PERTURBATIONS OF NUCLEAR C*-ALGEBRAS

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Setup

- H a fixed separable Hilbert space.
- (almost) all our C*-algebras will be C*-subalgebras of $\mathbb{B}(H)$ — Write $C^*(H)$ for all these C*-subalgebras.

Objective

- Turn $C^*(H)$ into a metric space (in a natural way).
- Question: What happens with $d(A, B)$ is small?

Recall

- A C*-subalgebra A of $\mathbb{B}(H)$ is a subalgebra of $\mathbb{B}(H)$, which is self adjoint (i.e. if the operator x lies in A then so does x^*) and closed in the norm topology.
- We do not assume that these algebras are unital (contain an identity) or that an identity for A coincides with the identity operator of H.

Close operator algebras

Questions

Example

Take a unitary u on H. Then $d(uAu^*, A) \leq 2\|u - 1\|$.

Hard to think of other ways of getting two close algebras.

Some questions

1. Do sufficiently close operator algebras have the same structural properties?
2. Are sufficiently close operator algebras isomorphic?
3. If so, can one find an isomorphism with nice additional properties? For example can we demand an isomorphism θ with $\|\theta(x) - x\|$ small for all x in the unit ball? Can we demand that our isomorphism is implemented by a unitary? By a unitary close to 1?

Near containments

Definition

Write $A \subseteq_\gamma B$ if $\forall x \in (A)_1$, $\exists y \in (B)_1$ with $\|x - y\| \leq \gamma$

Can ask similar questions: If $A \subseteq_\gamma B$ can one embed A into B.

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Easy Proposition

Suppose $A \subset B \subseteq_\gamma A$ for $\gamma < 1$. Then $A = B$.

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Easy Proposition

Suppose $A \subset B \subseteq_\gamma A$ for $\gamma < 1$. Then $A = B$.

Recall

- $M \in C^*(H)$ is a von Neumann algebra iff M is wot closed.
- $m_i \to m$ in wot $\Leftrightarrow \langle (m_i - m)\xi, \eta \rangle \to 0$ for all $\xi, \eta \in H$.

Corollary

Let $M, N \in C^*(H)$ with $d(M, N) = \gamma < 1/2$ and M a von Neumann algebra. Then N is a von Neumann algebra.

Injective von Neumann algebras

Close injectives are isomorphic

Recall

A von Neumann algebra $M \subset \mathbb{B}(H)$ is injective iff there exists a norm 1-projection $\Phi : \mathbb{B}(H) \to M$.

This means that if $d(M, N)$ is small and N is injective we have a map

$$\Phi|_M : M \to N$$

which is close to the identity map $I|_M : M \to \mathbb{B}(H)$.

Theorem (Christensen 74)

If M, N are injective von Neumann subalgebras of $\mathbb{B}(H)$ with $d(M, N)$ sufficiently small, then $M \cong N$.

Plan: Adjust $\Phi|_M$ using injectivity of M to obtain a *-isomorphism.

Injective von Neumann algebras

More perturbation results

Theorem (Raeburn, Taylor 75)

Let $M, N \subset \mathbb{B}(H)$ have $d(M, N)$ small. If M is an injective vNa, then N is an injective vNa. ($d(M, N) < 1/101$ will do.)

$\Rightarrow$ Christensen's perturbation result only needs one algebra to be injective.

Injective von Neumann algebras

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$\Rightarrow$ Christensen's perturbation result only needs one algebra to be injective.

Theorem (Christensen 80)

Let M, N be vNas with M injective. If $M \subseteq_\gamma N$ for $\gamma < 1/100$, then $\exists$ unitary $u \in (M \cup N)''$ st $uMu^* \subseteq N$ and $\|u - 1\| \leq 150\gamma$.

Things work as nicely as possible for injective vNas. The focus of research in perturbation theory then switched to C*-algebras

C^*-PERTURBATION THEORY

NOT EVERYTHING WORKS SO NICELY

COUNTEREXAMPLE (CHOI, CHRISTENSEN)

For $\epsilon > 0$, there exist non-isomorphic C^*-algebras $A, B \subset \mathbb{B}(H)$ with $d(A, B) < \epsilon$.

- Examples are not vNas and not separable.
- In C^*-setting restrict to the separable case.

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EXAMPLE (JOHNSON)

For $\epsilon > 0$, there exist two faithful representations of $C([0,1], \mathbb{K})$ on H with images A, B s.t. $d(A, B) < \epsilon$, yet any isomorphism $\theta : A \to B$ has $\|\theta(x) - x\| \geq \|x\|/70$ for some $x \in A$.

- $C([0,1], \mathbb{K})$ a nice C^*-algebra. Will look for perturbation results which don't give isomorphisms uniformly close to I.

AF-C*-ALGEBRAS

DEFINITION

A C*-algebra A is AF (approximately finite dimensional) if it is separable and there are finite dimensional C*-algebras $A_1 \subset A_2 \subset \dots$ with $\overline{\bigcup A_n} = A$.

- Introduced by Bratteli, who gave local charcterisation: $\forall \epsilon > 0, x_1, \dots, x_n \in A,\ \exists$f.d. $A_0 \subset A$ with $d(x_i, A_0) < \epsilon\ \forall i$.
- Classified by ordered K-theory by Elliott.

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THEOREM (PHILLIPS, RAEBURN 79)

If $A, B \subset \mathbb{B}(H)$ are AF-algebras with $d(A, B)$ sufficiently small. Then $A = uBu^*$ for some unitary $u \in (A \cup B)''$.

- Used Elliott to get $A \cong B$, then Bratteli to deduce unitary conjugacy. (In fact $d(A, B) < 1/16$ will do).

C^*-PERTURBATION THEORY

MORE PERTURBATION RESULTS

THEOREM (CHRISTENSEN 80)

Suppose $d(A, B) < 10^{-9}$. Then A is AF iff B is AF.

A class of C^*-algebras $\mathcal{A}$ is perturbable if there exists $\gamma_0 > 0$ (depending on A) so that if $d(A, B) < \gamma_0$ and one algebra lies in $\mathcal{A}$ then A and B are unitarily conjugate. Examples:

- abelian C^*-algebras.
- AF-algebras.
- Stable, separable and continuous trace — e.g $C[0, 1] \otimes \mathbb{K}$, or unital separable and continuous trace. (Phillips + Raeburn).
- Certain extensions some of the classes above (Khoshkam).

Completely positive maps

Definition

- Given C*-algebra A identify $M_n(A)$ as a subalgebra of $\mathbb{B}(H^{\oplus n})$. In this way $M_n(A)$ is a C*-algebra.
- Given a map $\phi : A \to B$ between C^*-algebras define $\phi_n : M_n(A) \to M_n(B)$ by $\phi_n((x_{i,j})) = (\phi(x_{i,j}))$, i.e. do ϕ component wise.

Definition

$\phi : A \to B$ is completely positive iff each ϕ_n is positive.

cpc = completely positive and contractive (norm ≤ 1).

Recall

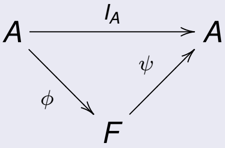
A linear map $\phi : A \to B$ is positive if $\phi(x) \geq 0$ whenever $x \geq 0$. Positive elements of C*-algebra those of form y^*y (same as positive as operator on H).

Nuclear C*-algebras

The completely positive approximation propety

Definition (Theorem of Choi, Effros, Lance)

A C*-algebra A is nuclear if, $\forall X \subset_{\text{fin}} A, \epsilon > 0$, $\exists F$ a finite dimensional C*-algebra and cpc maps $\phi : A \to F$, $\psi : F \to A$ s.t. $\psi \circ \phi \approx_{X,\epsilon} I_A$, i.e.



commutes on X upto ϵ.

Some Examples

- abelian and finite dimensional C*-algebras.
- $C^*_r(G)$ for a discrete amenable group G.
- Cuntz algebras $\mathcal{O}_n$.

Closed under: $\otimes$, quotients, extensions, $\oplus$, $\lim_{\to}$.

PERTURBATION RESULTS FOR NUCLEARS

SEPARABILITY THE ONLY OBSTRUCTION

THEOREM A (08)

Let $A, B \in C^*(H)$ have $d(A, B) = \gamma < 1/3686400$. Suppose A is separable and nuclear, then $A \cong B$. Furthermore, given $X \subset_{\text{fin}} (A)_1$ and $Y \subset_{\text{fin}} (B)_1$ there exists an isomorphism $\theta : A \to B$ with $\theta \approx_{X,50\gamma^{1/2}} I$ and $\theta^{-1} \approx_{Y,50\gamma^{1/2}} I$.

Second half of the theorem says that we can obtain some control on what the isomorphism does.

THEOREM B (09)

Let $A, B \in C^*(H)$ have $d(A, B) < 10^{-13}$ and suppose A is separable and nuclear. Then there exists a unitary $u \in (A \cup B)''$ with $uAu^* = B$.

NUCLEAR C*-ALGEBRAS ARE OPEN

PASS THROUGH THE BIDUAL

Every C*-algebra has a universal representation on some Hilbert space. The weak closure in this representation is isometrically isomorphic to the banach space bidual A^{**} which is then a von Neumann algebra.

THEOREM (CONNES + MANY HANDS)

A is nuclear $\Leftrightarrow$ A^{**} is injective.

COROLLARY (CHRISTENSEN)

Let A, B be C*-subalgebras of some C with $d(A, B) < 1/101$. Then A is nuclear iff B is nuclear.

Cpc maps between close nuclears

An application of Arveson's extension theorem

Recap: The injective von Neumann algebra situation

If $d(M,N)=\gamma$ and N is injective with conditional expectation $\phi:\mathbb{B}(H)\rightarrow N$, then $\phi|_M:M\rightarrow N$ has $\|\phi|_M-I\|\leq 2\gamma$.

Want something like this for nuclear C*-algebras.

CPC MAPS BETWEEN CLOSE NUCLEARS

AN APPLICATION OF ARVESON'S EXTENSION THEOREM

RECAP: THE INJECTIVE VON NEUMANN ALGEBRA SITUATION

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ARVESON'S EXTENSION THEOREM

Let $B\subset\mathbb{B}(H)$ be a C*-algebra and $\phi:B\to F$ be cpc with F finite dimensional. Then $\exists$ cpc $\tilde{\phi}:\mathbb{B}(H)\to F$ extending ϕ.

Our C*-version works in point-norm topology rather than the uniform norm.

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Lemma

Let $A,B\in C^*(H)$ have $A\subset_\gamma B$ with B nuclear. $\forall Z\subset_{fin}(A)_1$, $\exists\phi:A\to B$ cpc $\phi\approx_{Z,3\gamma} I$.

APPROXIMATE HOMOMORPHISMS

FROM $(\cdot, 9\gamma)$-HMS TO $(\cdot, \delta)$-HMS

DEFINITION

Given $X \subset_{\text{fin}} (A)_1$ and $\epsilon > 0$, $\phi : A \longrightarrow B$ is an (X, ϵ)-hm if it is cpc and $\|\phi(xx^*) - \phi(x)\phi(x^*)\| < \epsilon$ for $x \in X \cup X^*$.

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LEMMA

Let A be a nuclear C*-algebra. $\forall Y \subset_{\text{fin}} (A)_1, \delta > 0, \mu > 0$, $\exists Z \subset_{\text{fin}} (A)_1$ st:

If D is a C-algebra and $\phi : A \to D$ is a (Z, η)-hm for $\eta < 1/49$. Then $\exists$ (Y, δ)-hm $\psi : A \to D$ with $\|\psi - \phi\| < 16\eta^{1/2} + 8\mu$.*

- Lemma based on an 'on the nose' result for injective vNas.
- In this lemma, think of $\eta = 9\gamma$ as large.
- In some (precise) sense Z is a Følner set for (Y, δ).

Close nuclear C*-algebras

Ingredients of Theorem A

The previous lemma and the Arveson trick give:

Lemma A

Let $d(A,B) = \gamma$ is small with A nuclear. $\forall Y \subset_{\text{fin}} (A)_1,\ \delta > 0$, $\exists \phi : A \to B$ a (Y,δ)-hm with $\|\phi(y) - y\| \leq$ function of γ for $y \in Y$.

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Another nuclear analogue of a von Neumann result:

Lemma B

Let A be a nuclear C*-algebra. $\forall X \subset_{\text{fin}} (A)_1, \epsilon > 0$, $\exists Y \subset_{\text{fin}} (A)_1, \delta > 0$ st:

If D is a C^*-algebra and $\phi_1, \phi_2 : A \to D$ are (Y, δ)-approx hms with $\phi_1 \approx_{Y,\eta} \phi_2$ for $\eta < 1/24$, then $\exists$ unitary $u \in \tilde{D}$ with $\phi_1 \approx_{X,\epsilon} Ad(u) \circ \phi_2$ and $\|u - 1\| < 4\eta + 10\delta$.

Near inclusions

OK if large algebra nuclear

Question

Suppose we have $A \subseteq_\gamma B$ with A separable and nuclear. Do we have an injective *-hm $A \to B$?

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Suppose we have $A \subseteq_\gamma B$ with A separable and nuclear. Do we have an injective *-hm $A \to B$?

Only used $d(A, B)$ small to deduce that B is also nuclear. This allowed us to use Arveson to construct cpc maps $A \to B$. Other techniques only needed nuclearity of A.

Theorem

Suppose $A \subseteq_\gamma B$ for $\gamma < 1/3686400$ and A, B are nuclear and A is separable. Then there exists an embedding $A \hookrightarrow B$.

Question

If $A \subseteq_\gamma B$ and A is nuclear but B is not, can we find cpc maps $A \to B$ close to I on finite sets?

Near inclusions

A plan for finding cpc maps

Given $A \subseteq_\gamma B$ with A nuclear, fix $X \subset_{\text{fin}} (A)_1$.

Want

Want cpc map $\phi : A \to B$ with $\phi \approx_{X,f(\gamma)} I$. Some reasonable funtion f.

Take cpc $A \xrightarrow{\psi} M_n \xrightarrow{\phi} A$ with $\psi \circ \phi \approx_{X,\epsilon} I_A$. If we could find $\tilde{\phi} : M_n \to B$ with $\|\tilde{\phi} - \phi\|$ small, then $\psi \circ \tilde{\phi}$ would do.

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Facts

- $CP(M_n, A) \leftrightarrow M_n(A)^+$ via $\phi \mapsto (\phi(e_{i,j}))_{i,j}$.
- $M_n(A) \subseteq_{\gamma^2+2\gamma} M_n(B)$ — needs nuclearity of A.

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Facts

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- $M_n(A) \subseteq_{\gamma^2+2\gamma} M_n(B)$ — needs nuclearity of A.

Problem: cpc map ϕ corresponds to norm $\leq n$ of $M_n(A)^+$. Can get $\tilde{\phi}$ but $\|\tilde{\phi} - \phi\|$ depends on n. Need more structure on ϕ.

Order zero maps

Definition (Winter)

A linear map $\phi : A \to B$ is order zero if it is cpc and preserves orthogonality, i.e. $\phi(x)\phi(y) = 0$ whenever $x, y \geq 0$ in A have $xy = 0$.

These maps have nice structure:

Proposition (Winter)

Let F be a finite dimensional C^*-algebra. A linear map $\phi : F \to A$ is order zero iff there exists a *-hm $\pi : C_0(0,1] \otimes F \to A$ such that $\pi(\iota \otimes x) = \phi(x)$ for all $x \in F$.

Useful fact

Cones $CF = C(0,1] \otimes F$ are weakly semiprojective (in sense of Loring). In particular, $\forall X \subset_{\text{fin}} (CF)_1$, $\epsilon > 0$, $\exists Y \subset_{\text{fin}} (CF)_1$, $\delta > 0$ such that if $\phi : CF \to A$ is an (Y, δ)-hm, then there exists a *-hm $\psi : CF \to A$ with $\phi \approx_{X,\epsilon} \psi$.

Nuclear dimension
Definition

Definition (Winter, Zacharias (based on Kirchberg + Winter))

A nuclear C*-algebra A has nuclear dimension at most n ($\dim_{\mathrm{nuc}}(A) \leq n$) iff, for each $X \subset_{\mathrm{fin}} A, \epsilon > 0$, $\exists$ a factorisation through f.d. algebra $A \xrightarrow{\phi} F \xrightarrow{\psi} A$ which ϵ approximates I_A on X, with ϕ cpc and wrt to some decomposition $F = F_0 \oplus \cdots \oplus F_n$ each $\psi|_{F_i}$ is order zero.

Examples

- $\dim_{\mathrm{nuc}}(C_0(X)) = \dim X$ (the covering dimension of X).
- $\dim_{\mathrm{nuc}}(A) = 0 \Leftrightarrow A$ is AF.
- $\dim_{\mathrm{nuc}}(\mathcal{O}_n) = 1$ for all n.

Properties of nuclear dimension

Why is it a reasonable non-commutative dimension

Permanence properties

- $\dim_{\text{nuc}}(A) = \dim_{\text{nuc}}(A \otimes M_n) = \dim_{\text{nuc}}(A \otimes \mathbb{K})$.
- $\dim_{\text{nuc}}(A \oplus B) = \max(\dim_{\text{nuc}}(A), \dim_{\text{nuc}}(B))$.
- If $A = \lim_{n\to\infty} A_n$, then $\dim_{\text{nuc}}(A) \leq \liminf_{n\to\infty} \dim_{\text{nuc}}(A_n)$.
- If $B \subset A$ is a hereditary subalgebra of A, then $\dim_{\text{nuc}}(B) \leq \dim_{\text{nuc}}(A)$.
- If A is non-unital, $\dim_{\text{nuc}}(\tilde{A}) = \dim_{\text{nuc}}(A)$.
- $\dim_{\text{nuc}}(A \otimes B) \leq (\dim_{\text{nuc}}(A) + 1)(\dim_{\text{nuc}}(B) + 1) - 1$.
- $\dim_{\text{nuc}}(E) \leq \dim_{\text{nuc}}(J) + \dim_{\text{nuc}}(A) + 1$, whenever $0 \to J \to E \to A \to 0$ is exact.

Consequence (of above + Kirchberg, Phillips) $\dim_{\text{nuc}}(A) \leq 3$ whenever A is a classifiable Kirchberg algebra.

Perturbing order zero maps

Theorem (09)

For all $\epsilon > 0$, there exists $\delta > 0$ such that if F is a finite dimensional C^*-algebra, $A \subseteq_\delta B$ and $\phi : F \to A$ is order zero, then there exists $\psi : F \to B$ order zero with $\|\phi - \psi\| \leq \epsilon$.

Ingredients of proof: a trick of Christensen for simultaneously approximating n orthogonal elements, Structure theory of order zero maps, Cones CF weakly semiprojective, The two technical lemmas from earlier.

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Corollary

For each $n \geq 0$, there exists $\gamma_n > 0$ with the property that for any near inclusion $A \subseteq_{\gamma_n} B$ with $\dim_{nuc}(A) \leq n$, there is an embedding $A \hookrightarrow B$.

Can control the embeddings $A \hookrightarrow B$ on finite sets.

Direct limits

$A\mathbb{T}$-algebras

Definition

An $A\mathbb{T}$-algebra is a direct limit of algebras of the form $C(\mathbb{T}) \otimes$ finite dim.

Unlike the AF-case, the connecting maps need not be injective. e.g $\mathbb{C}$ is $A\mathbb{T}$.

Proposition (Elliott)

Let A be a separable C*-algebra. Then A is $A\mathbb{T}$ iff for all finite subsets X of the unit ball of A and $\epsilon > 0$ there exists a subalgebra A_0 of A with $X \subset_\epsilon A_0$ and A_0 is of the form

$$C(\mathbb{T}) \otimes F_1 \oplus C[0,1] \otimes F_2 \oplus F_3.$$

Not all direct limits have such local characterisations.

DIRECT LIMITS

A SAMPLE APPLICATION

We don't need to be able to approximate finite subsets arbitrarily well to get $A\mathbb{T}$ algebras.

THEOREM ('08)

There exists a constant $\gamma_0 > 0$ s.t. if A is a separable C^*-algebra for which given any finite subset X of the unit ball of A there exists a subalgebra A_0 of A with $X \subset_{\gamma_0} A_0$ and A_0 has the form

$$C(\mathbb{T}) \otimes F_1 \oplus C[0,1] \otimes F_2 \oplus F_3,$$

then A is $A\mathbb{T}$.

Variations on this theme for other direct limits of weakly semiprojective building blocks possible.

THE OBVIOUS QUESTION: WHAT IS γ_0?

Haven't worked through to find out (yet!), but very small.