Regimes of thermo-compositional convection and related dynamos in rotating spherical shells

James F. Mather¹ and Radostin D. Simitev²

School of Mathematics and Statistics, University of Glasgow – Glasgow G12 8QQ, UK

Convection and magnetic field generation in the Earth and planetary interiors are driven by both thermal and compositional gradients. In this work numerical simulations of finite-amplitude double-diffusive convection and dynamo action in rapidly rotating spherical shells full of incompressible two-component electrically-conducting fluid are reported. Four distinct regimes of rotating double-diffusive convection identified in a recent linear analysis (Silva et al., 2019, Geophys. Astrophys. Fluid Dyn., doi:10.1080/03091929.2019.1640875) are found to persist and can be followed significantly beyond the onset of instability while their regime transitions remain abrupt. In the semi-convecting and the fingering regimes characteristic flow velocities are small compared to those in the thermally- and compositionally-dominated overturning regimes, while zonal flows remain weak in all regimes apart from the thermally-dominated one. Compositionally-dominated overturning convection exhibits significantly narrower azimuthal structures compared to all other regimes while differential rotation becomes the dominant flow component in the thermally-dominated case as driving is increased. Dynamo action occurs in all regimes apart from the regime of fingering convection. While dynamos persist in the semi-convective regime they are very much impaired by small flow intensities and very weak differential rotation in this regime which makes poloidal to toroidal field conversion problematic. The dynamos in the thermally-dominated regime include oscillating dipolar, quadrupolar and multipolar cases not dissimilar to the ones known from earlier parameter studies. Dynamos in the compositionally-dominated regime exhibit subdued temporal variation and remain predominantly dipolar due to weak zonal flow in this regime. These results significantly enhance our understanding of the primary drivers of planetary core flows and magnetic fields.

Keywords: Double-diffusive convection; Buoyancy-driven instabilities; Dynamo action; Planetary cores;

1. Introduction

Convective flows and magnetic field generation in Earth’s fluid outer core are driven by a combination of thermal and chemical composition gradients (Kono 2002, Jones 2015, Wicht and Sanchez 2019). Rotating double-diffusive convection is also likely to occur in and affect the magnetic properties of other planets including Mercury (Breuer et al. 2007), Venus (Jacobson et al. 2017), Jupiter (Moll et al. 2017) and Saturn (Leconte and Chabrier 2012) as well as various stellar objects (Garaud 2018).

Numerical models of planetary and core convection and dynamos have been predominantly single-diffusive (Jones 2011), with the exception of the celebrated early geodynamo model of Glatzmaier and Roberts (1996). Either purely thermal convection was assumed or the so called “co-density” formulation introduced by Braginsky and Roberts (1995) was used. However, single-diffusive convection models fail to account for significant differences in the diffusivities of heat and chemical constituents as well as for essential differences in boundary conditions and sink/source distributions of the temperature and of the constituent concentration field (Jones 2015). In the last decade several authors have recognized these limitations and have sought to investigate explicitly double-diffusive themo-compositional effects on convection flows and
dynamo processes in rapidly-rotating spheres and shells. Using two separate equations for the temperature and for the concentration of light constituents, respectively, Breuer et al. (2010) observed an abrupt change in convective regime when the relative contribution of compositional driving exceeds 20%. In a similar model, Trümper et al. (2012) considered the effect of distinct boundary conditions for temperature and concentration and obtained preliminary results on the onset of convection using an initial value code. Simulating a double-diffusive model of Mercury’s dynamo, Manglik et al. (2010) observed that when thermal and compositional buoyancy are of equal intensity, a stratified outer layer is formed and is then penetrated by fingering convection that in turn enhances the poloidal magnetic field, a significant difference compared to co-density cases. Reporting some 20 numerical dynamo runs, Takahashi (2014) found that due to helicity increase magnetic fields have predominantly non-dipolar morphology when compositional buoyancy is less than 40% of the total driving and predominantly dipolar one otherwise. In short, significant thermo-compositional effects were found in all of the latter works. However, these studies were largely limited to numerical runs isolated in the configuration space and considered the case of when the thermal buoyancy and the compositional buoyancy are both destabilising.

Systematic parameter studies of the linear onset of double-diffusive convection in rotating spherical shells were undertaken by Net et al. (2012) and recently by Silva et al. (2019). Following Simitev (2011), the study of Silva et al. (2019) confirmed that due to distinct “double-diffusive” eigenmodes, critical curves for the onset of thermo-compositional instability are generally multi-valued and form what may be described as “pockets” of instability protruding regions of quiescence. Situations thus arise whereby increasing the thermal or the compositional driving leads to onset of instability at the entry of a pocket and a subsequent return to quiescence as driving is further increased to reach the exit of the pocket. The pockets of instability are closely related to transitions between four different regimes of rapidly-rotating double-diffusive convection that can be identified as semi-convection, fingering convection, thermally-dominated overturning convection and chemically dominated overturning convection. The onset of these regimes were mapped by Silva et al. (2019) who probed a significant portion of the configuration space by varying the values of all governing parameters.

These linear results were confirmed by Monville et al. (2019) in the case of a full sphere. The latter study also reports finite-amplitude simulations in the regime of fingering convection. However, further systematic study of nonlinear double-diffusive convection in rotating spherical shells, and especially of its dynamo action, across all regimes is desirable. The goal of our article is to provide results in this direction. To this end, we summarize below our observations of over new 80 finite-amplitude thermo-compositional convection simulations and over 30 dynamo solutions. In order to swipe all four double-diffusive regimes, we perform parameter continuations varying the thermal and compositional Rayleigh numbers (defined further below) but, for the sake of comparison, keep most other governing parameters fixed to values where purely thermal convection and dynamo solutions are well studied in the literature. In doing this, we wish to assess to what extent instability pockets survive nonlinear interactions; whether convective regime boundaries can be traced to finite-amplitudes; the structure and intensity of zonal and other flow components. Further it is of interest to verify whether dynamos can be obtained in all four convective regimes; to observe how magnetic field morphology and symmetry change across the regime boundaries; to characterize the time-dependent behaviour of dynamo solutions and to find out whether dynamos that are close to each other in the parameter space can exhibit widely different morphology and behaviour. These and other questions are addressed in the present article. They are relevant to understanding the geomagnetic field at present and in geological time as well as to understanding the zoo of other planetary and stellar magnetic fields.
2. Mathematical model

We follow standard mathematical formulations of the problem of rotating spherical dynamos e.g. (Simitev and Busse 2005) modified by introducing a separate equation for the chemical concentration of the form used in (Silva et al. 2019). In detail, we consider a two-component electrically conducting fluid confined to a spherical shell rotating with a fixed angular velocity \( \Omega \hat{k} \), where \( \hat{k} \) is the unit vector in the direction of the axis of rotation. The inner and outer spherical surfaces, \( r = r_i \) and \( r = r_o \), respectively, are kept at constant values of the temperature and of the concentration of the light element. The gravity field is assumed in the form \( g = -\gamma d \hat{r} \) where \( \hat{r} \) is the position vector with respect to the center of the sphere and \( r \) is its length measured in units of the thickness \( d \) of the spherical shell. Assuming volumetric sources/sinks of thermal and compositional buoyancy with constant densities \( \beta_T \) and \( \beta_C \), respectively, a static state exists with temperature and concentration profiles given by

\[
T_S = T_0 - \frac{\beta_T}{2} r^2, \quad C_S = T_0 - \frac{\beta_C}{2} r^2, \quad (1)
\]

respectively. Here \( \eta \) denotes the ratio of inner to outer radius of the shell, \( T_0 \) and \( C_0 \) are constant reference values of temperature and concentration. We employ the Boussinesq approximation in that all material properties of the fluid are assumed constant except the density which is taken to depend on temperature and concentration so the following truncated Taylor expansion near its reference value \( \rho_0 \) is used when it enters the buoyancy term

\[
\rho = \rho_0 (1 - \alpha_T \Theta - \alpha_C \Gamma),
\]

where \( \alpha_T \) and \( \alpha_C \) are the specific thermal and compositional coefficients of expansion/contraction, and \( \Theta \) and \( \Gamma \) are the deviations from the temperature and the concentration basic static states \( T_S \) and \( C_S \) given by equations (1). The length \( d \), the time \( d^2 / \nu \), the temperature \( \nu^2 / \gamma \alpha_T d^4 \) and composition \( \nu^2 / \gamma \alpha_C d^4 \) are used as scales for the dimensionless description of the problem where \( \nu \) denotes the kinematic viscosity. The dimensionless governing equations of momentum, temperature, concentration, magnetic induction and the conditions of incompressibility of the fluid and solenoidality of the magnetic field are then given by

\[
\partial_t \left( \mathbf{u} + \mathbf{u} \cdot \nabla \right) \mathbf{u} + \tau \mathbf{k} \times \mathbf{u} = -\nabla \pi + \left( \Theta + \Gamma \right) \mathbf{r} + \nabla^2 \mathbf{u} + \left( \nabla \times \mathbf{B} \right) \times \mathbf{B}, \quad (2a)
\]

\[
\nabla \cdot \mathbf{u} = 0, \quad (2b)
\]

\[
Pr \left( \partial_t \left( \mathbf{u} + \mathbf{u} \cdot \nabla \right) \right) \Theta = R_T \mathbf{r} \cdot \mathbf{u} + \nabla^2 \Theta, \quad (2c)
\]

\[
Sc \left( \partial_t \left( \mathbf{u} + \mathbf{u} \cdot \nabla \right) \right) \Gamma = R_C \mathbf{r} \cdot \mathbf{u} + \nabla^2 \Gamma, \quad (2d)
\]

\[
\partial_t \mathbf{B} = \nabla \times \left( \mathbf{u} \times \mathbf{B} \right) + \operatorname{Pm}^{-1} \nabla^2 \mathbf{B}, \quad (2e)
\]

\[
\nabla \cdot \mathbf{B} = 0, \quad (2f)
\]

where \( \mathbf{u} \) and \( \mathbf{B} \) are the velocity and the magnetic field vectors, respectively, and \( \pi \) denotes an effective pressure field representing all terms that can be expressed as a gradient. Appart from the radius ratio \( \eta \), six dimensionless numbers appear in the equations, namely, the thermal Rayleigh number \( R_T \), the compositional Rayleigh number \( R_C \), the Coriolis parameter \( \tau \), the Prandtl number \( Pr \), the Schmidt number \( Sc \) and the magnetic Prandtl number \( Pm \) defined as

\[
R_T = \alpha_T \gamma \beta_T d^6 / \nu \kappa, \quad R_C = \alpha_C \gamma \beta_C d^6 / \nu D, \quad \tau = \frac{2 \Omega d^2}{\nu}, \quad Pr = \frac{\nu}{\kappa}, \quad Sc = \frac{\nu}{D}, \quad Pm = \frac{\nu}{\lambda}, \quad (3)
\]

respectively. Here, \( \kappa \) denotes the thermal diffusivity, \( D \) denotes the mass diffusivity and \( \lambda \) denotes the magnetic diffusivity.

Since the velocity and the magnetic field are both solenoidal the general representation in
terms of poloidal and toroidal components can be used,
\[ u = \nabla \times (\nabla v \times r) + \nabla w \times r, \quad B = \nabla \times (\nabla h \times r) + \nabla g \times r. \]

By multiplying the \((\text{curl})^2\) and the curl of the momentum equation \((2a)\) by \(r\) we obtain two equations for the poloidal and toroidal scalar fields of the velocity, \(v\) and \(w\),
\[ ((\nabla^2 - \partial_t) L_2 + \tau \partial_\varphi) \nabla^2 v + \tau Q w - L_2(\Theta + \Gamma) = -r \cdot \nabla \times \nabla \times (u \cdot \nabla u - B \cdot \nabla B), \quad (4a) \]
\[ ((\nabla^2 - \partial_t) L_2 + \tau \partial_\varphi) w - \tau Q v = r \cdot \nabla \times (u \cdot \nabla u - B \cdot \nabla B). \quad (4b) \]

The temperature and the concentration equations may be written as follows
\[ \nabla^2 \Theta + R_T L_2 v = Pr(\partial_t + u \cdot \nabla) \Theta, \quad (4c) \]
\[ \nabla^2 \Gamma + R_G L_2 w = Sc(\partial_t + u \cdot \nabla) \Gamma. \quad (4d) \]

The equations for ploidal and toroidal scalar of the magnetic field, \(h\) and \(g\), are obtained by multiplication of \((2e)\) and of its curl by \(r\)
\[ \nabla^2 L_2 h = \text{Pm}(\partial_t L_2 h - r \cdot \nabla \times (u \times B)), \quad (4e) \]
\[ \nabla^2 L_2 g = \text{Pm}(\partial_t L_2 g - r \cdot \nabla \times (u \times B)). \quad (4f) \]

In the above, \(\partial_t\) and \(\partial_\varphi\) denote the partial derivatives with respect to time \(t\) and with respect to the angle \(\varphi\) of a spherical system of coordinates \(r, \theta, \varphi\) and the operators \(L_2\) and \(Q\) are defined by
\[ L_2 \equiv -r^2 \nabla^2 + \partial_r(r^2 \partial_r), \]
\[ Q \equiv r \cos \theta \nabla^2 - (L_2 + r \partial_r)(\cos \theta \partial_r - r^{-1} \sin \theta \partial_\theta). \]

Stress-free boundaries with fixed temperature and concentration values are assumed,
\[ v = \partial_{rr} v = \partial_r (w/r) = \Theta = \Gamma = 0 \quad \text{at} \quad r = r_i, \quad r = r_o, \quad (5a) \]
where \(r_i = \eta/(1 - \eta)\) and \(r_o = (1 - \eta)^{-1}\). For the magnetic field, electrically insulating boundaries are assumed in such a way that the poloidal function \(h\) is matched to the function \(h^{(e)}\) which describes the potential fields outside the fluid shell
\[ g = h - h^{(e)} = \partial_r (h - h^{(e)}) = 0 \quad \text{at} \quad r = r_i, \quad r = r_o. \quad (5b) \]

The mathematical problem is solved numerically as detailed in the next section.

3. Numerical methods and diagnostics

For the direct numerical integration of the problem defined by the scalar equations \((4)\) and the boundary conditions \((5)\) we use a pseudo-spectral method described by Tilgner (1999). A code used by us for a number of years (Busse et al. 2003, Busse and Simitev 2011, Simitev et al. 2015) and benchmarked for accuracy most recently in (Marti et al. 2014, Matsui et al. 2016) was adapted to include the additional equation for the chemical concentration of light elements \((4d)\). The code has been made open source (Silva and Simitev 2018). We briefly mention here that the spatial discretisation is based on an expansion of all dependent variables in spherical harmonics for the angular dependences and in Chebychev polynomials for the radial dependence, e.g. the expansion of the poloidal scalar function takes the form
\[ v(r, \theta, \varphi) = \sum_{l,m,n} V_{l,m,n}^m(t) T_n(x(r)) P_l^m(\theta) \exp(im\varphi), \quad (6) \]
with \( x(r) = 2(r - r_i) - 1 \), and analogous expressions for the other dynamical variables, \( w, h, g, \) \( \Theta \) and \( \Gamma \) are used. Here \( P_l^m \) denotes the associated Legendre polynomials of degree \( l \) and order \( m \) and \( T_n \) denotes the Chebychev polynomials of degree \( n \). The computation of nonlinear terms in spectral space is, however, expensive so all nonlinear products and the Coriolis term are computed in the physical space and then projected onto the spectral space at every time step. A hybrid combination of the Crank–Nicholson scheme for the diffusion terms and the second order Adams–Bashforth scheme for the nonlinear terms is used for time-stepping. Spherical harmonics truncation of \( N_l = N_m = 144 \) and up to \( N_r = 51 \) collocation points in radial direction have been used and a time step of the order \( 10^{-6} \) was typically required for the computations reported below.

To analyse the properties of the solution we typically monitor kinetic and magnetic energy components. In particular, we decompose the kinetic energy density into poloidal and toroidal parts (respectively denoted by subscripts as in \( X \) below) and where \( X \) denotes an appropriate quantity) and further into mean (axisymmetric) and fluctuating (nonaxisymmetric) components (respectively denoted by bars and tildes as in \( X \) and \( \tilde{X} \) below) and into equatorially-symmetric and equatorially-antisymmetric components (respectively denoted by superscripts as in \( X^s \) and \( X^a \) below),

\[
\begin{align*}
\tilde{E}_p &= \tilde{E}_p^s + \tilde{E}_p^a = \frac{i}{2} \left( \langle \nabla \times (\nabla (v^s + v^a) \times r) \rangle \right) , \\
\tilde{E}_t &= \tilde{E}_t^s + \tilde{E}_t^a = \frac{i}{2} \left( \langle \nabla r (w^s + w^a) \rangle \right) , \\
\tilde{E}_E &= \tilde{E}_E^s + \tilde{E}_E^a = \frac{i}{2} \left( \langle \nabla \times (\nabla (\tilde{v}^s + \tilde{v}^a) \times r) \rangle \right) , \\
\tilde{E}_t &= \tilde{E}_t^s + \tilde{E}_t^a = \frac{i}{2} \left( \langle \nabla r (\tilde{w}^s + \tilde{w}^a) \rangle \right),
\end{align*}
\]

where angular brackets \( \langle \rangle \) denote averages over the volume of the spherical shell. Since in our code the spectral representation of all fields \( X \) is given by the set of coefficients \( \{ X_l^m \} \) of their expansions in spherical harmonics \( Y_l^m \), it is easy to extract the relevant components, i.e. coefficients with \( m = 0 \) and with \( m \neq 0 \) represent axisymmetric and nonaxisymmetric components, respectively, while coefficients with even \( l + m \) and with odd \( l + m \) represent equatorially-symmetric and equatorially-antisymmetric components, respectively. The magnetic energy density is similarly decomposed into components.

### 4. The onset of rotating double-diffusive convection

A systematic investigation of the onset of double-diffusive convection in rotating spherical shells is reported in (Silva et al. 2019). In this section, we summarize very briefly some linear results from this study that are directly related to the discussion of finite-amplitude dynamics in the following sections.

Figure 1 shows the critical Rayleigh numbers \( R_T \) and \( R_C \), the preferred azimuthal wave number \( m \) and drift rate \( \omega \) for the onset of instability at other parameter values fixed to

\[
\begin{align*}
\text{Pr} = 1, \quad \text{Sc} = 25, \quad \tau = 10^4, \quad \eta = 0.35, \quad \text{Pm} = 0. 
\end{align*}
\]

To facilitate direct comparison with (Silva et al. 2019), in the left panel (a) of the figure a re-parametrisation is used where a ‘total effective’ Rayleigh number and a mixing angle \( \alpha \) are introduced as follows

\[
\begin{align*}
\text{Ra} = \sqrt{R_T^2 + R_C^2}, \quad \alpha = \text{atan2}(R_C, R_T).
\end{align*}
\]

Here the function \( \text{atan2}(y, x) \) is defined for \( x \in \mathbb{R}, \ y \in \mathbb{R} \) as the principal argument \( \text{Arg}(z) \)
of the complex number $z = x + iy$, a notation used in many programming languages. The more conventional representation of the critical curves in the $R_C - R_T$ plane is shown in the right panel figure 1(b). Four convective regimes are immediately discernible – semi-convection, thermally-dominated overturning convection, compositionally-dominated overturning convection and fingering convection. For convenience, these regimes are labeled by A, B, C and D, respectively in figures and discussion throughout. Regime A occurs for values of the mixing angle $\alpha$ between $-\pi/2$ and $-5\pi/16$. This regime was identified in (Silva et al. 2019) as semi-convection as it occurs at negative values of $R_C$ and positive values of $R_T$ which in our model corresponds to cooler and lighter fluid over warmer and heavier fluid. The linear analysis indicates that flows in this regime consist of large spatial structures with azimuthal wave number $m = 3$ drifting in retrograde direction ($\omega > 0$). An abrupt transition to a different regime B occurs at values of the mixing angle of about $\alpha \approx -5\pi/16$ and persists up to $\alpha \approx 5\pi/16$. Within this range the critical curve is smooth and includes as a particular case purely thermal convection $R_C = 0$ at $\alpha = 0$ so this regime can be identified as thermally-dominated overturning convection. At values of the other governing parameters given by (8) the flow takes the familiar form of columnar convection with $m = 9$ vertical columns arranged in a cartridge belt inside the cylinder tangent to the inner core and drifting in prograde direction ($\omega < 0$). A second abrupt transition takes place at $\alpha \approx 5\pi/16$ and brings the flow to a regime C that persist up to about $\alpha \approx 17\pi/32$. This range includes as a particular case purely compositional convection $R_T = 0$ at $\alpha = \pi/2$ and is, therefore, identified as compositionally-dominated overturning convection. It is interesting to note that there is a large jump in the azimuthal wave number to $m = 25$ which then monotonically decreases to $m = 8$ as the mixing angle $\alpha$ is increased. The convective patterns of regime C continue to drift in prograde direction. At $\alpha \approx 17\pi/32$ a last transition is observed leading to regime D that persist to $\alpha = 3\pi/4$. This
regime is identified as fingering convection as it largely occupies a region where $R_T < 0$ and $R_C > 0$ corresponding to a configuration of warmer and heavier fluid over cooler and lighter fluid. The wavenumber assumes a constant value of $m = 3$ and the flow pattern drifts in prograde direction. We note that while the critical curve is single-valued when plotted in the $Ra - \alpha$ plane, it is multivalued in certain regions when plotted in the $R_C - R_T$ plane. The linear results outlined below are robust in the sense that they persist in a similar form in a significant range of parameter values. For instance, Silva et al. (2019) report the same qualitative picture for values of the Coriolis parameter in the range $\tau \in [10^3, 10^6]$ (becoming more pronounced for larger values of $\tau$), for values of the Prandtl number in the range $Pr \in [10^{-5}, 10^3]$ (most pronounced at intermediate values of $Pr$), for Schmidt to Prandtl number ratio at least within the range $Sc/Pr \in [25, 100]$, and for shell radius ratio in the range $\eta \in [0.1, 0.7]$.

5. Finite amplitude thermo-compositional convection

In this section, we describe the properties of finite-amplitude thermo-compositional convection. To keep the volume of simulations at bay, we fix most of the parameter values as specified in equation (8). In order to trace out systematically the four distinct regimes identified by linear analysis, we now vary the values of the thermal and compositional Rayleigh numbers as it is best illustrated in figure 1(b). It is known from earlier studies of purely thermal convection e.g. (Sun et al. 1993, Christensen 2002, Simitev and Busse 2003b, Busse and Simitev 2005, Gillet et al. 2007) that immediately after onset convection assumes the form of shape-preserving columns parallel to the rotation axis and drifting in azimuthal direction with time-independent azimuthally averaged properties. From a frame of reference drifting together with the convection columns the entire pattern appears steady. Differential rotation is generated through the action of Reynolds stress caused by the spiralling cross section of the columns. Further away from the onset subsequent bifurcations break all available temporal and spatial symmetries of the problem and lead to flows with increasingly chaotic temporal and spacial structures and dependence.

With this in mind, we are interested to explore the transitions between the four double-diffusive regimes and outline features different from those of purely thermal convection. Figure 2 shows time-averaged values of the kinetic energy density components of the flows along several selected “slices” through the $R_C - R_T$ plane. With increasing the value of $R_T$ in panels 2(a) and (c) one observes the regimes of rotating fingering convection D, compositionally dominated overturning convection C and thermally dominated overturning convection B and the transitions between them. In figure 2(c), regime D appears as an isolated island of instability separated from regime C by a quiescent region. The amplitude of fingering convection motions attains a maximum at a value of $R_T$ situated within the interior of the island of instability and decreases towards its ends. At a somewhat larger value of $R_C$ depicted in figure 2(a), the domains of regimes D and C join up but the transition between them is well visible in the pronounced dip in kinetic energy components. An abrupt transition from regime C to regime B occurs at about $R_T = 2.7 \times 10^5$ in both panels (a) and (c) of figure 2. In figure 2(e), the transition between the regimes of rotating semi-convection A and thermally-dominated overturning convection B is shown. This transition is also abrupt and occurs at about $R_T = 2.8 \times 10^5$. The transition value $R_T = 2.8 \times 10^5$ appears to be situated along the continuation of the thermally-dominated overturning convection branch of the critical curve for the onset. Panels 2(b,d,f) show kinetic energy densities as functions of the compositional Rayleigh number $R_C$ for several selected values of the thermal Rayleigh number $R_T$. Panel (b) shows the energy density components of cases in regime B. A transition to a branch with dominant differential rotation is observed at about $R_C = 9 \times 10^5$. This is a transition internal to regime B, namely a transition to chaotic convection. In figure 2(d), regimes A and C are observed separated by
a wide quiescent region of no convection. A gradual increase of the amplitude of the flows is seen as convection is driven away from the critical curve of linear onset. Finally, figure 2(f) shows the energy density components of flows in regime D as they are driven away from the onset neutral curve.

The fluctuating toroidal energy \( \langle \tilde{E}_t \rangle_t \) and the fluctuating poloidal energy \( \langle \tilde{E}_p \rangle_t \) are kinetic energy components corresponding to flows in the azimuthal direction and flows within meridional planes, respectively. These two components show very similar behaviour in all four convective regimes as seen in figure 2 because locally any flow pattern consists of both azimuthal and radial motions. The fluctuating kinetic energies dominate the flows in the regimes of rotating semiconvection A, fingering convection D, and compositionally dominated overturning convection C as well as in the vicinity of the onset transition of the regime of thermally dominated overturning convection B. It may be observed in figure 2 that in all cases the approximate relation \( \langle \tilde{E}_p \rangle_t \approx 0.3 \langle \tilde{E}_t \rangle_t \) is satisfied regardless of the particular convection regime. The mean poloidal energy \( \langle E_p \rangle_t \) is the energy contained in the mean meridional circulation. Since the effects of rotation strongly suppress the motions in the axial direction this component re-

Figure 2. Time-averaged kinetic energy density components in the case \( \text{Pr} = 1, \text{Sc} = 25, \tau = 10^4, \eta = 0.3 \) and (a) \( R_C = 5 \times 10^5 \), (b) \( R_T = 3 \times 10^5 \), (c) \( R_C = 4.5 \times 10^5 \), (d) \( R_T = 2 \times 10^5 \), (e) \( R_C = -5 \times 10^5 \), (f) \( R_T = -2 \times 10^5 \) as functions of \( R_T \) or \( R_C \) as appropriate. The time-averaged mean poloidal component \( \langle E_p \rangle_t \), mean toroidal component \( \langle E_t \rangle_t \), fluctuating poloidal component \( \langle \tilde{E}_t \rangle_t \) and fluctuating toroidal component \( \langle \tilde{E}_t \rangle_t \) are indicated by a black circles, red squares, green plus signs and blue cross signs, respectively. Vertical dash-dotted lines indicate points of transition between convective regimes. The ranges over which distinct regimes are observed are indicated by arrows near the bottom abscissa. Arrowheads indicate that convection extends further and ticks at end of arrows indicate onset of convection. The locations in the \( R_C - R_T \) plane of some data points used here are also denoted by red solid circles in figure 1. (Colour online)
Figure 3. Flow components in the case $\eta = 0.3$, $\tau = 10^4$, $Pr = 1$, $Sc = 25$, $R_C = 4.5 \times 10^5$, and increasing values of $R_T = -2 \times 10^5$, $1 \times 10^5$, $2 \times 10^5$, $3 \times 10^5$, $5 \times 10^5$, from (a) to (f), respectively, (all part of the sequence shown in figure 2(c)). The first plot in each column shows isocontours of azimuthally-averaged $\bar{u}_\phi$ (left half) and streamlines $r \sin \theta \partial_r v = \text{const.}$ (right half) in the meridional plane. The second plot shows isocontours of $u_r$ at $r = r_i + 0.5$ mapped to the spherical surface using an isotropic Aitoff projection. The third plot shows isocontours of $r \partial_\phi v = \text{const.}$ in the equatorial plane. The isocontours are equidistant with positive isocontours shown by solid lines, negative isocontours shown by broken lines and the zeroth isocontour shown by a dotted line in each plot. All contour plots are snapshots at a fixed representative moment in time. Labels D, C and B denote the dominant regime of convection. (Colour online)

mains small, and this is seen on all convection branches in figure 2. The mean toroidal energy $\langle E_t \rangle_t$ is the kinetic energy contained in differential rotation. On all branches shown in figure 2 except in the regime of thermally-dominated overturning convection B, the time-averaged value of $\langle E_t \rangle_t$ is small and comparable to $\langle E_p \rangle_t$. There are a number of reasons for this. Firstly, compositional regimes A and D appear to be limited in range and transitions to the overturning regimes B and C occur before their energy components, including $\langle E_t \rangle_t$, can grow significantly. In addition, the two regimes A and D show broad convection structures that are not prone to spiralling, see further below. The regime of compositionally-dominated overturning convection C is characterised with a large value of the Schmidt number Sc and behaves much like purely thermal convection at large values of Pr where spiralling and consequently differential rotation is known to be weak (Simitev and Busse 2005). Differential rotation starts to grow only when the regime of thermally-dominated overturning convection B is entered. Since the toroidal fluid motions do not have a radial component, it is the poloidal motions which are directly associated with the transport of heat and material between the boundaries. The differential rotation, however, helps dynamo generation by the $\Omega$-effect.

Regarding time-dependence, compositional regimes A and D are time-independent since transitions to regimes B and C occur before their near-onset patterns can become unstable and develop chaotic structures. Regimes B and C are time dependent and exhibit the usual sequence from steady state via time-periodic oscillations to chaotic behaviour.

Examples of typical flow patterns across the four convective regimes are presented in figures 3 and 4. Figure 3, in particular, includes selected cases located along the slice of the $R_C - R_T$ plane shown in figure 2(c). The first column of 3 shows a typical case in the state of rotating fingering convection D. At a value $R_C = -2 \times 10^5$ this case is located nearly in the middle of the regime region. The equatorial cross section of the poloidal streamlines, shown in the lowermost plot of figure 3(a), reveals a pattern of wave number 5 consisting of pairs of a narrow column with anticlockwise flow and a broad column with clockwise flow. The broad column appears as composed of two structures with clockwise flow adjacent to each other and
Figure 4. Flow components in the case $\eta = 0.3$, $\tau = 10^4$, $Pr = 1$, $Sc = 25$, and values of the Rayleigh numbers as follows (a) $R_C = -4 \times 10^5$ and $R_T = 2 \times 10^5$, (b) $R_C = 8 \times 10^5$ and $R_T = 2 \times 10^5$, (c) $R_C = 8 \times 10^5$ and $R_T = 4 \times 10^5$ and (d) $R_C = 8 \times 10^5$ and $R_T = 6 \times 10^5$. The same flow components are shown as in figure 3. Labels A, C and B denote the dominant regime of convection. (Colour online)

with the one located closer to the inner core being more vigorous. The convective columns and their vertical morphology are visible in the plot of the radial velocity on a spherical surface located at the at the middle of the shell, shown in the middle row of figure 3(a). The plot of the radial velocity also reveals that there is little or no polar convection. Differential rotation and azimuthally averaged meridional circulation are plotted in the left and the right halves, respectively, of figure 3(a). The differential rotation profile is rather peculiar. Differential rotation is symmetric with respect to the equatorial plane and shows a prominent tube of retrograde flow situated at the equatorial region and extending to about $30^\circ$. On top of this tube there are two azimuthal jets of similarly strong prograde differential rotation. The shear layer between the jet of retrograde and the jets of prograde rotation is almost horizontal and the two prograde jets link to each other via a thin layer going behind the equatorial retrograde jet. Two weak retrograde jets are finally visible in the polar regions. The meridional circulation is antisymmetric with respect to the equatorial plane consists of cells three cells oriented parallel to the axis of rotation.

At $R_T = 10^5$ figure 3(b) illustrates a compositionally-dominated overturning convection case in regime C as expected from linear analysis. The case is in the vicinity of the linear onset and the convection pattern changes dramatically from the case just described. The poloidal streamlines plotted in the equatorial plane exhibit a dominant azimuthal wave number of 21. The convective columns are rather thin and adjacent to the outer spherical surface rather than to the inner core. There is no polar convection as rotation inhibits polar motions. Meridional circulation is antisymmetric with respect to the equator and consists of three cells. The convection columns are spiralling very weakly and thus only weak differential rotation is generated. The differential rotation is now prograde at the outer spherical surface and at the equatorial region. Pronounced retrograde jets appear at the poles while a pronounced prograde jet appears at the inner surface near the equatorial region.

At $R_T = 2 \times 10^5$ figure 3(c) shows a more chaotic compositionally-dominated overturning convection case in regime C. The equatorial streamlines show convection columns with a wave number of 20 near the outer surface which now extend all the way to the inner core where clockwise columns coalesce to a wavenumber of about 10. The columns extend to higher latitude as seen in the plot of the radial velocity on the spherical surface. The mean
meridional circulation is now five-cell and the differential rotation has become stronger with
the two large polar jets extending towards the equator and coalescing.

The last three columns of figure 3 exhibit three cases of thermally-dominated convection in
regime B that become increasingly chaotic with the increase of the value of \( R_T \). The poloidal
streamlines in the equatorial plane reveal a flow pattern of 7 pairs of convection columns that
are anchored at the inner core and strongly spiral outwards. The mean meridional streamlines
form a large one-cell antisymmetric pattern and the differential rotation is exhibits a strictly
goostrophic pattern constant on cylinders parallel to the tangent cylinder. The case with \( R_T = 2.8 \times 10^5 \) shown in figure 3(d) is situated very nearly at the transition from state C
which is evidenced by a weak modulation of the convective column tips near the outer spherical
surface. The case with \( R_T = 5 \times 10^5 \) shown in figure 3(f) is in a chaotic state known as a
localized convection where differential rotation is so strong that it shears-off the convective
columns so that convection is weak or suppressed in approximately half the shell volume.
These properties are essentially identical to the properties of purely thermal convection at
comparable parameter values (Simitev and Busse 2003b, Busse and Simitev 2005).

The last three columns of figure 4 also illustrate the transition from compositionally- to
thermally-dominated convection, i.e. from regime C to regime B, but in a more strongly
chaotic cases with a larger value of \( R_C = 8 \times 10^5 \). It is interesting that the C to B transition
and the corresponding patterns of convection are very similar to the ones just described despite
the significant increase in compositional driving.

Finally at \( R_C = -4 \times 10^6 \), the first column of figure 4 illustrates a typical rotating semi-
convection case in regime A. An azimuthal wave-number 3 is exhibited by the poloidal stream-
lines in the equatorial plane. The asymmetry between narrow columns with anticlockwise flow
and broad columns with clockwise flow featured in by fingering regime D is also present here.
The profiles of mean meridional flow and of the differential rotation are also very similar to
the corresponding patterns of the fingering regime but are in reversed direction, i.e. a simi-
lar in shape equatorial jet of differential rotation is seen however in prograde rather than in
retrograde direction.

In this section cases with small to moderate driving were used in order to clearly illustrate
the patterns of convection and the transitions between the four regimes of rotation double-
diffusive convection. More strongly driven convection cases are presented in the next section
where a sufficient intensity of the flow is a prerequisite for dynamo action.

6. Dynamos driven by thermo-compositional convection

Examples of dynamos generated by purely thermal convection as well as by thermo-
compositional convection are well documented in the literature, see e.g. (Busse and Simitev
2005) and (Takahashi 2014), respectively. Our simulations, indicate that flows in the rotating
semi-convective regime A are also capable of acting as dynamos. However, we were not able to
find a dynamo solution for flows in regime D of rotating fingering double-diffusive convection.
In this section we detail these findings.

A typical dynamo generated by rotating semi-convective flows in regime A at \( R_T = 2 \times 10^5 \)
and \( R_C = -6 \times 10^6 \) and a value of the magnetic Prandtl number \( Pm = 300 \) is presented in figure
5. This dynamo exhibits a chaotic time dependence even though snapshots at different times
remain rather similar to the one shown in figure 5. The ratio of toroidal to poloidal magnetic
energy is \( E_{tor}^{magn}/E_{pol}^{magn} = 9.98 \) so that much of the magnetic field is confined to the core.
The ratio of \( E_{dip}^{magn}/E_{quad}^{magn} = 3272.9 \) so that the dynamo is almost perfectly dipolar. However,
the ratio of total magnetic to total kinetic energy is negligibly small at \( E_{magn}/E_{kin} = 0.0042 \).
Because the magnetic field is so weak compared to the intensity of the flow, convection is
nearly unaffected by dynamo action. Indeed, the spatial patterns of the flow remain very
A

Figure 5. Flow and field structures of a dynamo in convective regime A at parameter values $\eta = 0.3$, $\tau = 10^4$, $Pr = 1$, $Sc = 25$, $R_T = 2 \times 10^5$, $R_C = -6 \times 10^6$, and $Pm = 300$. The first plot shows isocontours of $\mathbf{B}_\phi$ (left half) and meridional fieldlines $r \sin \theta \partial_\theta h = \text{const.}$ (right half). The second plot shows isocontours of radial magnetic field $B_r$ at $r = r_o + 0.1$ in isotropic Aitoff projection. The last three plots show the same flow components as in figure 3. All contour plots are snapshots at a fixed representative moment in time (Colour online).

similar to the ones described in relation to the corresponding case shown in figure 4(a). The most pronounced difference is that at $m = 6$ the azimuthal wave number is larger than in the case shown in the first column of 4(a) which is likely due to the smaller value of the compositional Rayleigh number $R_C$ used here. A plot of the azimuthally averaged toroidal fieldlines in the meridional plane are shown in the left half of the first plot of figure 5. The plot reveals a pair of toroidal flux tubes of opposite polarity located near the outer core above and below the equator. A second pair of elongated toroidal flux parallel to the axis of rotation are situated at the cylinder tangent to the inner core. This profile of the azimuthally averaged toroidal fieldlines correlates well with the profile of the differential rotation shown in the left half of the third plot in 5 which exhibits regions of shear near the equator and on the tangent cylinder. The azimuthally averaged poloidal fieldlines in the meridional plane are shown in the right half of the first plot of figure 5 and exhibit a typical dipolar structure. Besides the apparent large-scale dipolar field, the plot of the radial magnetic field at surface exhibits two small patches of inverted polarity near the equator. Because the ratio of total magnetic to total kinetic energy is very small while at the same time the value of the magnetic Prandtl number $Pm = 300$ needed to support dynamo action in this case is very large it is unlikely that dynamos in state A are of relevance in planetary and stellar magnetism.

We have been unsuccessful in our efforts to obtain a non-decaying dynamo in regime D of rotating fingering double-diffusive convection. It is more interesting to investigate how the sharp transition between regimes B to C affects dynamo behaviour and whether the properties of the magnetic field also exhibit an abrupt transition.

A example of a dynamo in the regime of thermally-dominated overturning convection B is presented in figure 6 at values of the magnetic Prandtl number $Pm = 15$, the compositional Rayleigh number $R_C = 1.2 \times 10^6$ and the thermal Rayleigh number $R_T = 3 \times 10^5$, respectively. This is a strong field dynamo with a ratio of total magnetic to kinetic energy $E_{\text{magn}}/E_{\text{kin}} = 1.01$. The predominant dynamo morphology is quadrupolar as evidenced also by the ratio of dipolar to quadrupolar energy components $E_{\text{dip}}^{\text{magn}}/E_{\text{quad}}^{\text{magn}} = 0.02$. The component detectable outside of the dynamo generating region is strong as measured by the ratio of toroidal to poloidal magnetic energy $E_{\text{tor}}^{\text{magn}}/E_{\text{pol}}^{\text{magn}} = 1.22$. As is typical for quadrupolar dynamos this example exhibits regular periodic oscillations in time. The oscillations are well visible in the plots of all magnetic field components and represent a dynamo wave propagating from the equator to the poles. Emergence of magnetic flux is initiated at the equator near the inner core boundary as can be best seen in the plots of the azimuthally averaged toroidal and poloidal fieldlines in the first column of figure 6. These new flux tubes proceed to grow and drift to higher latitudes pushing old flux tubes upwards in the northern hemisphere and downwards in the southern hemisphere. At high latitudes the magnetic flux tubes in question become weaker gradually dissipate and are replaced in a similar fashion by new flux of the opposite polarity. The dynamo wave exhibits a phase shift in azimuthal direction in the sense that at any given moment the pole-ward drifting flux tubes appear at various latitude for various azimuthal
Figure 6. Oscillation of a quadrupolar dynamo in convective regime B at parameter values \( \eta = 0.3, \tau = 10^4, \Pr = 1, \Sc = 25, \RT = 3 \times 10^5, \RC = 1.2 \times 10^6, \text{and } \Pm = 15 \). Snapshots are shown (a) at the beginning and (b) at the end of a half a period the oscillation with time lapse \( \Delta t = 5.0 \). The same flow and field components are shown as in figure 5. (Colour online)

angles thus forming a pair of V-shaped spiraling structures as seen in the contour plot of the radial magnetic field \( B_r \). The frequency of oscillations is proportional to the square roots of the magnetic Prandtl number and the kinetic helicity and to the fourth root of the differential rotation (Busse and Simitev 2006). Although this is a strong field dynamo, flow structures are weakly affected by the magnetic field and remain much as described in relation to figures 3 and 4 above. An exception to this is the strong effect of the magnetic field on differential rotation which is significantly reduced in amplitude and oscillates between a geostrophic and conical profile as seen in the left half of the third column of figure 6. The weaker differential rotation is as a result less able to shear-off the azimuthally drifting convective columns. This example is very similar to oscillating quadrupolar dynamos familiar from purely-thermal models e.g. (Busse and Simitev 2005). It is perhaps surprising that a quadrupolar dynamo occurs at a rather larger value of the magnetic Prandtl number than in the former case, e.g. see figure 3 of (Busse and Simitev 2005).

Two typical dynamos in regime C of rotating compositionally-dominated overturning convection are illustrated in figure 7 at a value of the magnetic Prandtl number \( \Pm = 30 \), a value of the compositional Rayleigh number \( \RC = 1.2 \times 10^6 \) and thermal Rayleigh numbers \( \RT = -2 \times 10^5 \) and \( \RT = 2 \times 10^5 \), respectively. Both dynamos appear very similar, indeed. They are strong field dynamos with a ratio of total magnetic to total kinetic energy \( E_{magn}/E_{kin} = 3.31 \) and 2.61 the dynamos with \( \RT = -2 \times 10^5 \) and \( \RT = 2 \times 10^5 \), respectively. The predominant dynamo morphology is dipolar as evidenced also by the ratio of dipolar to quadrupolar energy components which is \( E_{magn}^{\text{dip}}/E_{magn}^{\text{quadr}} = 28.2 \), and 8.87 for the first and the second dynamos, respectively. The externally observable components the dynamos are weak as measured by the ratio of toroidal to poloidal magnetic energy and are \( E_{magn}^{\text{tor}}/E_{magn}^{\text{pol}} = 0.32 \) and 0.56, for the first and the second dynamos, respectively. As is typical for dynamos dominated by dipolar components both dynamos are non-oscillatory and, while the time series of their kinetic and magnetic energy components are chaotic, the spacial patterns of the velocity and magnetic fields remain very similar to the ones shown in figure 7. The azimuthally averaged toroidal fieldlines shown in the left halves of the first column of figure 7 exhibit magnetic flux tubes antisymmetric with respect to the equator. In particular, pair of strong flux tubes appear in the polar regions adjacent to the axes of rotation. The case with \( \RT = -2 \times 10^5 \) shows an equally strong pair of “butterfly” shaped flux tubes at mid to low latitudes. A similar pair appears in the case \( \RT = 2 \times 10^5 \) but seems to be smaller and weaker in comparison to the polar flux tubes of this case. The plots of the radial magnetic field also exhibit similar equatorially antisymmetric patterns when projected onto the spherical surface of the shell.
Although both cases are strong field dynamos, their flow structures are not really affected by the magnetic field and remain much as described in relation to figures 3 and 4 above. Again an exception to this is the profound effect of the magnetic field on differential rotation which being weaker than the one of the cases in regime B, is further reduced in amplitude and is modified to assume a profile featuring a retrograde jet in the equatorial region.

It is interesting to note that an abrupt transition in magnetic field properties seem to exist when the boundary of the convective regimes B and C is traversed. Indeed, the relatively small change in the value of the thermal Rayleigh number $R_T$ needed to transition between the dynamos shown in figures 7 and 6 produces magnetic fields of very different intensity and morphology. In addition, there is a pronounced difference in the value of the magnetic Prandtl number required to obtain non-decaying dynamos in the two regimes in question.

7. Conclusion

In this paper, we report a numerical study of finite-amplitude double-diffusive convection in rapidly rotating spherical shells full of incompressible two-component electrically-conducting fluid and investigate the possibility of dynamo action and the features of the magnetic fields generated in the various convective regimes identified. In particular, guided by previous linear stability analysis of the onset of convection in this configuration (Silva et al. 2019), we find four distinct regimes of rotating double-diffusive convection, namely rotating semi-convection, rotating thermally-dominated overturning convection, rotating compositionally-dominated overturning convection and rotating fingering convection. Since the configuration space of the problem is now significantly larger that that of purely thermal convection, we restrict the attention to a set of carefully selected parameter values and model assumptions that allow for (a) a systematic access to all four double-diffusive regimes and at the same time (b) for direct comparison of results to purely thermal simulations readily available in the literature e.g. (Simitev and Busse 2003a, Busse and Simitev 2005; 2006). To this end, internally distributed heat and concentration sources, stress-free velocity and fixed temperature and concentration boundary conditions were used at the electrically insulating inner and outer spherical surface following the articles cited above. Both thermal and compositional Rayleigh numbers $R_T$ and $R_C$ were varied while all other parameters were fixed at relatively modest but computationally feasible values $Pr = 1$, $Sc = 25$, $\tau = 10^4$, $\eta = 0.35$, and the magnetic Prandtl number was selected so as to be just over the onset of dynamo action. We find that convection instability pockets identified in the linear stability analysis (Simitev 2011, Silva et al. 2019)
survive nonlinear interactions and that the boundaries of the four convective regimes can be traced towards finite-amplitudes in the configuration space for, at least, several times the critical values of $R_T$ and $R_C$. We confirm that boundaries remain sharp resulting in abrupt transitions between convective regimes as noted by Breuer et al. (2010) for the boundary between the thermally-dominated and the compositionally-dominated overturning convection at the parameter values they used. Typical velocities of flows in the semi-convecting and the fingering regimes remain relatively small compared to those in the two overturning regimes, and zonal flows remain weak in all regimes apart from the regime of thermally-dominated overturning convection. Finite-amplitude compositionally-dominated overturning convection exhibits significantly narrower azimuthal structures compared to all other regimes while the convective columns of the semi-convecting and the fingering regimes are particularly wide but with some notable asymmetry between clockwise and counter-clockwise vortices with the latter being narrower. The thermally-dominated overturning regime retain properties very similar to that of purely thermal convection, with differential rotation becoming the dominant flow component as driving is increased. We find that dynamo action occurs in all regimes apart from the regime of fingering convection. In the latter increasing the value of the magnetic Prandtl number and venturing, as far as feasible, further out to more strongly driven flows failed to sustain the initial magnetic field seeds. Dynamo action persists in the semi-convective regime but it is very much impaired by the small intensity of the flow and the by very weak differential rotation which makes conversion of poloidal to toroidal field problematic. Unrealistically, large values of the magnetic Prandtl number were required to obtain a non-decaying dynamo solution in this case. Both regimes of overturning convection easily support magnetic field generation. The dynamos in the thermally-dominated regime include oscillating dipolar, quadrupolar and multipolar cases not dissimilar to the ones known from our earlier parameter studies at comparable parameter values e.g. (Simitev and Busse 2003a, Busse and Simitev 2005; 2006; 2011). Due to the significantly weaker zonal flow dynamos in the compositionally-dominated regime show much more subdued temporal variation and remain predominantly dipolar as also reported by Takahashi (2014).

To meet the goals of this study, it was necessary to use parameter values removed from both geo- and planetary estimates. To rectify this, it is desirable to approach more ambitious values allowed by modern computing resources. In the configuration studied, we did not find a regime where stable quiescent layers or layers with a distinct flow properties form spontaneously below the core-mantle boundary and coexists with flow in the bulk. It is of significant interest (Jones 2015, Olson et al. 2018, Wicht and Sanchez 2019, Bouffard et al. 2019) to investigate whether such layers may form under different boundary conditions for the concentration and the temperature field and or different internal source-sink distributions. These issues remain interesting avenues for future work.

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