Convection-driven dynamos in rotating spherical shells: oscillations, reversals and bistability

R.D. Simitev
Department of Mathematics

F.H. Busse
Institute of Physics
Motivation: Applications of spherical dynamos

- Dynamos in rotating spherical shells

- Geomagnetism
- Planetary magnetism
- Solar and stellar magnetism
Outline of the talk

- Mathematical formulation of the problem
- Numerical methods of solution
- Overview of the basic effects controlling dynamo behaviour and types of dynamo solutions: a focus on low magnetic Prandtl number regime
- Oscillations of dipolar dynamos as a possible cause of geomagnetic excursions and reversals
- Bistability and hysteresis of fully nonlinear dympolar dynamos
- Conclusions
Convective spherical shell dynamos

Model equations & parameters

Boussinesq approximation

\[ \nabla \cdot \mathbf{u} = 0, \quad \nabla \cdot \mathbf{B} = 0, \]

\[
\partial_t \mathbf{u} + \mathbf{u} \cdot \nabla \mathbf{u} = -\nabla \pi - \tau \mathbf{k} \times \mathbf{u} + \Theta \mathbf{r} + \nabla^2 \mathbf{u} + \mathbf{B} \cdot \nabla \mathbf{B},
\]

\[ P \left( \partial_t \Theta + \mathbf{u} \cdot \nabla \Theta \right) = R \mathbf{r} \cdot \mathbf{u} + \nabla^2 \Theta, \]

\[ P_m \left( \partial_t \mathbf{B} + \mathbf{u} \cdot \nabla \mathbf{B} \right) = P_m \mathbf{B} \cdot \nabla \mathbf{u} + \nabla^2 \mathbf{B}. \]

\[
R = \frac{\alpha \gamma \beta d^6}{\nu \kappa}, \quad \tau = \frac{2 \Omega d^2}{\nu}, \quad P = \frac{\nu}{\kappa}, \quad P_m = \frac{\nu}{\lambda}
\]

Boundary Conditions

\[ \mathbf{r} \cdot \mathbf{u} = \mathbf{r} \cdot \nabla \mathbf{r} \times \mathbf{u} / r^2 = 0, \]

\[ \hat{\mathbf{e}}_r \cdot \mathbf{B}_{\text{int}} = \hat{\mathbf{e}}_r \cdot \mathbf{B}_{\text{ext}}, \]

\[ \hat{\mathbf{e}}_r \times \mathbf{B}_{\text{int}} = \hat{\mathbf{e}}_r \times \mathbf{B}_{\text{ext}}, \]

\[ \Theta = 0, \text{ at } r = r_i \equiv 2/3 \text{ and } r_o \equiv 5/3 \]

Basic state & scaling

\[ T_S = T_0 - \beta d^2 r^2 / 2 \]

\[ g = -d \gamma r \]

Length scale: \( d \)

Time scale: \( d^2 / \nu \)

Temp. scale: \( \nu^2 / \gamma \alpha d^4 \)

Magn. flux density: \( \nu (\mu_0)^{1/2} / d \)
Numerical Methods

3D non-linear problem:

Toroidal-poloidal representation

\[ \mathbf{u} = \nabla \times (\nabla \mathbf{v} \times \mathbf{r}) + \nabla \mathbf{w} \times \mathbf{r}, \quad \mathbf{B} = \nabla \times (\nabla \mathbf{h} \times \mathbf{r}) + \nabla \mathbf{g} \times \mathbf{r} \]

Spectral decomposition in spherical harmonics and Chebyshev polynomials

\[ x = \sum_{l,m,n} X_{l,n}^{m}(t) T_{n}(r) P_{l}^{m}(\cos \theta) e^{im\phi} \quad \text{where} \quad x = (v, w, \Theta, g, h)^T \]

Scalar equations

\[ \partial_t X_{l,n}^{m} = \hat{\mathbf{L}} X_{l,n}^{m} + N_{l,n}^{m}(X) \quad \text{where} \quad \hat{\mathbf{L}} X_{l,n}^{m}: \text{linear}, N_{l,n}^{m}(X): \text{non-linear} \]


\[ [X_{l,n}^{m}]^{k+1} = \left( 1 - \frac{\Delta t}{2} \hat{\mathbf{L}} \right)^{-1} \left\{ \left( 1 + \frac{\Delta t}{2} \hat{\mathbf{L}} \right) [F_{l,n}^{m}]^{k} + \frac{\Delta t}{2} \left( 3[N_{l,n}^{m}]^{k} - [N_{l,n}^{m}]^{k-1} \right) \right\} \]

Resolution: radial=33, latitudinal=193, azimuthal=96.

Linear problem: Galerkin spectral method for the linearised equations leading to an eigenvalue problem for the critical parameters.
Part I

**Basic effects controlling dynamo behaviour: a focus on low magnetic Prandtl number regime**


- Bounds on the value of the magnetic Prandtl number for dynamo action
  - Critical value of $Rm$,
  - Turbulent flux expulsion.
- Dynamo symmetry types as function of the magnetic Prandtl number.
- Temporal and spacial structures of low $Pm$ dynamos.
- Energetics
**Bounds on the value of $P_m$ for dynamo action**

(R-$P_m$ plane)

$P = 0.1$, $\tau = 10^5$

- **no dynamo due to low $R_m$**
- **no dynamo due to flux expulsion**

- **no dynamo**
- **chaotic dipole**
- **hemispherical**

Simitev & Busse, JFM, 2005
Bounds on the value of \( P_m \) for dynamo action
(general considerations)

Dynamo action is restricted by:

(a) **vigour of convection** - convection must be sufficiently vigorous to support dynamo action.

(b) **magnetic field diffusivity** – the magnetic diffusivity must be sufficiently low for the magnetic field to persist.

(c) **flux expulsion** - however, convection which is too vigorous can lead to expulsion of magnetic field from small eddies.

**Note:** It is convenient to measure (a) and (b) by introducing magnetic Reynolds number

\[
R_m = (2E)^{1/2} P_m
\]

Typically, dynamo action requires \( R_m > 80 \text{ to } 100 \) so for a given set of fixed values of Prandtl and Coriolis numbers a small value of \( P_m \) requires large kinetic energy and therefore a large value of the Rayleigh number.
Bounds on the value of $P_m$ for dynamo action
(R-$P_m$ plane)

$$P = 0.1, \tau = 10^5$$

Simitev & Busse, JFM, 2005
Decay of dynamo action due to flux expulsion

(A) \( \tau = 5 \times 10^3, P = P_m = 1 \)

Note: With the increase of the value of Rayleigh number at all other parameter values fixed the magnetic energy components saturate and ultimately decrease due to flux expulsion and increasingly filamentary structure of the magnetic field.

Note: Ohmic dissipation continues to increase with \( R \).

(B) \( P = P_m = 10 \)

\( R = 300000 \) - decay
\( 500000 \) - dynamo
\( 600000 \) - dynamo
\( 700000 \) - decay

Ex

Vx

Mx

Ox

Busse & Simitev, 2006
Bounds on the value of $P_m$ for dynamo action 
($R$-$P_m$ plane)

$P = 0.1$, $\tau = 10^5$

- No dynamo due to low $R_m$
- No dynamo due to flux expulsion
- Chaotic dipole
- Hemispherical

Simitev & Busse, JFM, 2005
Bounds on the value of Pm for dynamo action (P-Pm plane)

Note: The minimal value of Pm decreases as P decreases.

(As a rough rule the value of the critical Pm is of the same order as the ordinary Prandtl number)

\[ \tau = 5 \times 10^3 \]
Dynamo symmetry types as function of Pm

Dynamo solutions exhibit symmetry because rapidly-rotating convection remains equatorially-symmetric even in the turbulent regime.

**Dipolar**

\[ P = 0.1, \tau = 10^5 \]
\[ R = 2 \times 10^6, \ Pm = 1 \]

**Quadrupolar**

\[ P = 5, \tau = 5 \times 10^3 \]
\[ R = 8 \times 10^5, \ Pm = 3 \]

**Hemispherical**

\[ P = 0.1, \tau = 10^5 \]
\[ Pm = 0.11 \]
\[ R = 6 \times 10^6 \]
Types of dynamos in the $(R-Pm)$ plane

$P = 0.1$, $\tau = 10^5$

Simitev & Busse, JFM, 2005
There is little evidence that a generated magnetic field plays a role similar to externally imposed field and counteracts the Coriolis force.

Rather, the main effect of a generated field is to inhibit differential rotation and thereby increase amplitude of convection and its heat transport.
Part II

Oscillations of dipolar dynamos as a possible cause of geomagnetic excursions and reversals


- Examples of linear oscillations
- Parker's plane layer theory of dynamo wave
- Non-linear oscillations
- Mechanism of excursions and reversals
Non-oscillatory dynamos:

- exist if the dipolar component is strongly dominant,
- have large ratio of $P_m/P$, so that quadrupolar components are not strong,
- are not too turbulent for otherwise higher harmonics will enter

A  $P = 0.1$, $\tau = 10^5$, $R = 3 \times 10^6$, $P_m = 2$

B  $P = 1$, $\tau = 10^4$, $R = 3.5 \times 10^5$, $P_m = 10$

C  $P = 200$, $\tau = 5 \times 10^3$, $R = 10^6$, $P_m = 80$
Prototypical dipolar oscillations

Half period

\[ P = 1, \quad \tau = 3 \times 10^3, \quad R = 3 \times 10^5 \quad Pm = 3 \]

Meridional fieldlines

\[ r \sin \theta \partial_\theta \overline{h} = \text{const.} \]

right half: dipolar component
left part: all other components

ETH, Zurich, 11 June 2009
Prototypical quadrupolar oscillations

Half period

\begin{align*}
P &= 5, \quad \tau = 5 \times 10^3, \quad R = 8 \cdot 10^5, \quad P_m = 3 \\
\text{Meridional fieldlines} &\quad r \sin \theta \partial_\theta \overline{h} = \text{const.} \\
\text{right half: dipolar component} &\quad \text{left part: all other components}
\end{align*}
Example of a quadrupolar oscillation

\[ P = 5, \quad \tau = 5 \times 10^3 \]
\[ R = 8 \cdot 10^5, \quad P_m = 3 \]

One period

Mean meridional filedlines of constant \( \overline{B_\varphi} \) (left), \( r \sin \vartheta \partial_\vartheta \overline{h} \) (right) and radial magn. field.

Time series of toroidal \( G_1^0 \) and poloidal \( H_{1,2}^0 \) magn. coefficients.
Dipolar oscillations by increasing the Rayleigh number

\[ R = 1.4 \times 10^6, \quad \tau = 5 \times 10^3 \]
\[ P_m = P = 1 \]

One period

Mean meridional filedlines of constant \( \overline{B_\varphi} \) (left), \( r \sin \vartheta \partial_\theta \overline{h} \) (right) radial velocity (top) and radial magn. field (bottom).
Dipolar oscillation by reduction of the magnetic Prandtl number

\[ P = 1 \quad P_m = 4 \]
\[ \tau = 10^4, \quad R = 5.6 \cdot 10^5 \]

One period

Mean meridional filedlines of constant \( \overline{B_\varphi} \) (left), \( r \sin \vartheta \partial_\vartheta \overline{h} \) (right) and radial magn. field.

Time series of toroidal \( G_1^0 \) and poloidal \( H_1^0, H_2^0 \) magn. coefficients.
Effect of oscillations on convection

Mean meridional magn. fieldlines (clockwise)

Energy densities

\[ P = 0.1, \ \tau = 10^5, \ R = 6 \times 10^6, \ P_m = 0.11 \]
Linear Oscillations: Parker dynamo waves

Axisymmetric field:

\[ B = B_p + iB, \quad B_p = \nabla \times iA, \quad \mathbf{v} = iU + \hat{\mathbf{v}}, \]

Following Parker's (1955) plane layer analysis of dynamo waves:

\[
\frac{\partial}{\partial t} A = \hat{\alpha}B + \nabla^2 A/P_m, \quad \frac{\partial}{\partial t} B = B_p \cdot \nabla U + \nabla^2 B/P_m,
\]

Linear solution ansatz:

\[ (A, B) = (\hat{A}, \hat{B}) \exp[i\mathbf{q} \cdot \mathbf{x} + \sigma t] \]

Dispersion relation and growth rate

\[ p^2 = 2i\Gamma \equiv 2i(-\hat{\alpha}(\mathbf{q} \times \nabla U)_x/2) \quad \sigma = \begin{cases} 
-|\mathbf{q}|^2/P_m \pm \sqrt{\Gamma} \pm i\sqrt{\Gamma} & \text{for } \Gamma > 0, \\
-|\mathbf{q}|^2/P_m \pm \sqrt{|\Gamma|} \mp i\sqrt{|\Gamma|} & \text{for } \Gamma < 0
\end{cases} \]

Assuming pseudo isotropic turbulence the alpha-coefficient is related to the helicity

\[ \hat{\alpha} \equiv -\frac{1}{3P_m} \int \int \frac{\hat{q}^2 F(\hat{q}, \omega)}{\omega^2 + \hat{q}^4/P_m^2} d\hat{q}d\omega \approx -\frac{P_m}{3\hat{q}^2} \int \int F(\hat{q}, \omega) d\hat{q}d\omega \equiv -\frac{P_m}{3\hat{q}^2} \langle \hat{\mathbf{v}} \cdot \nabla \times \hat{\mathbf{v}} \rangle \]

**Period:**

\[ T \approx 4\pi^2 \left( P_m \frac{\pi}{3} \langle \hat{\mathbf{v}} \cdot \nabla \times \hat{\mathbf{v}} \rangle \sqrt{2E_t} \right)^{-1/2} \]
Period of oscillations: model vs. numerics

$P = 0.1$, $\tau = 10^5$

$P = 1$, $\tau = 10^4$

$P = 1$, $\tau = 3 \times 10^4$

$P = 1$

$P = 3$

$P = 5$

$\tau = 5 \times 10^3$

$R \times 10^{-5}$
Non-linear dynamo oscillations

\[ P = 0.1, \tau = 10^5, R = 4 \times 10^6, P_m = 0.5 \]

Mean meridional magn. fieldlines (clockwise)  

Energy densities

Busse & Similev, PEPI, 2008
Reversals cased by toroidal flux oscillations

\[ P = 0.1, \tau = 10^5 \]
\[ R = 4 \times 10^6, P_m = 0.5 \]

\[ P = 0.1, \tau = 3 \times 10^4 \]
\[ R = 850000, P_m = 1 \]

Oscillating FD dynamo

Busse & Simitev, PEPI, 2008
Reversals cased by toroidal flux oscillations

$P = 0.1, \tau = 10^5$
$R = 4 \times 10^6, P_m = 0.5$

$P = 0.1, \tau = 3 \times 10^4$
$R = 850000, P_m = 1$

Busse & Simitev, PEPI, 2008

Oscillating FD dynamo
Some similarities with geomagnetic observations

1) Reversed magnetic field appears first at low latitudes as also observed by Clement (Nature, 2004).

2) Average duration of a reversal event is ~20000yrs so roughly consistent with observations.

3) For each reversal we observe several excursion events.

4) Amplitude of the field increases more sharply after a reversal than it decays before the reversal.

\[ P = 0.1, \quad \tau = 10^5 \]
\[ R = 4 \times 10^6, \quad P_m = 0.5 \]
Part III

**Bistability and hysteresis of dipolar dynamos generated by chaotic convection in rotating spherical shells**


- Two types of dipolar dynamos generated by chaotic convection at identical external parameter values
- The transition between Mean Dipolar (MD) and Fluctuating Dipolar (FD) dynamos and the hysteresis phenomenon
- Contrasting properties of Mean Dipolar (MD) and Fluctuating Dipolar (FD) dynamos
Two types of dipolar dynamos generated by chaotic convection

Energy densities

- Fully chaotic (large-scale turbulent) regime.
- Two chaotic attractors for the same parameter values.
- Essential qualitative difference: contribution of the mean poloidal dipolar energy

<table>
<thead>
<tr>
<th></th>
<th>(ab)</th>
<th>(de)</th>
</tr>
</thead>
<tbody>
<tr>
<td>$R_m$</td>
<td>133.6</td>
<td>196.5</td>
</tr>
<tr>
<td>$M_{dip}/M_{tot}$</td>
<td>0.803</td>
<td>0.527</td>
</tr>
</tbody>
</table>

black.......mean poloidal
green.......fluctuating poloidal
red..........mean toroidal
blue..........fluctuating toroidal

$R = 3.5 \cdot 10^6$, $\tau = 3 \cdot 10^4$, $P = 0.75$ and $P_m = 1.5$
Regions and transition

Two types of dipolar dynamos

- Mean Dipolar (MD)
  \[ \widetilde{M}_p < \bar{M}_p \]

- Fluctuating Dipolar (FD)
  \[ \widetilde{M}_p > \bar{M}_p \]

- MD and FD dynamos correspond to rather different chaotic attractors in a fully chaotic system.
- The transition between them is not gradual but is an abrupt jump as a critical parameter value is surpassed.
- The nature of the transition is complicated.

### Ratio of fluctuating to mean poloidal magnetic energy

<table>
<thead>
<tr>
<th></th>
<th>MD</th>
<th>FD</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\text{Mdip}/\text{Mtot}$</td>
<td>(0.62,1)</td>
<td>(0.41,56)</td>
</tr>
</tbody>
</table>

\[ R = 3.5 \cdot 10^6, \quad \tau = 3 \cdot 10^4 \]
Bistability and hysteresis in the ratio of fluctuating poloidal to mean poloidal magn energy

(a) \( R = 3.5 \cdot 10^6 \quad P/P_m = 0.5 \)

(b) \( R = 3.5 \cdot 10^6, \quad P = 0.75 \)

(c) \( P = 0.75, \quad P_m = 1.5 \)

in all cases: \( \tau = 3 \cdot 10^4 \)

The coexistence is not an isolated phenomenon but can be traced with variation of the parameters.

\[
\begin{align*}
\text{P}_{MD} & = 2.2 \\
\text{P}_{FD} & = 0.5 \\
\sigma_{MD} & = 0.07 \\
\sigma_{FD} & = 1
\end{align*}
\]
The hysteresis is a purely magnetic effect

After magnetic field is suppressed both MD and FD dynamos equilibrate to statistically identical convective states (period of relaxation oscillations, clockwise)

Magnetic field is artificially suppressed, i.e. non-magnetic convection

\[ R = 3.5 \times 10^6, \tau = 3 \times 10^4 \quad P = 0.75 \quad \text{and} \quad P_m = 1.5 \]
A property comparison of MD and FD dynamos
(Spatial structures)

Both MD and FD dynamos appear dipolar from the outside
(radial magn field at r=ro+1.3, Earth's surface)

FD dynamos have a somewhat more irregular and small-scale internal structure
(Bphi and meridional fieldlines)

MD

FD

Snapshots of spatial structures

$R = 3.5 \times 10^6$, $\tau = 3 \times 10^4$

$P = 0.75$ and $P_m = 1.5$

The stronger magnetic field of MD dynamos counteracts differential rotation
(diff rotation, meridional streamlines)

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A property comparison of MD and FD dynamos
(Temporal variations)

- Mean Dipolar (MD) dynamos are non-oscillatory.
- Fluctuating Dipolar (FD) dynamos are oscillatory.

Half-period of oscillation in a FD dynamo (row-by-row)

\[ R = 3.5 \times 10^6, \quad \tau = 3 \times 10^4, \quad P = 0.75 \text{ and } P_m = 0.65 \]
Conclusion

A) Oscillations of dipolar dynamos are typical in thick spherical shells and may be the reason for geomagnetic excursions and reversals.

B) Two types of dipolar dynamos can be distinguished:
   * Mean dipolar dynamos (MD)
   * Fluctuating dipolar dynamos (FD)

with rather different properties.

Thank you!