

PERTURBATIONS OF NUCLEAR C^* -ALGEBRAS

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SETUP

KADISON & KASTLER 1972

- Fix (separable) Hilbert space H and consider the set $C^*(H)$ of all C^* -subalgebras of $\mathbb{B}(H)$.
- Equip $C^*(H)$ with a natural metric d .
- What happens if $d(A, B)$ is small?

RECALL

C^* -subalgebra A of $\mathbb{B}(H)$ is a norm closed vector subspace of $\mathbb{B}(H)$ which is also closed under multiplication (composition of operators), the adjoint operation $x \mapsto x^*$.

NB: A need not have a unit and if it does we **do not** assume that the unit of A is necessarily the identity operator H .

SOME QUESTIONS

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DEFINITION / RECALL

A unital C^* -algebra A has **real rank zero** if the invertible self-adjoint elements of A are dense in the self-adjoint elements of A .

- Suppose A, B share the same unit, $d(A, B) < \gamma$ and A has real rank zero. Fix $y = y^*$ in B with $\|y\| = 1$.
- Find $x = x^*$ in $(A)_1$ with $\|x - y\| < \gamma$.
- Take $\varepsilon > 0$ and find invertible $k = k^*$ in A with $\|x - k\| < \varepsilon$.

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- Find $x = x^*$ in $(A)_1$ with $\|x - y\| < \gamma$.
- Take $\varepsilon > 0$ and find invertible $k = k^*$ in A with $\|x - k\| < \varepsilon$.
- Find $h = h^*$ in B with $\|h - k\| < \gamma\|k\| \leq \gamma(1 + \varepsilon)$.
- If γ is small h is invertible.
- Have $\|h - y\| < \|h - k\| + \|k - x\| + \|x - y\| < \gamma(2 + \varepsilon) + \varepsilon$.

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We have just shown that every self-adjoint in the unit ball of B is within $2\gamma + \varepsilon$ of an invertible self-adjoint - the invertible self-adjoints are "2 γ -dense" in the self-adjoints. This does not show that B has real rank zero. Nevertheless:

PROPOSITION (CSSW, '08)

Suppose $d(A, B) < 0.008$. Then A has real rank zero if and only if B has real rank zero

This uses an alternative characterisation of real rank zero which is more amenable to perturbations.

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Does there exist $\gamma_0 > 0$ such that $d(A, B) < \gamma_0$ implies that $B = uAu^$ for some unitary $u \in (A \cup B)''$?*

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Does there exist $\gamma_0 > 0$ such that $d(A, B) < \gamma_0$ implies that $B = uAu^$ for some unitary $u \in (A \cup B)''$?*

Answer: No (to all the questions)! — Choi + Christensen 80's

Can ask similar questions about **near containments**, e.g. if $A \subseteq_\gamma B$ does A embed into B ?

VON NEUMANN ALGEBRAS

PROPOSITION (KADISON + KASTLER)

Take $A, B \in C^*(H)$. Then $d(\overline{A}^{\text{wot}}, \overline{B}^{\text{wot}}) \leq d(A, B)$.

Proof uses Kaplansky's density theorem: $\overline{(A)_1}^{\text{wot}} = (\overline{A}^{\text{wot}})_1$.

RECALL

- $M \in C^*(H)$ is a von Neumann algebra iff M is wot closed.
- $m_j \rightarrow m$ in wot $\Leftrightarrow \langle (m_j - m)\xi, \eta \rangle \rightarrow 0$ for all $\xi, \eta \in H$.

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COROLLARY

Let $M, N \in C^*(H)$ with $d(M, N) = \gamma < 1/2$. Suppose that M is a von Neumann algebra. Then N is a von Neumann algebra.

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Let $M, N \in C^*(H)$ with $d(M, N) = \gamma < 1/2$. Suppose that M is a von Neumann algebra. Then N is a von Neumann algebra.

EASY PROPOSITION

Suppose $A \subset B \subseteq_{\gamma} A$ for $\gamma < 1$. Then $A = B$.

RECALL

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INJECTIVE VON NEUMANN ALGEBRAS

DEFINITION

A von Neumann algebra $M \subset \mathbb{B}(H)$ is **injective** iff there exists a norm 1-projection $\Phi : \mathbb{B}(H) \rightarrow M$.

Injectivity has many equivalent definitions and is to von Neumann algebras what **amenability** is to groups. In particular one can use it (in a different form) to perform **averaging operations**.

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If $d(M, N)$ is small and N is injective we have a map $\Phi|_M : M \rightarrow N$ which is close to the identity map $I|_M : M \rightarrow \mathbb{B}(H)$. In particular $\Phi|_M$ is almost a $*$ -homomorphism.

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THEOREM (CHRISTENSEN 74)

If M, N are injective von Neumann subalgebras of $\mathbb{B}(H)$ with $d(M, N)$ sufficiently small, then $M \cong N$.

Essentially⁵ Erik used injectivity of M to **average** the map $\Phi|_N$ to a $*$ -hm from M onto N .

⁵I'm ducking some tricky technical issues regarding normality.

INJECTIVE VON NEUMANN ALGEBRAS II

THEOREM (RAEBURN, TAYLOR 75)

Let $M, N \subset \mathbb{B}(H)$ have $d(M, N)$ small. If M is an injective vNa , then N is an injective vNa . ($d(M, N) < 1/101$ will do.)

\Rightarrow Christensen's perturbation result only needs one algebra to be injective.

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THEOREM (CHRISTENSEN 80)

Let M, N be vNas with M injective. If $M \subseteq_{\gamma} N$ for $\gamma < 1/100$, then \exists unitary $u \in (M \cup N)''$ st $uMu^* \subseteq N$ and $\|u - 1\| \leq 150\gamma$.

OBJECTIVE OF REST OF TALK

Examine C^* -versions of these results.

TWO EXAMPLES

COUNTEREXAMPLE (CHOI, CHRISTENSEN)

For $\epsilon > 0$, there exist non-isomorphic C^* -algebras $A, B \subset \mathbb{B}(H)$ with $d(A, B) < \epsilon$.

- Examples are not vNas and not separable (so unreasonable)!
- In C^* -setting restrict to the separable case.

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EXAMPLE (JOHNSON)

For $\epsilon > 0$, there exist two faithful representations of $C([0, 1], \mathbb{K})$ on H with images A, B s.t. $d(A, B) < \epsilon$, yet any isomorphism $\theta : A \rightarrow B$ has $\|\theta(x) - x\| \geq \|x\|/70$ for some $x \in A$.

- $C([0, 1], \mathbb{K})$ a nice C^* -algebra. Will look for perturbation results which don't give isomorphisms uniformly close to I .

SOME POSITIVE RESULTS

A class of C^* -algebras \mathcal{A} is **perturbable** if there exists $\gamma_0 > 0$ (depending on \mathcal{A}) so that if $d(A, B) < \gamma_0$ and **one** algebra lies in \mathcal{A} then A and B are unitarily conjugate. Examples:

- abelian C^* -algebras.
- separable AF (approximately finite dimensional) algebras (Christensen)
- Stable, separable and continuous trace — e.g $C[0, 1] \otimes \mathbb{K}$, or unital separable and continuous trace. (Phillips + Raeburn).
- Certain extensions some of the classes above (Khoshkam).

A weaker perturbation result might allow the hypothesis $B \in \mathcal{A}$.

DEFINITION / RECALL

A C^* -algebra A is AF if it is a direct limit of finite dimensional C^* -algebras. In the separable case A is AF $\Leftrightarrow \forall \varepsilon > 0, x_1, \dots, x_n \in A, \exists$ finite dimensional C^* -subalgebra A_0 of A such that $d(x_i, A_0) < \varepsilon, \forall i$.

COMPLETELY POSITIVE MAPS

- Given C^* -algebra A identify $M_n(A)$ as a subalgebra of $\mathbb{B}(H^{\oplus n})$. In this way $M_n(A)$ is a C^* -algebra.
- Given a map $\phi : A \rightarrow B$ between C^* -algebras define $\phi_n : M_n(A) \rightarrow M_n(B)$ by $\phi_n((x_{i,j})) = (\phi(x_{i,j}))$, i.e. do ϕ component wise.

DEFINITION

$\phi : A \rightarrow B$ is **completely positive** iff each ϕ_n is positive.

cpc = completely positive and contractive (norm ≤ 1).

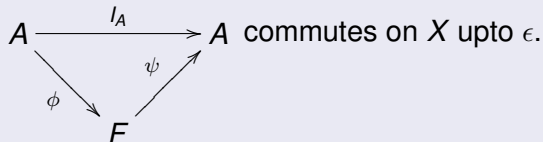
RECALL

A linear map $\phi : A \rightarrow B$ is **positive** if $\phi(x) \geq 0$ whenever $x \geq 0$. Positive elements of C^* -algebra are those of form y^*y (same as positive as operator on H).

NUCLEAR C^* -ALGEBRAS

DEFINITION (THEOREM OF CHOI, EFFROS)

A C^* -algebra A is **nuclear** if, $\forall X \subset_{\text{fin}} A, \epsilon > 0, \exists F$ a finite dimensional C^* -algebra and cpc maps $\phi : A \rightarrow F, \psi : F \rightarrow A$ s.t. $\psi \circ \phi \approx_{X, \epsilon} I_A$, i.e.



The original definition is in terms of tensor products: Given two C^* -algebras A, B we can put C^* -norms on the algebraic tensor product $A \odot B$. A is nuclear iff there is a unique way of doing this for all C^* -algebras B .

NUCLEAR C^* -ALGEBRAS II

SOME EXAMPLES OF NUCLEAR C^* -ALGEBRAS

- The perturbable classes discussed earlier.
- $C_r^*(G)$ for a discrete amenable group G .
- Cuntz algebras \mathcal{O}_n .
- Closed under: \otimes , quotients, extensions, \oplus , \lim_{\rightarrow} .

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THEOREM (CONNES + MANY HANDS)

A is nuclear $\Leftrightarrow A^{**}$ is injective.

A^{**} is the Banach space bidual which is a vNa. $A^{**} \cong \overline{A}^{wot}$ when A lies in its **universal representation**.

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COROLLARY (CHRISTENSEN)

Let A, B be C^* -subalgebras of some C with $d(A, B) < 1/101$. Then A is nuclear iff B is nuclear.

PERTURBATION RESULTS FOR NUCLEARS

SEPARABILITY THE ONLY OBSTRUCTION

THEOREM A (CSSWW '08)

Let $A, B \in C^*(H)$ have $d(A, B) = \gamma < 1/975157$. Suppose A is separable and nuclear, then $A \cong B$. Furthermore, given $X \subset_{\text{fin}} (A)_1$ and $Y \subset_{\text{fin}} (B)_1$ there exists an isomorphism $\theta : A \rightarrow B$ with $\theta \approx_{X, 50\gamma^{1/2}} I$ and $\theta^{-1} \approx_{Y, 50\gamma^{1/2}} I$.

Second half of the theorem says that we can obtain some control on what the isomorphism does (and answer Kadison and Kastler's question).

THEOREM B (CSSWW '09)

Let $A, B \in C^*(H)$ have $d(A, B) < 10^{-12}$ and suppose A is separable and nuclear and H is separable. Then there exists a unitary $u \in (A \cup B)''$ with $uAu^* = B$.

CPC MAPS BETWEEN CLOSE NUCLEARS

ARVESON'S EXTENSION THEOREM

Let $B \subset \mathbb{B}(H)$ be a C^* -algebra and $\phi : B \rightarrow F$ be cpc with F finite dimensional. Then \exists cpc $\tilde{\phi} : \mathbb{B}(H) \rightarrow F$ extending ϕ .

We can use this to get maps $A \rightarrow B$ when A, B are close and B is nuclear.

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LEMMA

Let $A, B \in C^*(H)$ have $A \subseteq_{\gamma} B$ with B nuclear. $\forall Z \subset_{fin} (A)_1$,
 $\exists \phi : A \rightarrow B$ cpc $\phi \approx_{Z, 3\gamma} I$.

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Let $A, B \in C^*(H)$ have $A \subseteq_\gamma B$ with B nuclear. $\forall Z \subset_{fin} (A)_1$, $\exists \phi : A \rightarrow B$ cpc $\phi \approx_{Z, 3\gamma} I$.

In particular we have $(X, 9\gamma)$ -hms $A \rightarrow B$ whenever $A \subseteq_\gamma B$ and B is nuclear.

DEFINITION

Given $X \subset_{fin} (A)_1$ and $\epsilon > 0$, $\phi : A \rightarrow B$ is an (Z, ϵ) -hm if it is cpc and $\|\phi(xx^*) - \phi(x)\phi(x^*)\| < \epsilon$ for $x \in X \cup X^*$.

AN AVERAGING LEMMA

Given $A \subseteq_\gamma B$ with B nuclear we have $(Z, 9\gamma)$ -hms $A \rightarrow B$ for all $Z \subset_{\text{fin}} (A)_1$. We now want to **average** these (as in the injective von Neumann situation) to get maps which are closer to being multiplicative.

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LEMMA

Let A be a nuclear C^* -algebra. $\forall Y \subset_{\text{fin}} (A)_1, \delta > 0, \mu > 0, \exists Z \subset_{\text{fin}} (A)_1$ st:

If D is a C^ -algebra and $\phi : A \rightarrow D$ is a (Z, η) -hm for $\eta < 1/49$.
Then $\exists (Y, \delta)$ -hm $\psi : A \rightarrow D$ with $\|\psi - \phi\| < 16\eta^{1/2} + 8\mu$.*

- In this lemma, think of $\eta = 9\gamma$ as large.
- In some (precise) sense Z is a Følner set for (Y, δ) .
- Key ingredients: nuclear \Rightarrow **amenable** (Haagerup) and approximate diagonals.

CLOSE NUCLEAR C^* -ALGEBRAS

STATE OF PLAY (LEMMA C)

Let $d(A, B) = \gamma$ is small with A nuclear. $\forall Y \subset_{\text{fin}} (A)_1$, $\delta > 0$,
 $\exists \phi : A \rightarrow B$ a (Y, δ) -hm with $\|\phi(y) - y\| \leq \text{function of } \gamma \text{ for } y \in Y$.

Want to take a limit (over larger finite sets and smaller tollarances δ) of these maps, but why should they converge?

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LEMMA D

Let A be a nuclear C^* -algebra. $\forall X \subset_{\text{fin}} (A)_1$, $\epsilon > 0$, $\exists Y \subset_{\text{fin}} (A)_1$, $\delta > 0$
 st:

If D is a C^ -algebra and $\phi_1, \phi_2 : A \rightarrow D$ are (Y, δ) -approx hms
 with $\phi_1 \approx_{Y, \eta} \phi_2$ for $\eta < 1/24$, then \exists unitary $u \in \tilde{D}$ with
 $\phi_1 \approx_{X, \epsilon} \text{Ad}(u) \circ \phi_2$ and $\|u - 1\| < 4\eta + 10\delta$.*

Careful use of these lemmas enables one to assemble a proof of Theorem A.

ABOUT THEOREM B

- When A is separable and nuclear and $d(A, B)$ is small can use Theorem A and Lemma D to find unitaries $(u_n)_{n=1}^{\infty}$ such that $\text{Ad}(u_n)$ converges in norm to an isomorphism from A onto B . If the sequence u_n converged (in strong*-topology say) to a unitary u then $uAu^* = B$ and we're done. There's no good reason why this should occur.
- Of course (u_n) has a weak*-accumulation point z , and if z is invertible then the unitary in its polar decomposition does the job. However z doesn't have to be non-invertible — it could be zero.

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PLAN

Carefully adjust the (u_n) so they do converge (in strong* topology).
Need H separable for this.

EASY FACT

If F is a finite dimensional subalgebra of A , then $\overline{F' \cap A}^{\text{wot}} = F' \cap \overline{A}^{\text{wot}}$.

ABOUT THEOREM B

EASY FACT

If F is a finite dimensional subalgebra of A , then $\overline{F' \cap A}^{wot} = F' \cap \overline{A}^{wot}$.

The approximate version of this lemma also uses "averaging".

LEMMA

Let A be a nuclear C^* -algebra non degenerately represented on H with $A'' = M$. $\forall X \subset_{fin} (A)_1, \epsilon, \mu > 0, \exists Y \subset_{fin} (A)_1, \delta > 0$ s.t:

If $k = k^* \in (M)_1$ has $\|[k, y]\| \leq \delta$ for $y \in Y$ and $S \subset_{fin} (H)_1$, then $\exists h = h^* \in A$ with $\|h\| \leq \|k\|, \|[h, x]\| \leq \epsilon$ for $x \in X$ and $\|(h - k)\xi\| < \mu$ for $\xi \in S$.

This enables us to adjust the unitaries so that they converge and so prove Theorem B.

WHEN BOTH ALGEBRAS ARE NUCLEAR

QUESTION

Suppose we have $A \subseteq_{\gamma} B$ with A separable and nuclear. Do we have an injective $$ -hm $A \rightarrow B$?*

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Used $d(A, B)$ small to deduce that B is also nuclear. This allowed us to use Arveson to construct cpc maps $A \rightarrow B$. If we already knew B was nuclear we'd be able to use the previous techniques to prove

THEOREM

Suppose $A \subseteq_\gamma B$ for $\gamma < 1/975157$ and A, B are nuclear and A is separable. Then there exists an embedding $A \hookrightarrow B$.

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QUESTION II

If $A \subseteq_\gamma B$ and A is nuclear **but B is not**, can we find cpc maps $A \rightarrow B$ close to I on finite sets?

A NAIVE PLAN

Given $A \subseteq_\gamma B$ with A nuclear let's use nuclearity of A to find cpc maps $A \rightarrow B$. Fix $X \in_{\text{fin}} (A)_1$.

WANT

Want cpc map $\phi : A \rightarrow B$ with $\phi \approx_{X, f(\gamma)} I$. Some reasonable function f .

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Take cpc $A \xrightarrow{\psi} M_n \xrightarrow{\phi} A$ with $\psi \circ \phi \approx_{X, \epsilon} I_A$. If we could find $\tilde{\phi} : M_n \rightarrow B$ with $\|\tilde{\phi} - \phi\|$ small, then $\psi \circ \tilde{\phi}$ would do.

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FACTS

- $CP(M_n, A) \leftrightarrow M_n(A)^+$ via $\phi \mapsto (\phi(e_{i,j}))_{i,j}$.
- $M_n(A) \subseteq_{\gamma^2 + 2\gamma} M_n(B)$ — needs nuclearity of A .^a

^aThis works more generally and has to do with Kadison's similarity problem.

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Problem: cpc map ϕ corresponds to positive element of norm $\leq n$ in $M_n(A)^+$. Can find $\tilde{\phi}$. **But $\|\tilde{\phi} - \phi\|$ depends on n .** This is not enough.

ORDER ZERO MAPS

WANT

Given $A \subseteq B$ and $\phi : M_n \rightarrow A$ cpc want $\tilde{\phi} : M_n \rightarrow B$ near ϕ .

DEFINITION (WINTER)

A linear map $\phi : A \rightarrow B$ is **order zero** if it is cpc and preserves orthogonality, i.e. $\phi(x)\phi(y) = 0$ whenever $x, y \geq 0$ in A have $xy = 0$.

These maps have nice structure (e.g. order zero + unital = *-homomorphism).

ORDER ZERO MAPS

WANT

Given $A \subseteq B$ and $\phi : M_n \rightarrow A$ cpc want $\tilde{\phi} : M_n \rightarrow B$ near ϕ .

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A linear map $\phi : A \rightarrow B$ is **order zero** if it is cpc and preserves orthogonality, i.e. $\phi(x)\phi(y) = 0$ whenever $x, y \geq 0$ in A have $xy = 0$.

These maps have nice structure (e.g. order zero + unital = *-homomorphism). This is enough extra structure to perturb them.

THEOREM (CSSWW '09)

For all $\epsilon > 0$, there exists $\delta > 0$ such that if F is a finite dimensional C^ -algebra, $A \subseteq_{\delta} B$ and $\phi : F \rightarrow A$ is order zero, then there exists $\psi : F \rightarrow B$ order zero with $\|\phi - \psi\| \leq \epsilon$.*

NUCLEAR DIMENSION

DEFINITION (WINTER, ZACHARIAS (BASED ON KIRCHBERG + WINTER))

A nuclear C^* -algebra A has **nuclear dimension** at most n ($\dim_{\text{nuc}}(A) \leq n$) iff, for each $X \subset_{\text{fin}} A$, $\epsilon > 0$, \exists a factorisation through f.d. algebra $A \xrightarrow{\phi} F \xrightarrow{\psi} A$ which ϵ approximates I_A on X , with ϕ cpc and wrt to some decomposition $F = F_0 \oplus \cdots \oplus F_n$ each $\psi|_{F_i}$ is order zero.

EXAMPLES

- $\dim_{\text{nuc}}(C_0(X)) = \dim X$ (the covering dimension of X).
- $\dim_{\text{nuc}}(A) = 0 \Leftrightarrow A$ is AF.
- $\dim_{\text{nuc}}(\mathcal{O}_n) = 1$ for all n .
- Nuclear dimension has nice permanence properties.
- Simple unital C^* -algebras of finite nuclear dimension are either stably finite or purely infinite.
- $\dim_{\text{nuc}}(A) \leq 3$ whenever A is a classifiable Kirchberg algebra.

NUCLEAR DIMENSION

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- $\dim_{\text{nuc}}(C_0(X)) = \dim X$ (the covering dimension of X).
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THEOREM (CSSWW ('09))

Fix $n \geq 0$. Given a near inclusion $A \subseteq_{\gamma} B$ with $\dim_{\text{nuc}}(A) \leq n$, A separable and $\gamma < 1/3920000(n+2)$, there is an embedding $A \hookrightarrow B$.

AT-ALGEBRAS

DEFINITION

An AT-algebra is a direct limit of algebras of the form $C(\mathbb{T}) \otimes$ finite dim.

Unlike the AF-case, the connecting maps need not be injective. e.g \mathbb{C} is AT.

PROPOSITION (ELLIOTT)

Let A be a separable C^* -algebra. Then A is AT iff for all finite subsets X of the unit ball of A and $\epsilon > 0$ there exists a subalgebra A_0 of A with $X \subset_\epsilon A_0$ and A_0 is of the form

$$C(\mathbb{T}) \otimes F_1 \oplus C[0, 1] \otimes F_2 \oplus F_3.$$

Not all direct limits have such local characterisations.

A SAMPLE APPLICATION

We don't need to be able to approximate finite subsets **arbitrarily** well to get AT algebras.

THEOREM (CSSWW '08)

There exists a constant $\gamma_0 > 0$ s.t. if A is a separable C^ -algebra for which given any finite subset X of the unit ball of A there exists a subalgebra A_0 of A with $X \subset_{\gamma_0} A_0$ and A_0 has the form*

$$C(\mathbb{T}) \otimes F_1 \oplus C[0, 1] \otimes F_2 \oplus F_3,$$

then A is AT .

Variations on this theme for other direct limits of weakly semiprojective building blocks possible.

THE OBVIOUS QUESTION: WHAT IS γ_0 ?

Haven't worked through to find out (yet)!