Close operator algebras and almost multiplicative maps

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1. Close operator algebras — invariants

This section of the report is concerned with close operator algebras. In [7], Kadison and Kastler equipped the collection of operator algebras acting on a fixed Hilbert space $\mathcal{H}$ with a metric d, which arises from restricting the Hausdorff metric on subsets of $\mathcal{B}(\mathcal{H})$ to the unit balls of C^*-subalgebras. Precisely, given C^* or von Neumann algebras $A, B \subset \mathcal{B}(\mathcal{H})$, we have

$$d(A,B) = \max\left(\sup_{\substack{a\in A\\ \|a\|\leq 1}} \inf_{\substack{b\in B\\ \|b\|\leq 1}} \|a-b\|, \sup_{\substack{b\in B\\ \|b\|\leq 1}} \inf_{\substack{a\in A\\ \|a\|\leq 1}} \|b-a\|\right).$$

The main focus of investigation is 'what can be said if $d(A,B)$ is small'?

- Can we show that sufficiently close operator algebras share the same properties and invariants?
- Must such algebras be isomorphic?

In the study of operator algebras, it is always natural to consider matrix amplifications. In our context this leads to the *completely bounded* version of the Kadison-Kastler metric:

$$d_{cb}(A,B) = \sup_n d(A\otimes M_n, B\otimes M_n),$$

where $A, B \subset \mathcal{B}(\mathcal{H})$, and the distance between $A\otimes M_n$ and $B \otimes M_n$ is measured in $\mathcal{B}(\mathcal{H})\otimes M_n \cong \mathcal{B}(\mathcal{H}^n)$. We also investigate the consequences of "complete closeness", i.e. what can be said when $d_{cb}(A,B)$ is small?

For example, if $d_{cb}(A,B)$ is small, then any projection $p \in A\otimes M_n$ can be suitably approximated by a projection $q \in B\otimes M_n$, leading an isomorphism $K_0(A) \to K_0(B)$ which maps $[p]_0$ to $[q]_0$. This strategy is due to Khoshkam [8], who used it to obtain an isomorphism $K_*(A)\cong K_*(B)$ when $d(A,B)$ is sufficiently small and A is nuclear.

The Cuntz semigroup $\mathrm{Cu}(A)$ of a C^*-algebra A is a highly refined invariant obtained from equivalence classes of positive elements of $A\otimes\mathcal{K}$. In contrast to projections, where the class in K_0 is invariant under small perturbations, the Cuntz class of a positive element $a\in A\otimes\mathcal{K}$ is sensitive to small modifications in norm. Nevertheless, it is possible to extend Khoshkam's isomorphism to this setting.

Theorem 1 (Perera, Toms, W, Winter, [11])**.** *Let $A, B \subseteq \mathcal{B}(\mathcal{H})$ be C^*-algebras with $d_{cb}(A,B) < 1/42$. Then $\mathrm{Cu}(A)\cong\mathrm{Cu}(B)$ in a scale preserving fashion.*

Given Khoshkam's work and Theorem 1 it is natural to ask when close algebras are automatically "completely close". This is resolved in the following theorem.

Theorem 2 (Cameron, C, Sinclair, Smith, W, Wiggins, [1] extending [4])**.** *The following are equivalent:*

(1) *d and d_{cb} are equivalent metrics;*
(2) *Kadison's similarity problem has a positive answer, i.e. any bounded homomorphism $A \to \mathcal{B}(\mathcal{H})$ whose domain is a C^*-algebra is similar to a *-homomorphism.*

This is also true locally: $d_{cb}(A, \cdot)$ and $d(A, \cdot)$ are equivalent if and only if A has the similarity property.

Using Theorem 2, we obtain the following corollaries of Theorem 1.

Corollary ([11])**.** *Let $A, B \subseteq \mathcal{B}(\mathcal{H})$ be C^*-algebras with $d(A, B)$ sufficiently small and suppose that one of A or B has stable rank one. Then A is stable if and only if B is stable.*

Idea of proof. When one algebra, say A, is stable, it has the similarity property. It follows that $d_{cb}(A, B)$ is small when $d(A, B)$ is small, so that $\mathrm{Cu}(A) \cong \mathrm{Cu}(B)$ when $d(A, B)$ is small enough. By a result of Rørdam and Winter, the Cuntz semigroup of a C^*-algebra with stable rank one enjoys a certain cancelation property; further, in the presence of this cancelation property, stability of the C^*-algebra can be determined from the Cuntz semigroup and its scale. □

Corollary ([11])**.** *Let $A, B \subseteq \mathcal{B}(\mathcal{H})$ be unital with $d_{cb}(A, B)$ sufficiently small (which happens when $d(A, B)$ is sufficiently small and $A \cong A \otimes \mathcal{Z}$). Then the natural affine isomorphism between $T(A)$ and $T(B)$ defined in* [4] *is a homeomorphism. Further, if A has the property that all bounded 2-quasitraces are traces, then so too does B.*

2. Close operator algebras — isomorphisms

We now turn to the question of whether sufficiently close operator algebras must be isomorphic. In the late 1970's EC, Johnson, Raeburn and Taylor established such results in the context of injective von Neumann algebras leading to the following theorem.

Theorem 3. *For all $\varepsilon > 0$, there exists $\delta > 0$ with the property that whenever $M, N \subseteq \mathcal{B}(\mathcal{H})$ are von Neumann algebras with M injective and $d(M, N) < \delta$, there exists a unitary operator $u \in (M \cup N)''$ with $uMu^* = N$ and $\|u - 1\| < \varepsilon$.*

The main strategy for obtaining an isomorphism between the algebras M and N in this theorem is performed in two main steps:

(1) Obtain a completely positive map $T : M \to N$ which is *almost multiplicative* in the sense that the bilinear map $T^\vee : M \times M \to N$ given by $T^\vee(x, y) = T(xy) - T(x)T(y)$ has small norm. If one additionally assumes that N is injective, one can take a conditional expectation $\Phi : \mathcal{B}(\mathcal{H}) \to N$ and set $T = \Phi|_M$.
(2) Prove that every almost multiplicative completely positive map $T : M \to N$ is close to a *homomorphism $M \to N$.

The second step is an "AMNM" (almost multiplicative near multiplicative) type result, as later formalised by Johnson in [6]. In the context of separable nuclear C^*-algebras, a point norm version of these AMNM techniques can be used to prove the theorem below.

Theorem 4 ([5])**.** *There exists a constant $\gamma_0 > 0$, such that if $A, B \subseteq \mathcal{B}(\mathcal{H})$ are C^*-algebras with A separable and nuclear, and $d(A, B) < \gamma_0$, then there is a unitary $u \in (A \cup B)''$ with $uAu^* = B$.*

A key difference between Theorem 3 and 4 is that the unitary in Theorem 4 can not in general be taken close to 1 due to the examples of Johnson. On the other hand, later work of Johnson shows that the unitary can be taken close to 1 when A is n-subhomogeneous, for some $n \in \mathbb{N}$.

Question. *Exactly which separable nuclear C^*-algebras A have the following property? For all $\varepsilon > 0$, there exists $\delta > 0$, such that whenever A is faithfully represented on $\mathcal{H}$ and $B \subseteq \mathcal{H}$ is another C^*-algebra with $d(A, B) < \delta$, then there exists an isomorphism $\theta : A \to B$ with $\|\theta(x) - x\| \leq \varepsilon\|x\|$ for all $x \in A$.*

Outside the amenable setting, the AMNM technique has not been successfully applied to close algebras — the major obstruction is the first step above. What are methods for obtaining bounded linear maps between close (or completely close) operator algebras?

Instead we consider an alternative approach to obtaining an isomorphism between close algebras. Let M be a II_1 factor. Recall that a *Cartan masa* in M, is a maximal abelian subalgebra $A \subset M$ such that $\mathcal{N}_M(A) := \{u \in \mathcal{U}(M) : uAu^* = A\}$ generates M as a von Neumann algebra. The canonical example arises from the group-measure space construction: if $\alpha : \Gamma \curvearrowright (X, \mu)$ is a free ergodic action of a countable discrete group on a probability space, then $L^\infty(X)$ is a Cartan masa in the II_1 factor $L^\infty(X) \rtimes \Gamma$. In their seminal work, Feldman and Moore classified inclusions $A \subset M$ of Cartan masas upto unitary equivalence. They associated to $A \subseteq M$ a measurable equivalence relation $\mathcal{R}_{A \subseteq M}$ and the (orbit of) a certain class in the cohomology group $H^2(\mathcal{R}_{A \subseteq M}, \mathbb{T})$ and showed that this information classifies the inclusion $A \subseteq M$. In the canonical example where we start from a free ergodic action $\Gamma \curvearrowright (X, \mu)$, the measurable equivalence relation $\mathcal{R}$ is the orbit equivalence relation on X ($x \sim y$ iff there exists $g \in \Gamma$ with $g \cdot x = y$) and the cohomology group $H^2(\mathcal{R}, \mathbb{T})$ is identified with $H^2(\Gamma, L^\infty(X, \mathbb{T}))$.

Theorem 5 (Cameron, C, Sinclair, Smith, W, Wiggins, [2, 3])**.** *Let $A \subset M$ be a Cartan masa in a II_1 factor. Suppose M is represented as a von Neumann algebra on $\mathcal{H}$ and N is another von Neumann algebra on $\mathcal{H}$ such that $d(M, N)$ is sufficiently small. Then there exists a Cartan masa $B \subset N$ with $d(A, B)$ small, and $\mathcal{R}_{A \subseteq M} \cong \mathcal{R}_{B \subset N} \cong \mathcal{R}$. Further one can choose representative cocycles $\omega_{A \subset M}$ for $A \subset M$ and $\omega_{B \subset N}$ for $B \subset N$ in $Z^2(\mathcal{R}, \mathbb{T})$ to be "uniformly close".*

Thus, given a II_1 factor M containing a Cartan masa A and N as in Theorem 5, if one can additionally find a method for showing that $\omega_{A \subset M}$ and $\omega_{B \subset N}$ are cohomologous, then N must be isomorphic to M. The easiest way of ensuring this happens is when the relevant cohomology group vanishes, yielding the following corollary.

Corollary. *There exists $\gamma_0 > 0$ with the property that whenever $A \subset M$ is a Cartan masa in a II_1 factor such that $H^2(\mathcal{R}_{A\subset M}, \mathbb{T}) = \{1\}$, N is another von Neumann algebra and M, N are represented on the same Hilbert space with $d(M,N) < \gamma_0$, we have $N \cong M$. In particular this happens when M arises from a free ergodic measure preserving action of a free group of rank at least 2.*

Another strategy for proving that $\omega_{A\subset M}$ and $\omega_{B\subset N}$ are cohomologous is to use uniqueness of Cartan results. A countable discrete group Γ is called *Cartan rigid* if any free ergodic probability measure preserving action $\Gamma \curvearrowright (X,\mu)$ has the property that $L^\infty(X)$ is the unique Cartan masa in $L^\infty(X)\rtimes\Gamma$ upto conjugation by an inner automorphism. Recent years have seen remarkable progress in producing examples of Cartan rigid groups — in particular Popa and Vaes have shown that non-elementary hyperbolic groups are Cartan rigid [10].

Corollary. *There exists a constant $\gamma_0 > 0$ with the following property. Let Γ be a Cartan rigid countable discrete group and let $M = L^\infty(X) \rtimes \Gamma$ be a II_1 factor obtained from a free ergodic probability measure preserving action $\Gamma \curvearrowright (X,\mu)$. Let $\Lambda \curvearrowright (Y,\nu)$ be a free ergodic probability measure preserving action of any countable discrete group and $N = L^\infty(Y) \rtimes \Lambda$. If M and N can be represented on the same Hilbert space with $d(M,N) < \gamma_0$, then $M \cong N$.*

Our third strategy uses the additional information in Theorem 5 that $\omega_{A\subset M}$ and $\omega_{B\subset N}$ can be chosen "uniformly close". Consider a free ergodic probability measure preserving action $\Gamma \curvearrowright (X,\mu)$ for which the bounded group cohomology group $H^2_b(\Gamma, L^\infty_{\mathbb{R}}(X)) = 0$ (for example, Monod has shown that any free ergodic probability measure preserving action of $SL_n(\mathbb{Z})$ for $n \geq 3$ enjoys this property [9]). One can use this information as in the first corollary to obtain an isomorphism between $M = L^\infty(X)\rtimes\Gamma$ and any other von Neumann algebra N sufficiently close to M. In fact more is true: under these circumstances an isomorphism $\theta : M \to N$ can be found with $\sup_{\substack{x\in M\\ \|x\|\leq 1}} \|\theta(x) - x\|$ small. In the Corollary below, we are able to extend this further and obtain the first examples of non-injective von Neumann algebras which satisfy essentially the same conclusions as Theorem 3. The purpose of the tensor copy of R is to ensure that the factor M_0 has the similarity property allowing us to use the ideas behind Theorem 2. Indeed, the first step in the proof of the corollary is essentially to remove the tensor copy of R, by showing that N_0 also factorises as $N_0 \cong N \mathbin{\overline{\otimes}} R$ in such a way that N is close to $L^\infty(X) \rtimes \Gamma$.

Corollary. *For all $\varepsilon > 0$, there exists $\delta > 0$ with the following property. Let $\Gamma \curvearrowright (X,\mu)$ be a free ergodic probability measure preserving action with $H^2_b(\Gamma, L^\infty_{\mathbb{R}}(X)) = 0$ and define $M_0 = (L^\infty(X)\rtimes\Gamma) \mathbin{\overline{\otimes}} R$, where R is the hyperfinite II_1 factor. Suppose M_0 is represented as a von Neumann algebra on $\mathcal{H}$ and N_0 is another von Neumann algebra on $\mathcal{H}$ with $d(M_0,N_0) < \delta$. Then there exists a unitary u on $\mathcal{H}$ with $uM_0u^* = N_0$ and $\|u - 1\| < \varepsilon$.*

3. Almost multiplicative maps

Motivated by the point-norm AMNM techniques used in Theorem 4, in ongoing joint work with Allan Sinclair and Roger Smith, we have become interested in general results for almost multiplicative maps between operator algebras. The prototype is the following theorem of Johnson, which in particular says that every almost multiplicative map from a nuclear C^*-algebra into a von Neumann algebra is close to a homomorphism.

Theorem 6 (Johnson [6])**.** *Let A be an amenable Banach algebra with amenability constant L and B a dual Banach algebra. For $\varepsilon > 0$ and $K > 0$, any bounded linear map $T : A \to B$ with $\|T\| < K$ and $\|T^\vee\| < \varepsilon/(4L + 8K^2L^2)$, there is a bounded homomorphism $S : A \to B$ with $\|S - T\| < \varepsilon$.*

At present we have the following two results, demonstrating that almost multiplicative maps on nuclear C^*-algebras are near to homomorphisms in the point norm topology. The restriction that $\|T\|$ is not too large is a way of ensuring that our maps are almost self-adjoint.

Theorem 7. *Let A be a separable unital nuclear C^*-algebra and B a unital C^*-algebra. Given a bounded unital map $T : A \to B$ with $\|T\| \leq 1.1$ and $\|T^\vee\| \leq 0.01$, then there is a *-homomorphism $\Psi : A \to B$ such that for any finite set X in the unit ball of A, there is a unitary operator $u \in B$ such that*

$$\|T(x) - u^*\Psi(x)u\| \leq 3\big(\|T^\vee\| + \|T\| - 1\big), \quad x \in X. \tag{1}$$

Theorem 8. *Let A be a separable unital C^*-algebra which is the direct limit of semiprojective unital and nuclear C^*-algebras and let B be a unital C^*-algebra. Given a bounded unital map $T : A \to B$ with $\|T^\vee\|(\|T\|^2 + 2) < 0.2$, then there is a *-homomorphism $\Psi : A \to B$ such that for every finite set X of the unit ball of A, there is an invertible positive operator $t \in B$ with $(2\|T\|)^2)^{-1}1_B \leq t \leq (2\|T\|^2)1_B$ such that*

$$\|T(x) - t^{-1}\Psi(x)t\| \leq 4\|T\|\|T^\vee\|, \quad x \in X. \tag{2}$$

The key difference between these results is that under the additional assumption that A is the limit of semiprojective nuclear C^*-algebras in Theorem 7, we do not require a universal bound on $\|T\|$; one can still obtain estimates for maps T of relatively large norm, provided $\|T^\vee\|$ is correspondingly small.

References

[1] J. Cameron, E. Christensen, A. Sinclair, R. Smith, S. White, A. Wiggins. A remark on the similarity and perturbation problems. *C. R. Math. Rep. Acad. Sci. Canada*, 35(2), 70–76, 2013.

[2] J. Cameron, E. Christensen, A. Sinclair, R. Smith, S. White, A. Wiggins. Kadison-Kastler stable factors. arXiv:1209.4116.

[3] J. Cameron, E. Christensen, A. Sinclair, R. Smith, S. White, A. Wiggins. Type II_1 factors satisfying the spatial isomorphism conjecture. *Proc. Natl. Acad. Sci. USA*, 109(5), 20338–20343, 2012.

[4] E. Christensen, A. Sinclair, R. Smith, S. White. Perturbations of C^*-algebraic invariants. *Geom. Funct. Anal.*, 20, 368–397, 2010.

[5] E. Christensen, A. Sinclair, R. Smith, S. White, W. Winter. Perturbations of nuclear C^*-algebras. *Acta. Math.*, 230(3), 93–150.

[6] B. Johnson. Approximately multiplicative maps between Banach algebras. *J. London Math. Soc. (2)*, 37, 294–316, 1988.

[7] R. Kadison, D. Kastler. Perturbations of von Neumann algebras I Stability of type. *Amer. Math. J.*, 94(1), 38–54, 1972.

[8] M. Khoshkam. Perturbations of C^*-algebras and K-theory. *J. Oper. Thy*, 12(1):89–99, 1984.

[9] N. Monod. On the bounded cohomology of semi-simple groups, S-arithmetic groups and products. *J. reine agnew. Math.*, 640, 167–202, 2010.

[10] S. Popa, S. Vaes, Unique Cartan decomposition for arbitrary actions of hyperbolic groups. *J. reine agnew. Math.*, in press, arXiv:1201.2824.

[11] F. Perera, A. Toms, S. White, W. Winter. The Cuntz semigroup and stability of close C^*-algebras. *Anal. PDE*, to appear, arXiv:1210.4533.