Convection-driven dynamos in rotating spherical shells - basic phenomenology

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Motivation: Applications of spherical dynamos

Dynamos in rotating spherical shells

Geomagnetism

Planetary magnetism

Solar and stellar magnetism
Outline of the talk

- Mathematical formulation of the problem.
- Numerical methods of solution.
- Typical convection and dynamo features. Turbulence.
- Overview of the basic effects controlling dynamo behaviour and types of dynamo solutions.
- Oscillations of dipolar dynamos as a possible cause of geomagnetic excursions and reversals.
- Bistability and hysteresis of fully nonlinear dipolar dynamos.
- Conclusions.
Model remarks

Our main motivation has been to model planetary dynamos and the Geodynamo. Accordingly, we have made a number of appropriate assumptions:

- **Boussinesq approximation** – constant material properties; variation of density is only included in the gravity term.
- **Incompressible fluid** – the velocity field is solenoidal.
- **Rapid rotation** – we strive to increase the rotation rate as our main motivation has been to model the Geodynamo.
- **Relatively thick spherical shells** - our main motivation has been to model the Geodynamo.
- **Direct numerical simulations** – no assumptions for the eddy diffusivities.
- **Self-sustained magnetic field** – we look for dynamo solutions and we do not impose external magnetic fields.

These may not always be appropriate in the Solar context but we believe they capture the basic physics of the dynamo process.

**OUR APPROACH:** systematic study of parameter dependencies and careful extrapolation to astrophysical objects – Earth, planets, stars.
Convective spherical shell dynamos

Model equations & parameters

Boussinesq approximation

\[ \nabla \cdot \mathbf{u} = 0, \quad \nabla \cdot \mathbf{B} = 0, \]
\[ \partial_t \mathbf{u} + \mathbf{u} \cdot \nabla \mathbf{u} = -\nabla \pi - \tau \mathbf{k} \times \mathbf{u} + \Theta \mathbf{r} + \nabla^2 \mathbf{u} + \mathbf{B} \cdot \nabla \mathbf{B}, \]
\[ P (\partial_t \Theta + \mathbf{u} \cdot \nabla \Theta) = R \mathbf{r} \cdot \mathbf{u} + \nabla^2 \Theta, \]
\[ P_m (\partial_t \mathbf{B} + \mathbf{u} \cdot \nabla \mathbf{B}) = P_m \mathbf{B} \cdot \nabla \mathbf{u} + \nabla^2 \mathbf{B}. \]

Basic state & scaling

\[ T_S = T_0 - \beta d^2 r^2 / 2 \]
\[ \mathbf{g} = -d \gamma \mathbf{r} \]

Length scale: \( d \)
Time scale: \( d^2 / \nu \)
Temp. scale: \( \nu^2 / \gamma \alpha d^4 \)
Magn. flux density: \( \nu (\mu_0)^{1/2} / d \)

Boundary Conditions

\[ \mathbf{r} \cdot \mathbf{u} = \mathbf{r} \cdot \nabla \mathbf{r} \times \mathbf{u} / r^2 = 0, \]
\[ \hat{\mathbf{e}}_r \cdot \mathbf{B}_{\text{int}} = \hat{\mathbf{e}}_r \cdot \mathbf{B}_{\text{ext}}, \]
\[ \hat{\mathbf{e}}_r \times \mathbf{B}_{\text{int}} = \hat{\mathbf{e}}_r \times \mathbf{B}_{\text{ext}}, \]
\[ \Theta = 0, \text{ at } r = r_i \equiv 2/3 \text{ and } r_o \equiv 5/3 \]
Numerical Methods

3D non-linear problem:

Toroidal-poloidal representation

\[ u = \nabla \times (\nabla v \times r) + \nabla w \times r, \quad B = \nabla \times (\nabla h \times r) + \nabla g \times r \]

Spectral decomposition in spherical harmonics and Chebyshev polynomials

\[ x = \sum_{l,m,n} X_{l,n}^m(t) T_n(r) P_l^m(\cos \theta) e^{im\phi} \quad \text{where } x = (v, w, \Theta, g, h)^T \]

Scalar equations

\[ \partial_t X_{l,n}^m = \hat{\mathcal{L}} X_{l,n}^m + N_{l,n}^m(X) \quad \text{where } \hat{\mathcal{L}} X_{l,n}^m: \text{ linear, } N_{l,n}^m(X): \text{ non-linear} \]


\[ [X_{l,n}^m]^{k+1} = \left(1 - \frac{\Delta t}{2} \hat{\mathcal{L}}\right)^{-1} \left\{ \left(1 + \frac{\Delta t}{2} \hat{\mathcal{L}}\right) [F_{l,n}^m]_k + \frac{\Delta t}{2} \left(3[N_{l,n}^m]_k - [N_{l,n}^m]_{k-1}\right) \right\} \]

Resolution: radial=49, latitudinal=96, azimuthal=193.

Linear problem: Galerkin spectral method for the linearised equations leading to an eigenvalue problem for the critical parameters.
Part I

Typical properties of dynamo solutions.

Turbulence

- Spectra and spatial features
- Temporal behaviour
- Values of the magnetic Reynolds number
- States of convection
- Dynamo symmetries
Typical time dependence: turbulence

\[ P = 0.1, \quad \tau = 3 \times 10^4, \quad R = 1.2 \times 10^6, \quad Pm = 0.5 \]
Spectra and separation of scales

\[ P = 0.75, \tau = 3 \times 10^4 \]

\[ R = 3.5 \times 10^6, P_m = 0.75 \]
**Typical magnetic Reynolds number for dynamo onset**

**Figure 7.** Magnetic Reynolds numbers $Rm$ for the onset of dynamo action as a function of $P_m$ in the cases $P = 0.01$, $\tau = 10^5$ (stars), $P = 0.025$, $\tau = 10^5$ (crosses), $P = 0.1$, $\tau = 10^5$ (circles), $P = 1$, $\tau = 3 \times 10^4$ (triangles up), $P = 1$, $\tau = 10^4$ (squares) and $P = 5$, $\tau = 5 \times 10^3$ (diamonds) and $P = 10$, $\tau = 5 \times 10^3$ (triangles down). The open symbols in the cases with $P \geq 0.1$ are based on decaying dynamos, the filled symbols are based on the lowest non-decaying solutions.
Finite-amplitude columnar convection

Simitev & Busse, NJP, 2003

$P = 0.1, \tau = 3 \times 10^4$

$R = 5 \times 10^5$

$R = 3.5 \times 10^5$

$R = 2.8 \times 10^5$

$R = 3 \times 10^5$

Simitev & Busse, NJP, 2003
Dynamo solutions exhibit symmetry because rapidly-rotating convection remains equatorially-symmetric even in the turbulent regime.

**Dipolar**

\[ P = 0.1, \ \tau = 10^5 \]
\[ R = 2 \times 10^6, \ Pm = 1 \]

**Quadrupolar**

\[ P = 5, \ \tau = 5 \times 10^3 \]
\[ R = 8 \times 10^5, \ Pm = 3 \]

**Hemispherical**

\[ P = 0.1, \ \tau = 10^5 \]
\[ Pm = 0.11 \]
\[ R = 6 \times 10^6 \]
Part II

**Basic effects controlling dynamo behaviour**


Grote et al., 2003.

- **Bounds on the region of dynamo action**
  - Critical value of $Rm$,
  - Turbulent flux expulsion.
- **Dynamo symmetry types as function of the magnetic Prandtl number.**
- **Effect of magnetic field on convection.**
Bounds on the region for dynamo action (R-Pm plane)

Dynamo action is restricted by:

(a) **vigour of convection** - convection must be sufficiently vigorous to support dynamo action.

(b) **magnetic field diffusivity** – the magnetic diffusivity must be sufficiently low for the magnetic field to persist.

(c) **flux expulsion** - however, convection which is too vigorous can lead to expulsion of magnetic field from small eddies.

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Simitev & Busse, JFM, 2005
Decay of dynamo action due to flux expulsion

**Note:** With the increase of the value of Rayleigh number at all other parameter values fixed the magnetic energy components saturate and ultimately decrease due to flux expulsion and increasingly filamentary structure of the magnetic field.

**Note:** Ohmic dissipation continues to increase with $R$.

(A) \[
\tau = 5 \times 10^3, \quad P = P_m = 1
\]

(B) \[
P = P_m = 10, \quad \tau = 5 \times 10^5
\]

- $R=300000$ - decay
- $500000$ - dynamo
- $600000$ - dynamo
- $700000$ - decay

East Kilbride, 28 Oct 2011
Types of dynamos in the parameter space

- **No dynamo**
- Regular and chaotic non-oscillatory dipolar dynamos (at large $Pm/P$ and not far above dynamo onset)
- Oscillatory dipolar dynamos (at values of $R$ larger than those of non-oscillatory dipoles)
- **Hemispherical dynamos** – always oscillatory
- **Quadrupolar dynamos** – always oscillatory

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Bounds on the value of $P_m$ for dynamo action (P-$P_m$ plane)

Note: The minimal value of $P_m$ decreases as $P$ decreases.

(As a rough rule the value of the critical $P_m$ is of the same order as the ordinary Prandtl number)

$$\tau = 5 \times 10^3$$
Effect of self-sustained magnetic field on convection

There is little evidence that a generated magnetic field plays a role similar to externally imposed field and counteracts the Coriolis force. Rather, the main effect of a generated field is to inhibit differential rotation and thereby increase amplitude of convection and its heat transport.
Part III

Oscillations of dipolar dynamos as a possible cause of geomagnetic excursions and reversals


• Examples of linear oscillations
• Parker's plane layer theory of dynamo wave
• Non-linear oscillations
• Mechanism of excursions and reversals
Non-oscillatory dynamos:

- exist if the dipolar component is strongly dominant,
- have large ratio of $Pm/P$, so that quadrupolar components are not strong,
- are not too turbulent for otherwise higher harmonics will enter

**A**  $P = 0.1, \tau = 10^5, R = 3 \times 10^6, Pm = 2$

**B**  $P = 1, \tau = 10^4, R = 3.5 \times 10^5, Pm = 10$

**C**  $P = 200, \tau = 5 \times 10^3, R = 10^6, Pm = 80$
Example of a quadrupolar oscillation

\[ P = 5, \quad \tau = 5 \times 10^3 \]
\[ R = 8 \times 10^5, \quad P_m = 3 \]

One period

Mean meridional filedlines of constant \( \overline{B_\varphi} \) (left),
\( r \sin \vartheta \partial_\theta \overline{h} \) (right)
and radial magn. field.

Time series of toroidal \( G_1^0 \) and poloidal \( H_1^0, H_2^0 \) magn. coefficients.
An example of a dipolar oscillation

\[ R = 3.5 \cdot 10^6, \quad \tau^* = 3 \cdot 10^4, \quad P^* = 0.75 \text{ and } P_m = 0.65 \]

Half-period of oscillation (column-by-column)
An example of a dipolar oscillation

\[ R = 3.5 \cdot 10^6, \quad \tau = 3 \cdot 10^4, \quad P = 0.75 \text{ and } P_m = 0.65 \]

http://www.maths.gla.ac.uk/~rs/res/B/anim.bm.gif
http://www.maths.gla.ac.uk/~rs/res/B/anim.radmagn_2.gif

Half-period of oscillation (column-by-column)
Effect of oscillations on convection

Mean meridional magn. fieldlines (clockwise)

Energy densities

$P = 0.1, \tau = 10^5, R = 6 \times 10^6, P_m = 0.11$

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Linear Oscillations: Parker dynamo waves

Axisymmetric field:

\[ B = B_p + iB, \quad B_p = \nabla \times iA, \quad \mathbf{v} = i\mathbf{U} + \mathbf{\tilde{v}}, \]

Following Parker's (1955) plane layer analysis of dynamo waves:

\[
\frac{\partial}{\partial t} A = \hat{\alpha} B + \nabla^2 A / P_m, \quad \frac{\partial}{\partial t} B = B_p \cdot \nabla U + \nabla^2 B / P_m,
\]

Using a linear wave solution ansatz:

\[(A, B) = (\hat{A}, \hat{B}) \exp[i\mathbf{q} \cdot \mathbf{x} + \sigma t]\]

we can obtain an expression for the growth rate

\[
\sigma = \begin{cases} 
\frac{-|\mathbf{q}|^2}{P_m} \pm i\sqrt{\Gamma} & \text{for } \Gamma > 0, \\
\frac{-|\mathbf{q}|^2}{P_m} \pm i\sqrt{|\Gamma|} & \text{for } \Gamma < 0
\end{cases}
\]

\[\Gamma \equiv -\hat{\alpha}(\mathbf{q} \times \nabla U)_x / 2\]

\[\hat{\mathbf{q}} \approx 2\pi / d\]

Assuming pseudo isotropic turbulence the alpha-coefficient is related to the helicity

\[\hat{\alpha} \equiv -\frac{1}{3P_m} \int \int \frac{\hat{q}^2 F(\hat{q}, \omega)}{\omega^2 + \hat{q}^4 / P_m^2} d\hat{q} d\omega \approx -\frac{P_m}{3\hat{q}^2} \int \int F(\hat{q}, \omega) d\hat{q} d\omega \equiv -\frac{P_m}{3\hat{q}^2} \langle \mathbf{\tilde{v}} \cdot \nabla \times \mathbf{\tilde{v}} \rangle\]

\[T \approx 4\pi^2 \left( \frac{P_m}{3} \langle \mathbf{\tilde{v}} \cdot \nabla \times \mathbf{\tilde{v}} \rangle \sqrt{2 E_t} \right)^{-1/2}\]
Period of oscillations: model vs. numerics

$P = 0.1, \tau = 10^5$

$P = 1, \tau = 10^4$

$P = 1, \tau = 3 \times 10^4$

$P = 1$

$\tau = 5 \times 10^3$

$P = 3$

$P = 5$

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Non-linear dynamo oscillations

\[ P = 0.1, \tau = 10^5, R = 4 \times 10^6, P_m = 0.5 \]

Mean meridional magn. fieldlines (clockwise)        Energy densities

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Reversals cased by toroidal flux oscillations

\[ P = 0.1, \ \tau = 10^5 \]
\[ R = 4 \times 10^6, \ \rho_m = 0.5 \]

Oscillating FD dynamo

\[ P = 0.1, \ \tau = 3 \times 10^4 \]
\[ R = 850000, \ \rho_m = 1 \]

Busse & Simitev, PEPI, 2008
Reversals cased by toroidal flux oscillations

\[ P = 0.1, \tau = 10^5 \]
\[ R = 4 \times 10^6, P_m = 0.5 \]

\[ P = 0.1, \tau = 3 \times 10^4 \]
\[ R = 850000, P_m = 1 \]

Busse & Simitev, PEPI, 2008
Some similarities with geomagnetic observations

$$P = 0.1, \quad \tau = 10^5$$
$$R = 4 \times 10^6, \quad P_m = 0.5$$

1) Reversed magnetic field appears first at low latitudes as also observed by Clement (Nature, 2004).

2) Average duration of a reversal event is ~20000yrs - roughly consistent with observations.

3) For each reversal we observe several excursion events.

4) Amplitude of the field increases more sharply after a reversal than than it decays before the reversal.

$$P = 0.1, \quad \tau = 3 \times 10^4$$
$$R = 850000, \quad P_m = 1$$
Part IV

**Bistability and hysteresis of dipolar dynamos generated by chaotic convection in rotating spherical shells**


- Two types of dipolar dynamos generated by chaotic convection at identical external parameter values
- The transition between Mean Dipolar (MD) and Fluctuating Dipolar (FD) dynamos and the hysteresis phenomenon
- Contrasting properties of Mean Dipolar (MD) and Fluctuating Dipolar (FD) dynamos
- Oscillations of Fluctuating Dipolar (FD) dynamos and reversals
Two types of dipolar dynamos generated by chaotic convection

Energy densities

- Fully chaotic (large-scale turbulent) regime.
- Two chaotic attractors for the same parameter values.
- Essential qualitative difference: contribution of the mean poloidal dipolar energy

<table>
<thead>
<tr>
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<th>(ab)</th>
<th>(de)</th>
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<tbody>
<tr>
<td>Rm</td>
<td>133.6</td>
<td>196.5</td>
</tr>
<tr>
<td>Mdip/Mtot</td>
<td>0.803</td>
<td>0.527</td>
</tr>
</tbody>
</table>

black......mean poloidal
green......fluctuating poloidal
red...........mean toroidal
blue...........fluctuating toroidal

\[ R = 3.5 \cdot 10^6, \quad \tau = 3 \cdot 10^4, \quad P = 0.75 \quad \text{and} \quad P_m = 1.5 \]
Regions and transition

Ratio of fluctuating to mean poloidal magn energy

Two types of dipolar dynamos

- **Mean Dipolar (MD)**
  \[ \tilde{M}_p < \bar{M}_p \]

- **Fluctuating Dipolar (FD)**
  \[ \tilde{M}_p > \bar{M}_p \]

- MD and FD dynamos correspond to rather **different chaotic attractors** in a fully chaotic system
- The transition between them is not gradual but is an **abrupt jump** as a critical parameter value is surpassed.
- The nature of the transition is complicated.

<table>
<thead>
<tr>
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<th>MD</th>
<th>FD</th>
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<tbody>
<tr>
<td>( R = 3.5 \cdot 10^6 ), ( \tau = 3 \cdot 10^4 )</td>
<td>(0.62,1)</td>
<td>(0.41,56)</td>
</tr>
</tbody>
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**Bistability and hysteresis in the MD \(\leftrightarrow\) FD transition**

**Bistability and hysteresis** in the ratio of fluctuating poloidal to mean poloidal magn energy

![Graphs showing bistability and hysteresis](image)

(a) \(R = 3.5 \cdot 10^6\) \(P/P_m = 0.5\)

(b) \(R = 3.5 \cdot 10^6, P = 0.75\)

(c) \(P = 0.75, P_m = 1.5\)

in all cases: \(\tau = 3 \cdot 10^4\)

The coexistence is **not an isolated phenomenon** but can be traced with variation of the parameters.

\(P_{MD} = 2.2\) \(\sigma_{MD} = 0.07\)

\(P_{FD} = 0.5\) \(\sigma_{FD} = 1\)
The hysteresis is a purely magnetic effect

After magnetic field is suppressed both MD and FD dynamos equilibrate to statistically identical convective states (period of relaxation oscillations, clockwise)

Magnetic field is artificially suppressed, i.e. non-magnetic convection

\[ R = 3.5 \times 10^6, \quad \tau = 3 \times 10^4 \quad P = 0.75 \quad \text{and} \quad P_m = 1.5 \]
A property comparison of MD and FD dynamos
(Spatial structures)

Both MD and FD dynamos appear dipolar from the outside (radial magn field at $r = r_0 + 1.3$, Earth's surface)

The stronger magnetic field of MD dynamos counteracts differential rotation (diff rotation, meridional streamlines)

FD dynamos have a somewhat more irregular and small-scale internal structure (Bphi and meridional fieldlines)

Snapshots of spatial structures

$R = 3.5 \times 10^6$, $\tau = 3 \times 10^4$

$P = 0.75$ and $P_m = 1.5$
A property comparison of MD and FD dynamos
(Temporal variations)

- **Mean Dipolar** (MD) dynamos are **non-oscillatory**.
- **Fluctuating Dipolar** (FD) dynamos are **oscillatory**.

Half-period of oscillation in a FD dynamo (row-by-row)

\[
R = 3.5 \cdot 10^6, \quad \tau = 3 \cdot 10^4, \quad P = 0.75 \text{ and } P_m = 0.65
\]
Conclusion

- We have described typical properties of self-consistent spherical dynamos in rotating spherical shells.

- Dynamo oscillations are typical temporal behaviour,

- Oscillations may lead to aperiodic reversals similar to geomagnetic reversals.

- Co-existence of chaotic attractors.