Parameter Dependences of Convection Driven Spherical Dynamos

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Abstract. Recent results are presented for convection driven dynamos in rotating spherical fluids shells which are obtained from computations carried out at the Stuttgart Supercomputing Center. Studies of the dependence on the Prandtl number $P$ indicate that dynamo action disappears with increasing $P$ unless the magnetic Prandtl number $P_m$ is also increased. Relaxation oscillations of convection coupled to magnetic torsional oscillations are found at low Prandtl numbers and various types of reversals of dipolar dynamos have been identified.

1 Introduction

Considerable progress has been made in the past years in simulating numerically the process of the generation of magnetic fields by convection in rotating spherical fluid shells. While in earlier work often particular values of the parameters of the problem have been chosen which seemed to provide the optimal compromise between applicability to the Earth’s core and computational efficiency, it now appears that extrapolations to conditions of planetary cores are best obtained on the basis of known dynamo properties over an extended domain in the parameter space. In this connection it should be mentioned that the often employed assumption that all diffusivities are replaced by the same eddy diffusivity caused by turbulent motions at small numerically unresolved scales is too simple. From experimental measurements (Ahlers and Xu, 2001) as well as from theoretical considerations (Eschrich and Rüdiger, 1983) it is evident that even in highly turbulent systems diffusivity ratios such as the Prandtl number are not necessarily equal to unity.

In this paper we thus intend to analyze the dependence of convectively driven dynamos on the Prandtl number $P$ and on the magnetic Prandtl number $P_m$ as well as on the Coriolis number and on the Rayleigh number which are the more commonly considered parameters. One of the major goals of studying parameter dependences is the possible discovery of approximate scaling relationships which would permit the elimination of one or more parameters from the problem. In the magnetostrophic approximation the Prandtl number dependence is dropped which is justified in the case of large $P$ (Glatzmaier and Roberts, 1995). For large $P_m$ the magnetostrophic approximation can also be expected to hold as we shall discuss in section 6. More interesting from a geophysical point of view are situations with small
values of \( P \) and \( P_m \) which, unfortunately, are also difficult to explore from a numerical point of view. Some new phenomena found in this parameters regime will be reported in this paper.

We start in section 2 with a brief review of the mathematical formulation of the problem and the numerical methods used for its solution. Computational aspects are discussed in section 3. Results on convection and convection driven dynamos with Prandtl numbers of the order unity or larger are presented in section 4, whereas the situation at low Prandtl numbers is described in section 5. Possibilities for the magnetostrophic approximation are discussed in section 6 and special topics such as reversals are addressed in section 7. An outlook on future research is given in section 8.

## 2 Mathematical Description of the Problem and Numerical Methods Employed for Its Solution

The analysis of finite amplitude convection in rotating spherical shells and its dynamo action is based on the standard formulation used in earlier work by the authors (Busse et al., 1998; Grote et al., 1999, 2000b, 2001). Instead of the general temperature distribution used in the last of these papers we shall focus on the homogeneously heated sphere as the basic static state of the problem. The gravity field is given by 

\[
g = -\gamma \mathbf{d}r
\]

where \( \mathbf{r} \) is the position vector with respect to the center of the sphere and \( r \) is its length measured in units of thickness \( d \) of the shell. In addition to \( d \), the time \( d^2/\nu \), the temperature \( \nu^2/\kappa d^4 \) and the magnetic flux density \( \nu(\mu \Omega)^{1/2}/d \) are used as scales for the dimensionless description of the problem where \( \nu \) denotes the kinematic viscosity of the fluid, \( \kappa \) its thermal diffusivity, \( \rho \) its density and \( \mu \) its magnetic permeability. The density is assumed to be constant except in the gravity term where its temperature dependence given by \( \alpha \equiv (d\rho/dT)/\rho = \text{const.} \) is taken into account. The general representation in terms of poloidal and toroidal components can be used for the velocity field \( \mathbf{u} \) and the magnetic flux density \( \mathbf{B} \),

\[
\mathbf{u} = \nabla \times (\nabla v \times \mathbf{r}) + \nabla w \times \mathbf{r}, \quad (1a)
\]

\[
\mathbf{B} = \nabla \times (\nabla h \times \mathbf{r}) + \nabla g \times \mathbf{r}. \quad (1b)
\]

As in the earlier work 5 equations for \( v, w, h, g \) and the deviation \( \Theta \) of the temperature from the static distribution can be obtained,

\[
[(\nabla^2 - \partial_t)L_2 + \tau \partial \varphi] \nabla^2 v + \tau Q w - L_2 \Theta = -\mathbf{r} \cdot \nabla \times [\nabla \times (\mathbf{u} \cdot \nabla \mathbf{u} - \mathbf{B} \cdot \nabla \mathbf{B})] \quad (2a)
\]

\[
[(\nabla^2 - \partial_t)L_2 + \tau \partial \varphi] w - \tau Q v = \mathbf{r} \cdot \nabla \times (\mathbf{u} \cdot \nabla \mathbf{u} - \mathbf{B} \cdot \nabla \mathbf{B}) \quad (2b)
\]

\[
\nabla^2 \Theta + RL_2 v = P(\partial_t + \mathbf{u} \cdot \nabla) \Theta \quad (2c)
\]

\[
\nabla^2 L_2 h = P_m[\partial_t L_2 h - \mathbf{r} \cdot \nabla \times (\mathbf{u} \times \mathbf{B})] \quad (2d)
\]

\[
\nabla^2 L_2 g = P_m[\partial_t L_2 g - \mathbf{r} \cdot \nabla \times (\nabla \times (\mathbf{u} \times \mathbf{B}))] \quad (2e)
\]
where \( \partial_t \) and \( \partial_\varphi \) denote the partial derivatives with respect to time \( t \) and with respect to the angle \( \varphi \) of a spherical system of coordinates \( r, \theta, \varphi \) and where the operators \( L_2 \) and \( Q \) are defined by

\[
L_2 \equiv -r^2 \nabla^2 + \partial_r (r^2 \partial_r )
\]

\[
Q \equiv r \cos \theta \nabla^2 - (L_2 + r \partial_r )(\cos \theta \partial_r - r^{-1} \sin \theta \partial_\theta )
\]

The Rayleigh number \( R \), the Coriolis parameter \( \tau \), the Prandtl number \( P \) and the magnetic Prandtl number \( P_m \) are defined by

\[
R = \frac{\alpha \gamma \beta d^6}{\nu \kappa}, \quad \tau = \frac{2 \Omega d^2}{\nu}, \quad P = \frac{\nu}{\kappa}, \quad P_m = \frac{\nu}{\lambda}
\]

where \( \lambda \) is the magnetic diffusivity. We assume stress-free boundaries with fixed temperatures,

\[
v = \partial_{rr}^2 v = \partial_r (w/r) = \Theta = 0 \quad \text{at} \quad r = r_i \equiv \eta/(1-\eta)
\]

and at \( r = r_o = (1-\eta)^{-1} \) (4a)

where \( \eta \) is the radius ratio of the spherical shell. Throughout this paper the case \( \eta = 0.4 \) will be assumed. For the magnetic field electrically insulating boundaries are usually used such that the poloidal function \( h \) must be matched to the function \( h^{(e)}(e) \) which describes the potential fields outside the fluid shell

\[
g = h - h^{(e)} = \partial_r (h - h^{(e)}) = 0 \quad \text{at} \quad r = r_i \text{ and } r = r_o.
\]

(4b)

But computations with a conducting inner core, \( 0 < r < r_i \), have also been done. The numerical integration of equations (2) together with boundary conditions (4) proceeds with the pseudo-spectral method as described by Tilgner and Busse (1997) which is based on an expansion of all dependent variables in spherical harmonics for the \( \theta, \varphi \)-dependences,

\[
v = \sum_{l,m} V_l^m (r, t) P_l^m (\cos \theta) \exp\{im\varphi\}
\]

with analogous expressions for \( w, \Theta, h \) and \( g \). \( P_l^m \) denotes the associated Legendre functions. For the \( r \)-dependence expansions in Chebychev polynomials are used. For further details see also Busse et al. (1998).

The standard numerical resolution is given by 33 collocation points in the radial direction and spherical harmonics up to the order 64. But cases up to 65 collocation points and spherical harmonics up to the order 128 are often used.
3 Implementation of the Numerical Algorithm on the Stuttgart Supercomputer

The functions $V_l^m(r,t), W_l^m(r,t), T_l^m(r,t), H_l^m(r,t)$ and $g_l^m(r,t)$ corresponding to the fields $v, w, \Theta, h$ and $g$, respectively, are being stored in $r, l, m$-space. In order to allow fast transforms from normal space to Chebychev expansion and vice versa, the $N$ collocation points in the radial direction are chosen to lie at $x_n = \cos \left( \frac{n\pi}{N-1} \right)$ where $x$ is defined by $x = 2r - (r_o + r_i)$. The dynamic equations are converted into a system of ODEs in time through the enforcement of the full equations at every collocation point. The decomposition in terms of Chebychev polynomials is thus merely used to compute radial derivatives.

The main reason for this “pseudo-spectral” method is the fact that in a pure spectral method the computation of the nonlinear terms would be very expensive both in terms of CPU time and in memory consumption. On the other hand, spectral methods provide a better convergence behavior than finite difference or finite element methods. Thus, the goal is to benefit from both techniques as much as possible. In our preference for the pseudo spectral method we feel confirmed by the comparison done by Fornberg and Merrill (1997) and by the predominance of the pseudo-spectral approach in a recent dynamo benchmark study (Christensen et al., 2001).

Time stepping is performed by a combination of an Adams-Bashforth second order scheme treating all the right hand sides of equations (2) explicitly, whereas the terms at the left hand sides are included in an implicit Crank-Nicolson step. At the beginning of a time integration an Euler step is used to start up the scheme.

In order to take advantage of the multiprocessor architecture the code is parallelized in azimuthal direction, i.e., all coefficients sharing a common index $m$ are stored at the same processor. This means that for a typical calculation usually 64 processors are being used. All communications between them are done with the help of the Message Passing Interface (MPI).

At the beginning of each time step all fields and their first and second derivatives are given in $r, l, m$-space. At this stage the spectral coefficients of $v, w, \Theta, g, h$ are stored such that coefficients with identical $m$ are stored at the same processor.

The calculation of the fields $v, w, \Theta, g, h$ requires adding associated Legendre functions and performing a Fourier transform. The summation over $l$ in (5) is implemented as matrix vector multiplications and obviously parallelizes over $m$. The summation over $m$ is the Fourier transform which requires interprocessor communication. Before the actual execution of the FFT, data are redistributed such that individual processors contain all data with a given index $l$. The fast Fourier transform algorithm can then be executed locally and in parallel for separate $l$ and $r$. The nonlinear terms are now easily obtained since they only involve multiplications of local data. The transformation back
Read parameters of the run. Load initial data from disc.

Transform to physical space. Compute right hand sides. (parallel over radial index)

Take one time step and obtain solutions for the fields. (parallel in m)

Compute and invert matrices required in the implicit timestep. Fill tables with values of Legendre functions and their derivatives.

Compute first and second derivatives of the fields.

When necessary: compute observables, save data to disk.

**Fig. 1.** Diagram of steps in the computational process for solving equations (2) in time.

into the $r, l, m$-space is performed with a FFT followed by a Gauss quadrature. These are technically the same operations as for the first transformation. At the end of the FFT the original data distribution is restored, i.e. all variables at a given $m$ are collected in the storage of individual processors. The Gauss quadrature is again expressed in terms of matrix vector multiplications which run independently on all processors for different $m$.

Once the nonlinear terms have been obtained, they can be combined as required by the Adams-Bashforth scheme and added to the terms of the implicit time step involving the variables at the present moment in time only. To complete the time step, a set of $N$ linear equations must be solved for every $l, m$. Boundary conditions are also included in this set of equations. The coefficients in these equations are independent of $m$ and are collected in separate matrices which are inverted during initialization and multiplied with vectors containing the spectral coefficients of $v, w, \Theta, g, h$ during the actual time step. These multiplications separate again in $m$ and involve only local data for each processor. The discretized equations are formulated such that the updated fields are obtained in the $n, l, m$-space where the radial derivatives can be conveniently computed. A fast cosine transform brings the variables back into the $r, l, m$-space ready for use in the next time step. For further details we refer to the analogous numerical treatment of the problem of non-magnetic convection by Tilgner and Busse (1997). The diagram of Fig. 1 provides an overview of these computational steps. In summary, the computational burden lies mostly in matrix-vector multiplications, followed by fast cosine and Fourier transforms. The matrix vector multiplications car-
ried out at each processor are of course readily vectorized. However, with the resolution used so far, each vector is relatively short (usually 64 elements). Only the FFT needs to shuffle data between processors. Interprocessor communication thus contributes little to the CPU time expenditure.

4 Prandtl Number Dependence of Dynamos

Before entering the discussion of dynamos we shall briefly outline the influence of the Prandtl number $P$ on convection without a magnetic field. In Fig. 2 typical examples of convection for $P = 15$ and $P = 1$ are shown side by side.

While the columnar form of the convection eddies do not differ much the differential rotation generated by convection is most strikingly different. In the case $P = 1$ the differential rotation is generated by the Reynolds stresses

![Fig. 2. Convection in rotating spherical fluid shells in the cases $\tau = 5 \cdot 10^3, R = 8 \cdot 10^5, P = 15$ (left column) and $\tau = 10^4, R = 4 \cdot 10^5, P = 1$ (right column). Lines of constant mean azimuthal velocity $\bar{u}_\phi$ are shown in the left halves of the upper circles and isotherms of $\Theta$ are shown in the right halves. The plots of the middle row show streamlines, $r \partial v / \partial \phi = \text{const.}$, in the equatorial plane. The lowermost plots indicate lines of constant $u_r$ in the middle spherical surface, $r = r_i + 0.5$.](image)
of convection and obeys the geostrophic balance perfectly, i.e. it varies only with distance from the axis. On the other hand, in the case $P = 15$ the differential rotation is much weaker and no longer satisfies the geostrophic balance. Instead it obeys primarily a thermal wind relationship as can be seen by comparison with the plot of the axisymmetric component of $\Theta$. The strong decay of the energy density $\tilde{E}_t$ of the differential rotation is caused primarily by the general decrease of the amplitude of the convection velocity with increasing $P$. The energy densities of different components of the velocity field are defined by

$$\tilde{E}_p = \frac{1}{2} \langle |\nabla \times (\nabla v \times r)|^2 \rangle, \quad \tilde{E}_t = \frac{1}{2} \langle |\nabla \tilde{w} \times r|^2 \rangle$$  \hspace{1cm} (6a)

$$\tilde{E}_p = \frac{1}{2} \langle |\nabla \times (\nabla \tilde{w} \times r)|^2 \rangle, \quad \tilde{E}_t = \frac{1}{2} \langle |\nabla \tilde{w} \times r|^2 \rangle$$  \hspace{1cm} (6b)

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**Fig. 3.** Dependences of energy densities of the axisymmetric toroidal (red), axisymmetric poloidal (black), non-axisymmetric toroidal (blue) and non-axisymmetric poloidal (green) components of motion on $R$ and $P$ in the case of $\tau = 5 \times 10^3$. The energy densities have been multiplied by $P^2$ and thus are measured in terms of the thermal scaling.
where the angular brackets indicate the average over the fluid shell and where the bar indicates the average over the azimuthal coordinate \( \varphi \). Thus \( v = \bar{v} + \hat{v} \) holds where \( \hat{v} \) denotes the non-axisymmetric component of \( v \).

If thermal scaling \( \kappa/d \) is used instead of the viscous scaling \( \nu/d \) the fluctuating component of the velocity field show much less variation with \( P \) as shown in Fig. 3. But the differential rotation still decreases with increasing \( P \) because the tilt of the streamlines of the convection columns as shown in the equatorial plane plots of Fig. 2 is much weaker for larger Prandtl numbers.

An overview of the Prandtl number dependence of convection driven dynamos is given in Fig. 4. The results displayed here have been obtained for the particular value \( 5 \cdot 10^3 \) of the Coriolis parameter \( \tau \). But from earlier work (see, for example, Busse et al., 1998) and several computations carried out for \( \tau = 10^4 \) we expect that these results are representative for a fairly large regime of the parameter \( \tau \). A most important result is the property that for \( P \geq 5 \) the value of the magnetic Prandtl number \( P_m \) must always exceed a value of the order \( P \) in order that dynamo action can be achieved. The

![Fig. 4. Dynamo solutions indicated by red (dipolar), blue (quadrupolar), green (hemispherical) and purple (mixed symmetry) balls in the \( R - P - P_m \) parameter space. No dynamo solution could be obtained for values of \( P, P_m \) in the shaded region.](image)
numerous attempts to obtain dynamos in the darkly shaded region of Fig. 4 have not been displayed. But the obvious difficulty to reach high Rayleigh numbers with sufficient numerical resolution is not the cause of disappearing dynamo action as is evident from the results obtained in borderline cases such as $P = P_m = 10$. It is found in this case that dynamos are obtained for intermediate values of the Rayleigh number $R$ of the order $5 \cdot 10^5$. At such a value of $R$ the magnetic Reynolds number is high enough to permit dynamo action, but is not so high that flux expulsion from the convection eddies becomes a dominant effect. The cause for the increase of the critical value of $P_m$ with increasing $P$ lies in the decline of the differential rotation with increasing $P$. Indeed, it has been pointed out frequently in previous studies (Grote et al., 2001; Grote and Busse, 2001a) that the interaction between the differential rotation generated by the Reynolds stresses of convection and the magnetic field is the major feature of convection driven dynamos with Prandtl numbers of the order unity or less. Speaking in terms of mean-field-magnetohydrodynamics (Krause and Rädler, 1980), we typically find in this regime that the dynamos are $\alpha\omega$-dynamos for which the differential rotation plays an essential role. For Prandtl numbers larger than unity the nomenclature of mean-field-magnetohydrodynamics is no longer useful since the

![Fig. 5. Magnetic (heavy symbols) and kinetic (light symbols) energies on the left side and corresponding dissipations (right side) have been plotted for $P = 1$, 5 and 8. Energy densities and dissipation densities of axisymmetric field components are indicated by circles (poloidal) and squares (toroidal) while plus-signs and lying crosses denote the corresponding quantities for fluctuating poloidal and toroidal components, respectively.](image-url)
axisymmetric large scale components of the magnetic field become small in comparison with the small scale non-axisymmetric components as can be seen in Fig. 5 where energies of different components have been plotted. For the magnetic energies $\tilde{M}_p$, $\tilde{M}_t$, $\tilde{M}_d$, $\tilde{M}_q$ definitions analogous to relationships (6) are used with $h$ and $g$ replacing $v$ and $w$. Another feature that can be noticed in Fig. 4 is the preference for dipolar dynamos with increasing Prandtl number. The property that mixtures of dipolar and quadrupolar components become predominant for high Rayleigh number dynamo is just a consequence of the onset of convection in the polar regions of the fluid shell. As a result the convection velocity field loses its approximate symmetry with respect to the equatorial plane and thus dynamos can no longer be clearly separated into those of dipolar and those of quadrupolar symmetry.

5 Low Prandtl Number Regime

At low Prandtl numbers higher values of the Coriolis parameter $\tau$ can be reached since the stabilizing effect of rotation increases with $P\tau$. But the increasing fluctuations in time cause numerical difficulties because of the small time steps that are required. Therefore the systematic study of convection with and without dynamo action has not proceeded as far as it has done in the regime of Prandtl numbers of the order unity and larger. A typical example of convection without magnetic field is shown in Figs. 6a and 6b. The time dependence of $\tilde{E}_t$ exhibits the relaxation oscillations well known from earlier studies at values of $P$ of the order one (Grote et al., 2000; Grote and Busse, 2001a; Grote et al., 2002). The energies $\tilde{E}_t$ and $\tilde{E}_p$ of the fluctuating components of the velocity field do not decay to zero, however, in the interval where $\tilde{E}_t$ is large because the Rayleigh number for onset of convection in the polar regions is exceeded. Only the convection columns outside the tangent cylinder touching the inner core at its equator participate in the relaxation oscillations. Since the polar convection dominates the heat transport, the Nusselt numbers which are defined by

$$
Nu_{i,o} = 1 - \frac{P}{r_{i,o}} \frac{\partial \bar{\theta}}{\partial r} \bigg|_{r=r_i,r_o}
$$

(7)

exhibit a weaker influence of the relaxation oscillations than at $P \approx 1$. In particular the Nusselt number $Nu_o$ measured at the outer boundary is nearly constant in time.

The highly time dependent nature of dynamic processes in convecting low Prandtl number fluids is also reflected in the convection driven dynamos. A typical example is shown in Fig. 7a where the intermittent character of dynamo action is evident. From a strongly convective state in which the predominantly dipolar magnetic field suppresses the differential rotation the system changes into a state of weak convection with strong differential rotation. The magnetic field is still predominantly dipolar as can be seen from
Fig. 6. a) Relaxation oscillation of convection in the case $\tau = 2 \cdot 10^4$, $R = 6 \cdot 10^5$, $P = 0.1$. A section of a much longer time series is shown with the energies $\tilde{E}_l$ (short dashed line), $\tilde{E}_i$ (long dashed line) and $\tilde{E}_p$ (dotted line) displayed in the upper graph and the Nusselt number $Nu_i$ (solid line) and $Nu_o$ (dashed line) in the lower graph.

The dynamo oscillations become much more pronounced as the Coriolis parameter $\tau$ and the Rayleigh number $R$ are increased. In Fig. 8a the strong nearly periodic oscillations are clearly seen. They represent a striking coherent structure of a highly turbulent system. The oscillations are similar to the relaxation oscillations of convection in the absence of a magnetic field in that the convection columns can grow in amplitude only if the differential rotation is sufficiently weak. While viscous diffusion leads to the decay of
Fig. 6. b) Time sequence of equidistant plots (top to bottom) covering the time span from $t = 0.057$ to $t = 0.073$ of the time series of Fig. 6a. The left half of the left circle in each row indicates lines of constant $\bar{u}_\phi$, the right half displays meridional streamlines, $r \sin \theta \partial \nu / \partial \theta = \text{const}$. The middle circle shows streamlines, $r \partial \nu / \partial \phi = \text{const}$., in the equatorial plane and the plot on the right side shows lines of constant $u_r$ on the spherical surface $r = r_i + 0.5$. 
Fig. 7. a) Time series of a convection driven dynamo with $\tau = 3 \cdot 10^4$, $R = 8.5 \cdot 10^5$, $P = 0.1$, $P_m = 1$. The first, second and third plot from the top show energy densities of the dipolar and quadrupolar components of the magnetic field and of the velocity field, respectively. The mean toroidal and fluctuating toroidal energy densities are indicated by solid and dashed lines, respectively. The lowermost plot shows the Nusselt number $N_u(t)$ (solid line) and $N_{u_0}$ (dashed line).

$\tilde{E}_t$ in the non-magnetic case, the axisymmetric poloidal component of the magnetic field brakes the differential rotation in the case of Fig. 8a. Thus a period of about 0.02 is seen instead of the period 0.1 resulting from the viscous decay. From Fig. 8b it is apparent that the radial dependence of the differential rotation and not only its amplitude changes throughout the oscillation period. Similarly the distribution of the azimuthal component $\tilde{B}_\phi$
Fig. 7. b) Time sequence of equidistant plots (top to bottom) covering the time span from $t = 2.24$ to $t = 2.69$ of the time series of Fig. 7a. The left half of the first circle in each row indicates lines of constant $\bar{u}_\varphi$ while the right half displays meridional streamlines, $r \sin \theta \partial v / \partial \theta = \text{const.}$. The second circle in each row shows streamlines, $r \partial v / \partial \varphi = \text{const.}$, in the equatorial plane. The oval plot exhibits lines of constant $u_r$ on the surface $r = r_i + 0.5$. The last circle in each row indicates lines of constant $\bar{B}_\varphi$ in its left half and meridional field lines, $r \sin \theta \partial h / \partial \theta = \text{const.}$, in its right half.
Fig. 8. a) Time series of magnetic (upper graph) and kinetic (middle graph) energy densities and of Nusselt numbers (lower graph) in the case $\tau = 10^5, R = 2 \cdot 10^6, P = 0.1, P_m = 1$. Axisymmetric poloidal (solid line) and toroidal (short dashed lines) components and non-axisymmetric poloidal (dotted lines) and toroidal (long dashed line) components of the energy densities are shown. The component $E_p$ is too small to be noticeable in the middle graph. $N u_i$ (solid line) and $N u_o$ (dashed line) are displayed in the lower graph.

of the magnetic flux density changes owing to the stretching of the meridional field lines by the differential rotation. The period of 0.02 corresponds roughly to that expected for a standing torsional Alfvén wave. There thus appears to be a resonance between the convection oscillation and a torsional Alfvén wave. A more detailed investigation of the torsional oscillation will be of geophysical interest since it has long been believed (Braginsky, 1980) that they play an important role in the geomagnetic secular variation.

6 On the Magnetostrophic Approximation

Among the various advection terms in the basic equation (2) for convection driven dynamos the advection of momentum appears to be the least important. The advection of the temperature provides the essential nonlinearity
Fig. 8. b) Time sequence of equidistant plots (top to bottom) covering the time span from $t = 0.234$ to $t = 0.258$ of figure 8a. The left half of the first circle in each row indicates lines of constant $\bar{u}_\psi$, while the right half shows meridional streamlines, $r \sin \theta \partial \bar{v} / \partial \theta = \text{const.}$. The middle circle exhibits streamlines, $r \partial \nu / \partial \nu = \text{const.}$, in the equatorial plane. The right circle shows lines of constant $\bar{B}_\psi$ in its left half and meridional field lines, $r \sin \theta \partial \bar{h} / \partial \theta = \text{const.}$, in its right half.

for the dependence of the convection amplitude on the Rayleigh number and the advection of the magnetic flux density represents an intrinsic part of the equation of magnetic induction. But in the equation of motion the Coriolis force term usually exceeds by far the other inertial terms and it is not surprising that this property is used as an argument for the neglect of the term $\partial \mathbf{u} / \partial t + \mathbf{u} \cdot \nabla \mathbf{u}$. This so-called magnetostrophic approximation is formally justified in limit of large Prandtl numbers. When the thermal time
scale $d^2/\kappa$ is used instead of the viscous one and when the magnetic flux density is scaled with $\sqrt{\mu_0 \kappa \sqrt{P/d}}$, the basic equations assume the form

$$P^{-1} \frac{Du}{Dt} + \tau k \times u = -\nabla \pi + \Theta r + \nabla^2 u + (\nabla \times B) \times B$$  \hspace{1cm} (8a)$$

$$\frac{D\Theta}{Dt} = Ru \cdot r + \nabla^2 \Theta$$ \hspace{1cm} (8b)$$

$$\frac{\kappa}{\lambda} \left( \frac{DB}{Dt} - B \cdot \nabla u \right) = \nabla^2 B$$ \hspace{1cm} (8c)$$

where $D/Dt$ is the material derivative, $D/Dt = \partial/\partial t + u \cdot \nabla$. The temperature has been scaled by $\beta_0/R$ in this case. In the magnetostrophic approximation the term $P^{-1} Du/Dt$ is neglected and thus one of the dimensionless parameters of the problem has disappeared.

There is another way in which a magnetostrophic approximation can be obtained. Through the use of the magnetic diffusion time scale, $d^2/\lambda$, and the scale $\sqrt{\mu_0 \lambda \sqrt{P_m/d}}$ for the magnetic flux density the basic equation can be transformed into

$$P_m^{-1} \frac{Du}{Dt} + \tau k \times u = -\nabla \pi + \Theta r + \nabla^2 u + (\nabla \times B) \times B$$ \hspace{1cm} (9a)$$

$$\frac{\lambda}{\kappa} \frac{D\Theta}{Dt} = Ru \cdot r + \nabla^2 \Theta$$ \hspace{1cm} (9b)$$

$$\frac{DB}{Dt} - B \cdot \nabla u = \nabla^2 B$$ \hspace{1cm} (9c)$$

For large $P_m$ it appears to be justified to drop $P_m^{-1} Du/Dt$ from equation (9a) whereby a magnetostrophic approximation is obtained with one parameter less than in the original equations. While convection without magnetic field seems to become nearly independent of the Prandtl number $P_m$ for $P \approx 10$, this does not seem to be the case for convection driven dynamo which depend rather sensitively on $P$ even for $P > 10$. Although the differential rotation decreases in amplitude rapidly with increasing $P$ it continues to exert important influence on the dynamo process. In the case of equations (9) the magnetostrophic approximation appears to have a similarly restricted range of validity.

7 Oscillatory Dipolar Dynamos and Reversals

Dipolar dynamos with low magnetic field strength usually do not possess the oscillatory character that is characteristic for quadrupolar and hemispheric dynamos (Grote et al., 2000). But there are some situations where a cyclical behavior can also be realized in the case of dipolar dynamos. At high Rayleigh numbers after convection has also appeared in the polar regions of the spherical shell quadrupolar dynamos are replaced by predominantly
dipolar dynamos which, however, tend to continue to exhibit the oscillatory behavior of the quadrupolar dynamo as shown in figure 17 of Busse (2002). Since $R$ is of the order 20 times its critical value the dynamo process is highly chaotic. But except for the different symmetry with respect to the equatorial plane the oscillations resemble those of the quadrupolar case. The dynamo
Fig. 10. A time sequence (top to bottom with $\Delta t = 0.06$) of plots for a partially oscillating dipolar dynamo with $\tau = 5 \cdot 10^3$, $R = 6 \cdot 10^5$, $P = P_{n0} = 5$. The left halves of the circles in the left column show lines of constant $B_\phi$, the right half indicates field lines, $r \sin \theta \partial \tilde{h} / \partial \theta = \text{const.}$. The right column displays lines of constant $B_r$ on the spherical surface $r = r_0 + 0.3$. 
oscillation of Fig. 9 is quite different. While the flux tubes of alternating polarity of the mean azimuthal component $\bar{B}_\varphi$ of the magnetic field propagate towards higher latitude as in the quadrupolar case, an oscillation cannot be detected in the axisymmetric component of the poloidal field outside the sphere. This property is caused by the predominance of the $m = 1$ component of the field as shown on the right hand side of Fig. 9. The oscillatory process is nearly $180^\circ$ out of phase on opposite sides of the sphere. At larger distances from the sphere an equatorial dipole slowly drifting relative to the rotating frame of reference will be the main feature of the field.

Another type of oscillatory dynamo is exhibited in Fig. 10. As is typical for dynamos at Prandtl numbers of the order 5 or larger, strong magnetic flex tubes surround the axis in the polar regions. These flux tubes together with the axisymmetric component of the poloidal field do not change much in time while the mean azimuthal magnetic field $\bar{B}_\varphi$ in the equatorial region outside the tangent cylinder executes the usual nearly periodic oscillations as shown in the figure. Little information about this oscillation can be gained by watching the poloidal field from the outside which exhibits a steady dipole with only a minor oscillatory modulation at low latitudes.

Besides these more or less regular oscillatory dipolar dynamos numerous chaotic dynamos have been found which show much more irregular oscillations such that only occasionally a reversal is observed. Further computations will be needed to obtain a clearer picture of the dependence of the statistics of reversals on the parameters of the problem.

8 Concluding Remarks

The computational results presented in this report represent highlights of a systematic exploration of convection driven dynamos in rapidly rotating spherical shells. The attention has been focused on the influence of Prandtl numbers different from unity. It has become evident that even a slight increase of $P$ to values of the order 5 has a profound effect on the differential rotation and thereby on the dynamo process. Similarly, a lowering of the Prandtl number to values of the order 0.1 gives rise to new forms of coherent phenomena of the turbulent systems which still need to be explored more systematically. Of particular interest are torsional oscillations for which some evidence exists in the geomagnetic secular variation.

Further studies are also needed for the determination of the parametric dependence of aperiodic reversals, their typical properties and the frequency with which they occur. It appears that from these studies and those of other groups a much better understanding of the geodynamo will soon become available.
References


