap. As an illustration of this Fig. 6 is presented, for $\eta = 0.6$, $m = 6$, $P = 0.0001$. Here an interesting way of conforming to the wall-attachment can be noticed in the first plot.

Figure 8: The ratio between the frequency $\omega$ and the rotation parameter $\tau$ of the retrograde(negative values) and prograde modes(positive values) as a function of $\eta$ in the case $P = 10^{-5}$ for three different wavenumbers as indicated in the plot.
Having examined in some detail the transitions between the various modes preferred at onset, we turn our attention to the other questions posed in the introductory section.

![Graph: The Rayleigh number $R$ as a function of $\tau$ in the case $P = 10^{-4}$, $m = 8$ for different values of $\eta$ as indicated in the plot. A power law fit of the $\eta = 0.4$ curve is plotted with a dashed line.]

Figure 9: The Rayleigh number $R$ as a function of $\tau$ in the case $P = 10^{-4}$, $m = 8$ for different values of $\eta$ as indicated in the plot. A power law fit of the $\eta = 0.4$ curve is plotted with a dashed line.

Fig. 7 represents an attempt of a precise determination of the border between the retrograde and prograde mode of the wall-attached convection in $P - \tau - \eta$ space. The construction of this border has been a major computational effort, since many of the points fall in the regions of very low Prandtl numbers and at the same time of very high rotation rates $\tau$, where increased numerical resolution and long computational times were required. As can be easily noticed, the surface is much smoother towards lower values of the radius ratio $\eta \leq 0.5$. This is not surprising since at higher values of $\eta$ the inner spherical boundary of the shell has a significant effect on the structures of the flow which start to show similarities to the case of plane layer convection and are much better described by the small-gap limit approximation. The availability of the border between the two modes opens a new possibility for tests of both, the equatorial and the inertial wave, approximations since they both pretend to be able to predict the preferred mode of convection through a selection of the lowest critical number of the various modes.

The next step towards numerical verification of the inertial mode approximation is shown on Fig. 8, where the computed values of the $\frac{2}{(1-\eta)^2} R$ ratio are plotted against the value predicted by the theory, for three different wavenumbers $m$. For values of the radius ratio $\eta \leq 0.5$ a perfect agreement can be observed. For values higher than that neither a good agreement nor a well-established dependency on the radius ratio $\eta$ is obvious. This is again no surprise since we do not expect the analytical approach to be valid for thin shells, since it is based on the assumption of a full fluid sphere.

Having studied the various preferred convection modes and verified some predictions of the theory, we would like to explore in more detail the dependence of the numerical values of the critical Rayleigh number on the radius ratio $\eta$ as well as on the rotation parameter $\tau$. In Fig. 9 the ratio $\frac{2}{(1-\eta)^2} R$ is plotted against $\tau$ for different values of the radius ratio $\eta$. For all values of $\eta \leq 0.5$ the Rayleigh number decreases with increasing $\eta$. As already discussed, this rule does not hold for $\eta \geq 0.5$. More interesting is the dependence on the
rotation parameter $\tau$. For $\eta \leq 0.5$ the $R(\tau)$ obeys an power low as is obvious from the logarithmic plot of Fig. 9. A power law fit to the curve with $\eta = 0.4$ is plotted with a dotted line and we can observe that it describes the $R(\tau)$ dependence very well. Fitting one can easily estimate the relationship,

$$R \sim \tau^{\frac{1}{40}} = \tau^{0.04}. \quad (5.39)$$

This relationship suggests that for the investigated parameter region of very low Prandtl numbers and very high rotation rates and radius ratios less or equal to one half, the critical Rayleigh number $R$ is almost independent on the rotation parameter $\tau$. This confirms once again the predictions of the inertial mode approximation, since according to (3.33) the critical Rayleigh number does not depend on $\tau$. Of course, at very low $\tau$ the assumption $\frac{1}{\tau} \rightarrow 0$ for the validity of the perturbation expansion is not satisfied, while at very high $\tau$ the convection leaves the wall-attached regime to enter the multi-columnar solution. This provides an explanation of why (5.39) holds only in the intermediate values of $\tau$ but deviates at very low and high rotation rates.

![Figure 10: The frequency $\omega$ as a function of $\tau$ in the case $\eta = 0.2$, $m = 4$ for different values of $P$ as indicated in the plot. Theoretical values are plotted with dashed lines.](image)

Finally, we would like to show how the wall-attached convection described by the equatorial and the inertial wave approximations is limited with respect to the variations of the Prandtl number. This question has, of course, been addressed in the earlier studies, but it is instructive to visit it again. Fig. 10 shows the $\frac{\omega}{\tau}$ ratio against $\tau$ for fixed values $m = 4$, $\eta = 0.2$ and for several Prandtl numbers. The theoretical prediction for $\frac{\omega}{\tau}$ has also been plotted with dotted lines. At $10^{-5} \leq P \leq 10^{-2}$ the theoretical prediction is exactly fulfilled, which indicates that these points are well in the regime of equatorially wall-attached convection. The higher values of $P \geq 10^{-1}$ are in the marginal region of the regime and the analytical prediction is only approached. Nevertheless, it is our belief that in the parameter region of the Prandtl numbers lower than 0.01 the qualitative scenario of the onset of convection outlined in this section of the report is followed.
6. **The Onset of Convection with Insulating Thermal Boundary Conditions**

In this section we present some preliminary results emerging from the numerical investigation of the onset of convection in rotating spherical shells with insulating thermal boundary conditions of type B. Since we believe that convection in the case of insulating boundary conditions is not fundamentally different from convection with conducting boundary conditions, for which many results have been accumulated, the main purpose of this part of the report is to outline briefly the similarities and the differences between the two cases.

![Figure 11: The Rayleigh number $R$ (upper plot) and the frequency $\omega$ (lower plot) as a function of $\tau$ in the case $P = 0.1$, $\eta = 0.4$ for values of the wavenumber $m = 4$ (solid line), $m = 6$ (long-dashed line) and $m = 8$ (dotted line), in the case of insulating (thick lines) and conducting (thin lines) thermal boundary conditions.](image)

Fig. 11 compares the critical Rayleigh numbers and frequencies of various modes of convection with different thermal boundary conditions. Towards low values of the rotation parameter, $\tau$, all modes of the flow with insulating boundary conditions have a lower value of the Rayleigh number than the corresponding modes with conducting temperature boundary conditions. At these low values of the rotation parameter the $R(\tau)$ curves of the two cases are well separated and it is easy to observe that they have roughly the same structure and form. This is apparent from the $\omega/\tau$ ratio plotted against $\tau$. At low values of the rotation parameter the two cases undergo the same types of transition from retrograde to prograde mode. A slight difference in the values of the $\omega/\tau$ may be noticed. A more important difference is that the transitions occur for much lower values of $\tau$ in the case of insulating
boundary conditions. The value $P = 0.1$ of the Prandtl number is on the border of the low Prandtl number region as it was described in section 4. Thus as the rotation parameter increases when we cannot follow so well all the various transitions occurring at very low values of the Prandtl number. After a certain value of the rotation rate $\tau$ has been reached the $\omega/\tau$ ratio starts to change almost continuously since a large number of modes take part in the competition for providing a minimal $R$ and only a very small number of tiny jumps can be noticed. During the last big discontinuous transition the critical Rayleigh number of the modes with insulating thermal boundary conditions becomes larger than the Rayleigh number of the modes in the conducting case. But soon thereafter when the variation of $\omega/\tau$ becomes almost continuous, the Rayleigh numbers of the modes with the different boundary conditions become roughly equal and have also the same structure as can be seen in Fig. 11.

![Figure 12: Same as Fig. 10 but in the case of $P = 10^{-4}$, $\eta = 0.3$, $m = 8$](image)

For low values of the Prandtl number the sequence of transitions with increasing rotation rate $\tau$ can be observed more clearly and in a more pure form. Fig. 12 shows again a comparison between convection obtained under the two different boundary conditions for the Prandtl number, $P = 0.0001$. The same qualitative features as in Fig. 11 can be observed as well as the same number and type of transitions as described in section 4 where the same case but with the conducting boundary conditions was discussed in much detail.

This observation poses the question whether this is always the case and whether in addition to the similar transitions we may hope to observe the same regimes of convective flow at the various Prandtl number and rotation rate regions. The answer to this question requires an enormous computational task which we do not attempt to undertake here. Our
Figure 13: The streamfunctions $r \sin \theta \partial_\theta v$ in the equatorial plane illustrating various regimes of convection in the $P - \tau$ space in the case of insulating thermal boundary conditions. From left to right and top to bottom the parameters $(P, \tau)$ are: $(100, 56.3)$, $(100, 5623.4)$, $(100, 562341.3)$, $(1, 100)$, $(1, 10000)$, $(1, 10^6)$, $(0.01, 316.2)$, $(0.01, 31622.8)$, $(0.01, 3162278.1)$ for $\eta = 0.4$ and $m = 5$.

efforts are restricted to the investigation of a number of distant points in the $P - \tau$ space where we believed the various regimes of convection may manifest themselves. As shown in earlier studies (Ardes et al. (1997)) the border between the wall-attached and the columnar type of flow obeys the approximate relation,

$$P^{1/4} \tau^2 = \text{const.}$$

(6.40)

Using this expression as a guideline we investigated several different cases with increasing Prandtl number, $P = 0.001$, $P = 1$, $P = 100$. Indeed as Fig. 13 indicates, all major regimes of convection may be observed for the case of insulating thermal boundary conditions as well. At low Prandtl number, $P = 0.001$ the wall-attached regime persist for a wide range of values of $\tau$. For high $P = 1, 100$ wall-attached pattern may also be found but only for a very limited range of low rotation rates. For moderate Prandtl numbers of the order of unity at high rotation rates the generic case of columnar convection takes place.
It can also be found in a large region in the \( P - \tau \) space and in the case of Fig. 13 we may notice how the pattern changes from elongated to shorter and thicker columns when the Prandtl number is increased from 1 to 100. On the other end at very high values of the rotation parameter we find the newly observed pattern of multi-columnar convection. A very remarkable and surprising discovery is that it persists even at very high values of the Prandtl number if sufficiently high values of \( \tau \) are reached.

7. Concluding Remarks

Rotating thermal convection has been a focus of intensive research for more than five decades now, but continues to be a constant source of open questions and possibilities for further developments. In the present report the linear properties of convection in rotating spherical shells have been studied from both analytical and numerical points of view. The most valuable result in this study seem to be the refinement and extension of the perturbation approach of Zhang (1994). Future efforts will be devoted to obtaining higher order analytical approximations that will enable us to explain some numerical results. On the numerical side many interesting features has been found and examined in considerable detail. Convection in the case of insulating thermal boundary conditions need to be further investigated.

Acknowledgements

I wish to express my deep gratitude and respect to our principal lecturer Friedrich Busse. I thank to him for his help, knowledge, ideas and a world of possibilities he has shared with me in the last two years being the adviser of my doctoral thesis project. He also supervised my GFD project this summer and many ideas presented here belong to him. His help with the analytical part of the report was invaluable.

I would like to thank the organizing committee of the GFD program for the opportunity to work in a challenging atmosphere, the members of the staff for sharing their knowledge and the rest of fellows for the nice time spent together.

It was a truly rewarding summer!

Appendix A. A Formula Involving the \( Q \) Operator

For the solution of the zeroth order equations (3.21) and (3.22) we need to evaluate the expression 
\( Q P_m^n(\cos \theta) f(r) \), where \( P_m^n(\cos \theta) \) is the associated Legendre polynomial of degree \( n \) and order \( m \).
and \( f(r) \) is an arbitrary radial dependence.

\[
Q P_n^m(\cos \theta) f(r) = \left( k \cdot \nabla - \frac{1}{2}(L_2 k \cdot \nabla + k \cdot \nabla L_2) \right) P_n^m(\cos \theta) f(r)
\]

\[
= \left[ \left( 1 - \frac{1}{2}L_2 \right) \left( \cos \theta \frac{\partial}{\partial r} - \frac{\sin \theta}{r} \frac{\partial}{\partial \theta} \right) - \frac{1}{2} \left( \cos \theta \frac{\partial}{\partial r} - \frac{\sin \theta}{r} \frac{\partial}{\partial \theta} \right) L_2 \right] P_n^m(\cos \theta) f(r)
\]

\[
= \left[ \left( 1 - \frac{1}{2}L_2 \right) \left( \cos \theta \frac{\partial}{\partial r} - \frac{\sin^2 \theta}{r} \frac{\partial}{\partial \theta} \right) - \frac{1}{2} \left( \cos \theta \frac{\partial}{\partial r} - \frac{\sin^2 \theta}{r} \frac{\partial}{\partial \theta} \right) L_2 \right] P_n^m(\cos \theta) f(r)
\]

\[
= \left( 1 - \frac{1}{2}L_2 \right) \left\{ f' \left( \frac{n+m}{2n+1} P_{n-1}^m + \frac{n-m+1}{2n+1} P_{n+1}^m \right) + \frac{f}{r} \left( \frac{(n+m)(n+1)}{2n+1} P_{n-1}^m - \frac{n(n-m+1)}{2n+1} P_{n+1}^m \right) \right\}
\]

\[
- \frac{1}{2} n(n+1) \left\{ f' \left( \frac{n+m}{2n+1} P_{n-1}^m + \frac{n-m+1}{2n+1} P_{n+1}^m \right) + \frac{f}{r} \left( \frac{(n+m)(n+1)}{2n+1} P_{n-1}^m - \frac{n(n-m+1)}{2n+1} P_{n+1}^m \right) \right\}
\]

\[
= \left( f' + \frac{n+1}{r} f \right) \left( \frac{1-n^2}{2n+1} \right) P_{n-1}^m + \left( f' - \frac{n}{r} f \right) \left( \frac{1-(n+1)^2}{2n+1} \right) P_{n+1}^m,
\]

(7.41)

where formula (8.5.3) of Abramowitz and Stegun (1964) has been used.

References


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