6 Convection in rotating spherical shells and its dynamo action

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1. Introduction

The evolution of celestial bodies such as the Earth is characterized by the transport of heat from the interior to outside. Typically the basic static (or nearly static) state in which heat is transported by conduction and radiation is unstable in all or in parts of the interior and convection flows occur. Unlike molecular conduction and radiation, convection flows are rather sensitive to the state of rotation of the body, unless the viscosity is very high as in the mantles of the terrestrial planets. The action of the Coriolis force on fluid motion usually inhibits the efficiency of the convective heat transport, and the ways in which oscillatory motions and turbulence may overcome the inhibiting influence of rotation pose some most interesting dynamical problems. A way chosen most frequently by nature to counteract the effects of strong rotation is the generation of a magnetic field. Through the Lorentz force a new participant enters the balance of forces and evidently facilitates a more efficient transport of energy.

In this chapter, we intend to discuss first the dynamical problems of convection and then turn to the roles that magnetic fields generated through the dynamo process may play in changing the structure of convection flows and their capacity for transporting heat. In Section 2 the basic equations and the numerical approach are introduced. In Section 3 we review briefly the onset of convection in rotating spherical fluid shells. The typical bifurcation scenarios that develop as the Rayleigh number increases and give rise to turbulent convection with its coherent structures are discussed in Section 4. In Section 5 the dynamo process in the presently computationally accessible parameter regime is discussed. Numerical solutions based on hyperdiffusivity schemes will not be considered since they tend to introduce artificial effects (Zhang and Jones, 1997; Grote et al., 2000a). The interaction between magnetic fields and convection flows in the presence of a dominant Coriolis force is considered in Section 6. Open problems of future research are mentioned in Section 8.

2. Mathematical formulation of the problem

For the description of finite amplitude convection in rotating spherical shells and its dynamo action, we follow the standard formulation used in earlier work by the authors (Bussé et al., 1998; Grote et al., 1999, 2000b). But we assume that a more general static state exists with the temperature distribution $T_S = T_0 - (\beta d^2 r^2 / 2) + \Delta T \eta r^{-1} (1 - \eta)^{-2}$ where $\eta$ denotes the ratio of inner to outer radius of the shell and $d$ is its thickness. $\Delta T$ is the temperature difference between the boundaries in the special case $\beta = 0$. The gravity field is given by $g = -\gamma dr$ where $r$ is the position vector with respect to the center of the sphere and $r$ is its length measured in units of $d$. In addition to the length $d$, the time $d^2 / \nu$, the temperature $\nu^2 / \gamma \alpha d^4$
and the magnetic flux density $\mathbf{v}(\mu Q)^{1/2}/d$ are used as scales for the dimensionless description of the problem where $\nu$ denotes the kinematic viscosity of the fluid, $\kappa$ its thermal diffusivity, $\varphi$ its density and $\mu$ is the magnetic permeability. The density is assumed to be constant except in the gravity term where its temperature dependence given by $a = (d\varphi/dT)/Q = \text{constant}$ is taken into account. Since the velocity field $\mathbf{u}$ as well as the magnetic flux density $\mathbf{B}$ are solenoidal vector fields, the general representation in terms of poloidal and toroidal components can be used

$$\mathbf{u} = \nabla \times (\nabla v \times \mathbf{r}) + \nabla w \times \mathbf{r},$$  \hspace{1cm} (1a)$$ $$\mathbf{B} = \nabla \times (\nabla h \times \mathbf{r}) + \nabla g \times \mathbf{r}.$$  \hspace{1cm} (1b)$$

By multiplying the $(\text{curl})^2$ and the curl of the Navier–Stokes equations in the rotating system by $\mathbf{r}$ we obtain two equations for $v$ and $w$:

$$[(\nabla^2 - \partial_t) L_2 + \tau \partial_\rho] \nabla^2 v + \tau Q w - L_2 \Theta = -\mathbf{r} \cdot \nabla [\nabla \times (\mathbf{u} \cdot \nabla \mathbf{u} - \mathbf{B} \cdot \nabla \mathbf{B})],$$  \hspace{1cm} (2a)$$

$$[(\nabla^2 - \partial_t) L_2 + \tau \partial_\rho] \nabla^2 v - \tau Q v = \mathbf{r} \cdot \nabla [\mathbf{u} \cdot \nabla \mathbf{u} - \mathbf{B} \cdot \nabla \mathbf{B}],$$  \hspace{1cm} (2b)$$

where $\partial_t$ and $\partial_\rho$ denote the partial derivatives with respect to time $t$ and with respect to the angle $\varphi$ of a spherical system of coordinates $r, \theta, \varphi$ and where the operators $L_2$ and $Q$ are defined by

$$L_2 = -r^2 \nabla^2 + \partial_r (r^2 \partial_r),$$

$$Q = r \cos \theta \nabla^2 - (L_2 + r \partial_r \cos \theta \partial_r - r^{-1} \sin \theta \partial_\theta).$$

The heat equation for the dimensionless deviation $\Theta$ from the static temperature distribution can be written in the form

$$\nabla^2 \Theta + \left[R_i + R_e \eta r^{-3}(1 - \eta)^{-2}\right] L_2 v = P(\partial_t + \mathbf{u} \cdot \nabla) \Theta$$  \hspace{1cm} (2c)$$

and the equations for $h$ and $g$ are obtained through the multiplication of the equation of magnetic induction and of its curl by $\mathbf{r}$:

$$\nabla^2 L_2 h = P_m [\partial_t L_2 h - \mathbf{r} \cdot \nabla \times (\mathbf{u} \times \mathbf{B})],$$  \hspace{1cm} (2d)$$

$$\nabla^2 L_2 g = P_m [\partial_t L_2 g - \mathbf{r} \cdot \nabla \times (\nabla \times (\mathbf{u} \times \mathbf{B}))].$$  \hspace{1cm} (2e)$$

The Rayleigh numbers $R_i$ and $R_e$, the Coriolis parameter $\tau$, the Prandtl number $P$ and the magnetic Prandtl number $P_m$ are defined by

$$R_i = \frac{\alpha \nu^2 \Omega^2}{\mu \kappa}, \hspace{1cm} R_e = \frac{\alpha \Delta T d^4}{\nu \kappa}, \hspace{1cm} \tau = \frac{2 \Omega d^2}{\nu}, \hspace{1cm} P = \frac{\nu}{\kappa}, \hspace{1cm} P_m = \frac{\nu}{\lambda},$$  \hspace{1cm} (3)$$

where $\lambda$ is the magnetic diffusivity. We assume stress-free boundaries with fixed temperatures:

$$v = \partial_r^2 v = \partial_r (w/r) = \Theta = 0 \hspace{1cm} \text{at} \hspace{1cm} r = r_1 \equiv \eta/(1 - \eta) \hspace{1cm} \text{and} \hspace{1cm} r = r_0 = (1 - \eta)^{-1}.$$  \hspace{1cm} (4a)$$

Throughout this chapter, the case $\eta = 0.4$ will be considered unless indicated otherwise. For the magnetic field electrically insulating boundaries are used such that the poloidal function $h$ must be matched to the function $h^{(e)}$ which describes the potential fields outside the fluid shell:

$$g = h - h^{(e)} = \partial_r (h - h^{(e)}) = 0 \hspace{1cm} \text{at} \hspace{1cm} r = r_1 \hspace{1cm} \text{and} \hspace{1cm} r = r_0.$$  \hspace{1cm} (4b)$$
The numerical integration of Eqs (2a)–(2e) together with boundary conditions (4a) and (4b) proceeds with the pseudo-spectral method as described by Tilgner and Busse (1997) which is based on an expansion of all dependent variables in spherical harmonics for the $\theta, \phi$-dependences, that is

$$v = \sum_{l,m} V_l^m(r, t)P_l^m(\cos \theta) \exp(i m \phi)$$

(5)

and analogous expressions for the other variables, $w$, $\Theta$, $h$ and $g$. $P_l^m$ denotes the associated Legendre functions. For the $r$-dependence expansions in Chebychev polynomials are used. For further details see also Busse et al. (1998).

For most of the computations to be reported in the following thirty-three collocation points in the radial direction and spherical harmonics up to the order 64 have been used.

3. Onset of convection in rotating spherical shells

Even the linear problem of the onset of convection described by Eqs (2a)–(2c) in the limit when all non-linear terms can be neglected is a demanding problem because the critical Rayleigh number depends on the Coriolis number $\tau$, the Prandtl number $P$, the radius ratio $\eta$ and on the way in which gravity and temperature distribution of the basic state of pure conduction depend on the radius. For the latter dependences, it is usually assumed that both, gravity and the negative temperature gradient increase in proportion to $r$ corresponding to the case $R_c = 0$ in Eqs (2a)–(2c). For the full sphere, $\eta = 0$, the asymptotic scaling of the problem in the limit of large $\tau$ has been derived by Roberts (1968) and the conditions for the onset of the columnar mode of convection have been determined by Busse (1970) on the basis of the annulus model of convection. A complete treatment of the asymptotic linear problem has been given only recently by Jones et al. (2000) following an earlier analytical study by Yano (1992) based on the model of a thick cylindrical annulus.

Besides the columnar mode of convection, there exists another mode of onset of convection which becomes preferred at sufficiently small Prandtl numbers (Zhang and Busse, 1987). This equatorially attached type of convection flow can be described as inertial waves which are modified by viscous friction and buoyancy forces (Zhang, 1994). Since most of the numerical analysis discussed in this chapter will be confined to more moderate Prandtl numbers we shall not consider equatorially attached convection any further.

The parameter dependence of the onset of convection becomes more complex as spherical shells of finite radius ratio $\eta$ are considered. Fortunately, the influence of $\eta$ is relatively weak if the Coriolis parameter is sufficiently high and $\eta$ does not approach unity too closely. A radius ratio of $\eta = 0.4$ has been used traditionally (Zhang and Busse, 1987) and extensive computations of the critical conditions for onset have been done (Zhang, 1992a; Ardes et al., 1997). Here, we just want to apply the asymptotic formulas derived for the rotating cylindrical annulus (Busse, 1970, 1986) to give a general impression of the approximate parameter dependence of the onset of convection.

Using the rotating annulus bounded by the coaxial cylinders with radii $r_i$ and $r_o$ as shown in Figure 1 and using the asymptotic relationships (3.9) of Busse (1986) which were derived for the small gap limit we obtain the following expressions for the critical Rayleigh number $R_{ic}$, the critical wavenumber $m_c$ and the corresponding angular frequency $\omega_c$ of wave
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Figure 1 Qualitative sketch of convection columns in the rotating cylindrical annulus.

\[ R_{ic} = 3 \left( \frac{P\tau}{1 + P} \right)^{4/3} (\tan \theta_m)^{8/3} r_m^{-1/3} 2^{-2/3}, \]  
\[ m_c = \left( \frac{P\tau}{1 + P} \right)^{1/3} (r_m \tan \theta_m)^{2/3} 2^{-1/6}, \]  
\[ \omega_c = \left( \frac{2^2}{(1 + P)^2 P} \right)^{1/3} 2^{-5/6} (\tan^2 \theta_m / r_m)^{2/3}. \]

Here \( r_m \) refers to the mean radius of the fluid shell, \( r_m = (r_i + r_o)/2 \), and \( \theta_m \) to the corresponding colatitude, \( \theta_m = \arcsin (r_m(1 - \eta)) \). The basic temperature gradient of the annulus model has been identified with the \( r \)-derivative of the temperature in the equatorial plane at the distance \( r_m \) from the axis. Similar expressions with the same dependence on \( \tau \) and \( P \) can be obtained when the general case (2c) of the basic temperature distribution is used instead of the special case \( R_c = 0 \). It is obvious that the expressions (6a)–(6c) cannot be expected to yield more than an order of magnitude estimate for the accurate parameter values characterizing the onset of convection. But even this rough guideline is useful for the orientation in the vast parameter space. Expression (6a) suggests that \( R_c \) does not change when \( P = 0.5 \) instead of \( P = 1 \) is used while
Figure 2 The critical values $R_{ic}$ of the Rayleigh number for the onset of convection in the case $\tau = 10^4$, $P = 1$, $Re = 0$ for different values of $m$ (indicated by circles). Also shown are values $R_{ic}$ (multiplied by $[P\tau/(1+P)]^{4/3}$) and corresponding values of $m$ (multiplied by $[P\tau/(1+\tau)]^{1/3}$) in the cases $\tau = 10^4$, $P = 0.5$ (triangles), $\tau = 10^4$, $P = 2$ (diamonds), $\tau = 1.5 \times 10^4$, $P = 0.5$ (squares) and $\tau = 3 \times 10^4$, $P = 1$ (crosses).

$\tau$ is increased by 50%. In Figure 2, examples of this kind have been displayed which indicate that the scaling of expressions (6a)–(6c) is only approximately valid. At least the critical wavenumber $m_c$ appears to obey the scaling Eq. (6b) fairly well as is evident from the nearly identical location of the minima of Figure 2. In the limit of low $P$ the critical Rayleigh number in particular departs rather rapidly from the value given by expression (6a). As shown by Yano (1992) $R_c$ is amplified by a factor of the order $P^{-1}$ from the value (6a) in that limit, owing to the influence of the radial dependence of the convection flow.

The above relationships (6a)–(6c) do not take into account the radial dependence of the convection columns at onset. In fact, it is the property that the onset of convection in the rotating cylindrical annulus becomes independent of the gap width in the asymptotic limit of high $\tau$ that make relationships of this kind applicable to spherical shells and other axisymmetric containers. While the annulus model with straight conical end boundaries in the axial direction as shown in Figure 1 yields a phase of the convection columns or thermal Rossby waves independent of radius, the surfaces of constant phase are tilted in the prograde direction with increasing distance from the axis when convexly curved end boundaries are used. The opposite sense of spiralling is found when concavely curved boundaries are applied. This feature is a consequence of the property that the more strongly (weakly) inclined part of the
boundary imparts onto the thermal Rossby waves a tendency to propagate faster (slower) which must be compensated by viscous stresses (Busse and Hood, 1982).

The spiralling nature of thermal Rossby waves has an important consequence at finite convection amplitudes: The azimuthally averaged advective of azimuthal momentum does no longer vanish as in the case of a radially independent phase. But instead a differential rotation is generated by the Reynolds stresses of the convection columns. This differential rotation could not be a very strong effect if a feedback process did not come in as well: The steady differential rotation created by the balance of Reynolds and viscous stresses tends to increase the tilt of the convection cells and thus enhances its own source. This feedback mechanism is responsible for the instability that occurs in the case of thermal Rossby waves in the absence of a curved boundary. While this mean flow instability (Busse, 1986; Or and Busse, 1987) can develop in the annulus with either sign, that is, outward transports of prograde as well as of retrograde momentum are possible depending on the sign of the disturbances, only the prograde transport occurs in the case of a spherical shell. The generation of differential rotation by convection in spherical shells could thus be regarded as an imperfect bifurcation of the mean flow instability.

4. Convection in rotating spherical shells at finite amplitudes

The main difficulty faced by the convection flow as it evolves with increasing Rayleigh number is the fact that its columnar structure is not well suited to transport heat from and to the curved boundaries. Since the term describing the transport of heat is basically the same as the term describing the release of potential energy, it is clear that the amplitude of convection cannot grow rapidly with $R_i$ or $R_e$ if the heat transport to the walls is impeded. It is thus not surprising that the amplitude of the steadily drifting convection columns tends to saturate with increasing $R$ as has been demonstrated, for example, in Figure 7 of Ardes et al. (1997). Convection modes with a strong time dependence tend to replace the steadily drifting columns because of their ability to enhance the heat transport. Most common are vacillating convection columns which either oscillate coherently or incoherently in that a $m = 1$ azimuthal modulation sets in. Vacillating convection can be found in the rotating cylindrical annulus (Or and Busse, 1987), in the spherical case at infinite Prandtl number (Zhang, 1992b), at $P = 1$ (Sun et al., 1993; Grote and Busse, 2001) or at low Prandtl numbers (Ardes et al., 1997). But there are also other ways in which bifurcations from the steadily drifting columns occur, for example through a subharmonic bifurcation as shown in Figure 3. Every second convection column is stretched in the azimuthal direction until the outer tip of the spiraling column snaps off. In the meantime the stretching of the neighboring columns has begun and the process is repeated in a time periodic fashion. If one defines the period of this process as the time in which the same pattern is obtained except for a rotation about the axis, one finds that the period decreases from 0.032 at $R_i = 3.1 \times 10^5$ to 0.021 at $R_i = 3.3 \times 10^5$. At the slightly higher Rayleigh number of $R_i = 3.5 \times 10^5$ a transition to a quasi-periodic time dependence has already occurred which is characterized by a $m = 1$ modulation of the pattern. With increasing $R_i$ the convection pattern becomes increasingly chaotic as is evident from the time dependence of the energies of various components of motion shown in Figure 4. The energy of the axisymmetric toroidal component of motion grows especially rapidly with $R_i$ because of the strong differential rotation created by the Reynolds stresses.
Figure 3  Sequence in time (upper row left to right then lower row right to left) of plots of streamlines \((r\partial v/\partial \phi = \text{constant})\) in the equatorial plane in the case \(R_i = 3.2 \times 10^5\), \(\tau = 1.5 \times 10^4\), \(P = 0.5\), \(R_e = 0\). The equidistant \((\Delta t = 0.005)\) plots cover a period such that the last plots closely resembles the first plot except for a shift in azimuth.

The kinetic energy can be separated into four different kinds:

\[
E_p^m = \frac{1}{2} \langle |\nabla \times (\nabla \bar{v} \times \bar{r})|^2 \rangle, \quad E_t^m = \frac{1}{2} \langle |\nabla \tilde{w} \times \bar{r}|^2 \rangle, \quad (7a)
\]
\[
E_p^f = \frac{1}{2} \langle |\nabla \times (\nabla \tilde{v} \times \bar{r})|^2 \rangle, \quad E_t^f = \frac{1}{2} \langle |\nabla \tilde{w} \times \bar{r}|^2 \rangle. \quad (7b)
\]

where the brackets \(\langle \cdots \rangle\) indicate the average over the spherical shell, \(\bar{v}\) denote the axisymmetric component of \(v\) and \(\tilde{v} \equiv v - \bar{v}\) indicates the non-axisymmetric component of \(v\).

Also presented in Figure 4 are time series of the Nusselt number \(Nu\), which measures the convective heat transport and which is defined by

\[
Nu = 1 - \frac{P}{r_i} \frac{\partial \Theta}{\partial r} \bigg|_{r = r_i},
\]

where \(\bar{\Theta}\) indicates the average of \(\Theta\) over the surface \(r = \text{constant}\). The energy \(E_t^f\) has not been plotted because it closely parallels \(E_t^m\) even though it is about twice as large. The energy of the axisymmetric poloidal component has also not been included because it is several orders of magnitude smaller than the other energies. As \(R_i\) is increased to \(5 \times 10^5\) the activity of convection tends to be localized on one side of the sphere similarly as in the case of \(P = 1\), \(\tau = 10^4\), \(R_i = 7 \times 10^5\) studied by Grote and Busse (2001). Because of the strong increase in the differential rotation, the radial shear inhibits convection over most parts of the spherical fluid shell. With increasing \(R_i\), the inhibition of convection by the differential...
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rotation finally becomes so strong that convection is only possible after the shear has decayed sufficiently in a time interval of near absence of convection. At the end of this interval when the differential rotation reaches its minimum value convection starts to grow for a short moment until the differential rotation has been accelerated by the Reynolds stresses to its previous level and begins to shear off again the convection columns. The resulting relaxation oscillations are characterized by a surprisingly well-defined period in spite of the chaotic nature of the fluctuating part of the velocity field as can be seen from the uppermost lines in each of the three plots of Figure 4. The period of the order of 0.1 corresponds to the viscous decay time of the differential rotating and has been found to be rather independent of $P, R_i$.
Figure 5 Time series of the mean energy densities of the axisymmetric toroidal component (upper plot) and of the non-axisymmetric poloidal component of motion (middle plot) and of the Nusselt number (lower plot) in the case $\tau = 10^4$, $P = 1$, $R_i = 0$ for $R_e = 6 \times 10^5$ (dash dotted lines), $8 \times 10^5$ (long dash lines), $1.2 \times 10^6$ (short dash lines), $1.4 \times 10^6$ (dotted lines) and $1.7 \times 10^6$ (solid lines).

or $\tau$. For other examples of these oscillations, we refer to Grote et al. (2000b) and Grote and Busse (2001).

Here we wish to demonstrate that the coherent structures of turbulent convection in rotating spherical shells are not limited to the case of internal heating. When $R_e$ is used as a parameter instead of $R_i$ rather similar phenomena are observed. An overview of the dependence of convection on $R_e$ is provided by Figure 5. After onset of convection a transition to a quasi-periodic form occurs rather quickly with increasing Rayleigh number. In Figure 6, the time dependence of the convection columns is visualized in a sequence of plots. Similar plots for different parameter values are shown by Sun et al. (1993). In the absence of internal heating, $R_i = 0$, the temperature gradient near the inner boundary is especially strong and the amplitude of the spiralling convection columns decays more strongly with increasing $r$ than in the case $R_e = 0$. For this reason, the disruption of the convection columns and
Figure 6 Time sequence (left to right, then top to bottom) of equidistant \((\Delta t = 5 \times 10^{-3})\) plots of streamlines, \(r \partial v / \partial \phi = \text{constant}\) in the equatorial plane in the case \(\tau = 10^4, R_e = 5.5 \times 10^5, R_i = 0, P = 1\). The kinetic energies exhibit oscillations with the period 0.018, similar to those shown in the case of \(R_e = 6 \times 10^5\) in Figure 5.

their subsequent reconnection as shown in the figure occurs more rapidly after onset. At \(R_e = 8 \times 10^5\) convection flow has become chaotic as can be seen from the energy of the fluctuating poloidal component of motion. A visualization of the flow at a particular time can be gained from Figure 7. The flow at \(R_e = 10^6\) which is also shown in this figure, indicates the tendency towards localized convection although this feature does not seem to develop as dramatically as in the case exhibited in the paper of Grote and Busse (2001). At \(R_e = 1.2 \times 10^6\), the relaxation oscillation begins to set in which becomes fully developed as we reach the highest Rayleigh number, \(R_e = 1.7 \times 10^6\), of Figure 5. The amplitude of the oscillation of the differential rotation is not quite as large as in the case of Figure 4 and the period is slightly shorter. But the phenomenon itself does not depend on the detailed form of the basic temperature distribution.
Figure 7 Lines of constant radial component of the velocity on the mid-surface of the shell, \( r = \frac{(1 + \eta)}{2(1 - \eta)} \), (upper plots) and streamlines \( (r \partial v/\partial \varphi = \text{constant}) \) in the equatorial plane for \( R_e = 8 \times 10^5 \) (left side) and \( R_e = 10^6 \) (right side) in the case \( \tau = 10^4, P = 1, R_i = 0 \).

5. Convection-driven dynamos

All convection flows realized in sufficiently rapidly rotating spherical shells appear to be capable of acting as dynamos if only the magnetic Prandtl number \( P_m \) is sufficiently large, such that a critical magnetic Reynolds number of the order 100 is exceeded even for small amplitudes of convection. A typical diagram is shown in Figure 8, where the open symbols characterize the onset of dynamo action in the \( R_i - P_m \)-space for the Coriolis parameter \( \tau = 10^4 \) and the closed symbols do the same for \( \tau = 3 \times 10^4 \). In the latter case, the values of \( R_i \) must be multiplied by the factor 10 as indicated on the right ordinate. This figure is an extension of a similar one presented by Grote et al. (2000b). Analogous diagrams as a function of \( R_e \) instead of \( R_i \) have been obtained by Christensen et al. (1999). There are some typical differences in that dynamos other than dipolar ones are the exception in the cases considered by Christensen et al. (see also Kutzner and Christensen, 2000) while in the case of Figure 8 dynamos of predominantly dipolar character are found only for relatively high values of \( P_m \). They are replaced by hemispherical dynamos as \( P_m \) decreases and finally at low values of \( P_m \), quadrupolar dynamos are obtained. But the meaning of “high” and “low” values of \( P_m \) varies with \( \tau \) in that the transition from dipolar to hemispherical dynamos shifts towards lower \( P_m \) with increasing \( \tau \). In the case of \( \tau = 3 \times 10^4 \) it has not yet been possible to reach the regime of quadrupolar dynamos.

All hemispherical and quadrupolar dynamos exhibit an oscillatory character in that a dynamo wave propagates from the equator to the pole or to the poles, respectively.
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Figure 8 Existence of convection dynamos of different types as a function of the Rayleigh number $R_i$ and the magnetic Prandtl number $P_m$ in the cases $\tau = 10^4$ (open symbols and crosses) and $\tau = 3 \times 10^4$ (filled symbols). In the latter case, the values of $R_i$ are given by the right ordinate. The Prandtl number is $P = 1$ in all cases.

An example is shown in Figure 9. The emergence of magnetic flux with a new polarity appears to be initiated at the equator of the inner boundary as can be best seen in the plots of $B_\phi$ in the figure. The magnetic energy also changes in phase with the oscillation and reaches its maximum when the axisymmetric flux tubes with opposite signs of $B_\phi$ reach about equal amplitude. The perfect correlation between the amplitudes of spherical harmonics with $l - m = \text{odd}$ and with $l - m = \text{even}$, which characterizes an ideal hemispherical dynamo is nearly approached in the case $m = 0$ as can be seen from the upper plot of Figure 10. In the case of the non-axisymmetric components, the correlation is not as good, but is still quite remarkable.

A clear distinction of dynamos with different symmetry is usually only possible as long as convection occurs predominantly outside the tangent cylinder and still exhibits an approximate symmetry about the equator. Even in the case of hemispherical dynamos only minor deviation from this symmetry are caused by the action of the Lorentz force. As soon as the critical Rayleigh number for onset of convection in the polar regions is exceeded, the influence of equatorial symmetry diminishes and mixtures of quadrupolar and dipolar components of the magnetic field are generated without the nearly perfect correlation that characterizes hemispherical dynamos. Usually, the quadrupolar part of the magnetic field is still strongest outside the tangent cylinder while the dipolar components appear to be generated primarily in the polar regions. With increasing Rayleigh number convection grows more strongly in the polar regions than outside the tangent cylinder and as a consequence the magnetic field becomes more dipolar. In contrast to the dipolar dynamos indicated in Figure 8, the high
Figure 9 Oscillating hemispherical dynamo in the case $R_i = 2.7 \times 10^6$, $\tau = 3 \times 10^4$, $P_m = 0.5$, $P = 1$. The plots represent a sequence in time (from top to bottom in equidistant steps with $\Delta t = 0.0012$) with the column displaying the velocity field with lines of constant $\overline{u}_\phi$ in the upper left quarter, streamlines $r \sin \theta \partial \overline{E}_r / \partial \theta = \text{constant}$ in the upper right quarter and streamlines $r \partial \overline{E}_r / \partial \phi = \text{constant}$ in the lower half of each circle. The middle column shows lines of constant $B_r$ at the outer surface, $r = (1 - \eta)^{-1}$, and right column shows lines of constant $B_\phi$ in the left half and meridional field lines $r \sin \theta \overline{E}_r / \partial \theta = \text{constant}$ in the right half of each circle.
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Figure 10 Coefficients $H_{m}^{0}(r, t)$ and $G_{m}^{0}(r, t)$ (defined for the variables $h, g$ in analogy to definition (5) for $v$) as a function of time at the position $r = r_i + 0.5$. The upper plot shows $H_{0}^{0}(t)$ (dotted line), $H_{2}^{0}(t)$ (solid line) and $G_{1}^{0}(t)$ (dashed line). The lower plot shows $G_{1}^{1}(t)$ (dotted line) and $G_{2}^{1}(t)$ (solid line).

Rayleigh number dipolar fields still exhibit the oscillations that characterize the predominantly quadrupolar fields at lower values of $R_i$. An example of an oscillatory mainly dipolar dynamo is shown in Figure 11.

6. Interaction of magnetic fields and convection

The problem of convection in rotating systems in the presence of an imposed magnetic field has a long history. It can be formulated as a linear problem when the conditions for the onset of convection are of primary interest. Early work on the subject is reviewed in Chandrasekhar’s (1961) monograph. Here it has been demonstrated that the stabilizing effect of the Coriolis force on the onset of convection in a horizontal fluid layer heated from below and rotating about a vertical axis can be partly released when a vertical magnetic field is imposed. Even an imposed horizontal field may lead to reduction of the critical Rayleigh number $R_c$ for onset of convection if the rotation rate is high enough (Eltayeb, 1972). For a recent study of this problem which includes the possibility of oscillatory onset we refer to Roberts and Jones (2000). The optimal strength $B_0$ of the magnetic field for lowering $R_c$ is usually given by
an Elsasser number $\Lambda$ of the order unity where $\Lambda$ is defined by

$$\Lambda = B_0^2 / \Omega \mu \lambda.$$  \hfill (9)

Here $\Omega$ denotes the absolute value of the angular velocity of rotation.
Convection in rotating spherical shells

In the rotating cylindrical annulus as shown in Figure 1, an axisymmetric azimuthal magnetic field will also lead to a reduction of \( R_c \) if the parameter \( \tau \) is high enough (Busse, 1976; Busse and Finocchi, 1993). The same result has been obtained for an axisymmetric radial field and for the case of convection in the presence of an azimuthal magnetic field in a rotating sphere (Fearn, 1979a,b). Whenever inhomogeneous magnetic fields are imposed such as a curved azimuthal one in the case of the sphere, the possibility of magnetic instability must be taken into account. We do not want to enter the discussion of this subject here and instead refer to the recent paper by Zhang (1995) and the references cited therein. Except for this latter possibility the optimal field strength for the reduction of \( R_c \) corresponds to an Elsasser number of the order unity and the characteristic wavenumber \( \alpha \) of convection is reduced to a value of the order \( \pi \) or less. While in the absence of a magnetic field the inhibiting effect of the Coriolis force is counteracted by the viscous friction associated with high wavenumber convection rolls, a second minimum of the Rayleigh number in the \( R(\alpha) \)-relationship typically develops at low values of \( \alpha \) when the Lorentz force becomes sufficiently strong. Viscous dissipation is replaced by ohmic dissipation in this case and convection flows in low Prandtl number liquid metals exhibits features which are typical for convection in fluids with much higher Prandtl numbers (Busse et al., 1997; Petry et al., 1997). Non-linear properties of convection subject to both, the effects of rotation and of an imposed azimuthal magnetic field, have also been studied by Olson and Glatzmaier (1995) and by Zhang (1999) in the case of a spherical shell and by Cardin and Olson (1995) in the case of a rotating cylindrical annulus with the same radius ratio as the liquid outer core of the Earth. Besides the imposed azimuthal magnetic field, the case of an imposed uniform magnetic field parallel to the axis of rotation has received much attention (Sarson et al., 1999; Sakuraba and Kono, 2000). The influence of the onset of convection is minimal in this case since the dependence of columnar convection on the coordinate in the axial direction is rather weak. But a change in the non-linear properties of convection is found in the above mentioned papers when \( \lambda \) exceeds a value of the order \( \pi \) or less.

The interaction of convection with magnetic fields generated by its own dynamo action appears to be quite different from that with an imposed magnetic field. No reduction of the critical Rayleigh number \( R_c \) for onset of convection has yet been found in the case of convection-driven dynamos and significant reductions of the characteristic azimuthal wavenumber of columns have not been noticed in the case of the dynamos discussed in Section 5. The major effect of the generated magnetic field on convection in fluids with \( P \) of the order unity or less is the braking of the differential rotation. Through the Lorentz force the magnetic field drains energy from the differential rotation and destroys the relaxation oscillations mentioned in Section 4. The radial extent of the convection columns is increased and the transport of heat is enhanced. These effects are clearly demonstrated in Figure 12, where the relaxation oscillation with an increased energy of the differential rotation returns after the magnetic field has decayed as often happens when the Rayleigh number is just below the value needed for sustained dynamo action.

The main question of the interaction between convection and the magnetic field generated by its own dynamo action is the question of the mean amplitude of the magnetic field. With this in mind, extensive computations have been performed to produce the dependences on \( R_i \) of the energies of the various components of magnetic and velocity fields as shown in Figure 13. Although the energies are obtained from averages over several ohmic or viscous decay times, the influence of the statistical fluctuations cannot be eliminated entirely and the curves shown in the figure are not as smooth as can be expected if computations would be continued over much longer periods in time. While the magnetic field exhibits a pure
quadrupolar symmetry at low Rayleigh numbers, $R_i \lesssim 5 \times 10^5$, the dipolar component sets in for $R_i \gtrsim 5 \times 10^5$ which corresponds to the Rayleigh number where convection in the polar regions becomes possible. There is actually an uncertainty about the onset of the dipolar component of the magnetic field because it corresponds to a subcritical bifurcation. As demonstrated in Figure 14, a purely quadrupolar dynamo state as well as one with a small but finite dipolar component can be realized in the neighborhood of $R_i = 5 \times 10^5$. At $R_i = 4.8 \times 10^5$ only a purely quadrupolar state is obtained with about 3/4 of the average energy densities found for $R = 5 \times 10^5$.

With increasing $R_i$, the dipolar component grows slightly faster than the quadrupolar component such that it tends to exceed the latter in energy for $R_i \gtrsim 1.4 \times 10^9$. We have already mentioned the example of an oscillating predominantly dipolar magnetic field as shown in Figure 11. The axisymmetric components of both, the dipolar and the quadrupolar parts of the magnetic field, clearly saturate with increasing $R_i$. They actually tend to decay with a further increase of $R_i$ because flux expulsion from the convection eddies becomes a dominant process. Even the fluctuating components are affected by this process which limits their energies and leads to a highly filamentary structure of the magnetic field. This latter property is also evident from the ohmic dissipation which does not exhibit the saturation of the magnetic energies, but instead parallels the growth of viscous dissipation albeit at a lower level.
Since the Elsasser number can be written in the form:

\[ \Lambda = 2 M P_m \tau^{-1} \]  

(10)

where \( M \) denotes the average density of the total magnetic energy, we conclude from Figure 13 that the value \( \Lambda = 1 \) is approached by the saturating magnetic field. While such a value makes
Figure 14  Energies of quadrupolar (dipolar) components of the magnetic field are shown as a function of time in upper (lower) plots for $\tau = 5 \times 10^3, R = 5 \times 10^5, P = P_m = 1$. The left side corresponds to a dynamo of mixed parity, while the right side presents a purely quadrupolar dynamo. The toroidal energy densities $M_m^T, M_f^T$ (definitions are analogous to those given by expressions (7a) and (7b)) and poloidal energy densities $M_m^P, M_f^P$ are indicated by solid and dashed lines, respectively. Thick (thin) lines indicate energy densities of axisymmetric (non-axisymmetric) components.

sense on the basis of considerations discussed above, the open question remains, why this value is not reached at lower values of the Rayleigh number. The fact that the ohmic dissipation amounts to a certain fraction of the viscous dissipation could be a more typical feature of the interaction between magnetic field and convection in the parameter regime that has been investigated so far. Further explorations of the parameter dependences of convection-driven dynamos are clearly needed.

7. The problem of reversals
We have already mentioned that in contrast to dynamos with hemispherical or quadrupolar structure, dynamos generating dipolar fields do not exhibit oscillatory behavior in general. Only in the case of high Rayleigh numbers when the dipolar component of the magnetic field is supported by convection flows in the polar regions can oscillatory behavior be recognized as shown in Figure 11. Reversals of a predominantly dipolar component of the magnetic field can be seen, however, as a more randomly occurring event in cases of high Rayleigh number.
Figure 15 Time sequence (from top to bottom) with $\Delta t = 0.02$ of a reversing dynamo in the case $R = 7 \times 10^6$, $\tau = 10^4$, $P = 1$, $P_m = 6$. The plots on the left side show lines of constant $B_{\varphi}$ in the left half and meridional field lines $r \sin \theta \partial \vec{B}/\partial \theta = $ constant in the right half of each circle. The plots on the right side show lines of constant $B_r$ on the surface $r = 2r_o$.

when turbulent convection generates strongly time dependent fields. An example is shown in Figure 15.

Because the magnetic field is already highly filamentary we have plotted the component $B_r$ on the surface of the sphere with twice the radius of the fluid shell. The plots on the right-hand
side of Figure 15 thus emphasize the large-scale structures and indicate substantial deviations from the structure of an axial dipole. If we had plotted lines of constant $B_r$ on the surface of the fluid shell, a filamentary structure of the magnetic field would have been apparent similar to that shown in the plots of the middle column in Figure 11.

Although the transition to the reversed polarity does not occur in an oscillatory manner, the same mechanism still seems to be working. The magnetic field with the new polarity first appears near the equator of the inner sphere and propagates outward and to higher latitude. While this process can be seen most clearly in the mean zonal flux, $B_\phi$, in the case of the oscillatory dynamo of Figure 11, it is now more clearly evident in the structure of the meridional field. The flux tubes of $B_\phi = \text{constant}$ have assumed such a small scale in Figure 15 that their propagation can no longer be easily seen.

While the present computations have been continued for several viscous diffusion times, longer integration times are needed to obtain a statistically valid sample of reversals. It will also be of interest to study the effect of an electrically conducting inner core on the reversal process in its oscillatory and its more random manifestations. The computations done by Hollerbach and Jones (1993) have indicated a strong influence of the inner core, but our preliminary computations do not indicate a significant effect in the case of reversals with a highly turbulent dynamo.

Occasional reversals in dynamo simulations exhibiting dominant fluctuating components of the magnetic field have been found in previous work (see, e.g. Kida and Kitauchi, 1998) and appear to be a common phenomenon when the axisymmetric components of the magnetic field play a secondary role in comparison to the non-axisymmetric ones. This situation cannot be excluded for the Earth’s core and some paleomagnetic observations (Guyodo and Valet, 1999) indicate strongly fluctuating intensity variations. But because the parameter regime accessible to numerical dynamo simulations is still quite different from that of the Earth’s core, it is too early to draw definitive conclusions from similarities between computed and observed magnetic fields.

8. Concluding remarks
The dynamo studies discussed in this chapter have been motivated by geophysical applications. But it is still doubtful whether the mechanisms explored hitherto can provide a realistic description of the geodynamo. It has been difficult, for instance, to reach a magnetic Prandtl number much less than unity. It may not be necessary to attain the value $10^{-6}$ based on the molecular diffusivities expected to characterize the liquid outer core of the Earth. But the effects of small-scale turbulence are unlikely to raise the magnetic Prandtl number much beyond $10^{-2}$. Christensen et al. (1999) have suggested that the critical value of $P_m$ for dynamo action decreases like $750\tau^{-3/4}$ with increasing $\tau$. We are less optimistic in this respect since our computations suggest a much weaker dependence.

A common property of most computational dynamos is that ohmic dissipation is either less or not far above viscous dissipation. With decreasing Prandtl number the ratio of ohmic to viscous dissipation usually increases as can be seen from figure 7(b) of Grote et al. (2000b). It thus seems prudent to proceed in the direction of lower $P$ in order to increase that ratio, especially since low values of $P$ are typical for liquid metals like those in planetary cores. On the other hand, the effective Prandtl number must be large in the Earth’s core since large ratios between magnetic and kinetic energies can be expected only in the case of large $P$ (Glatzmaier and Roberts, 1995; Busse et al., 1998). This latter property can be used for arguing that the concentration of light elements with its low diffusivity rather than the temperature provides...
the buoyancy for driving convection. A combination of both effects may even lead to new effects which have not yet been fully explored.

References


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