KMS states on self-similar groupoid actions

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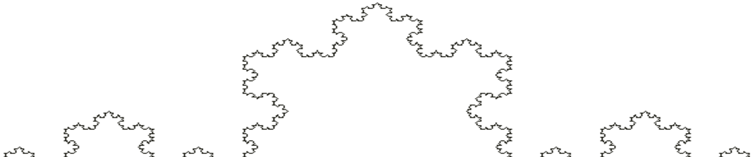
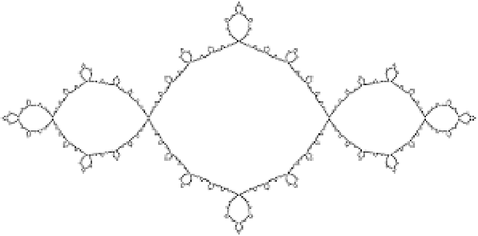


Plan

1. Self-similar groups
2. Self-similar groupoids
3. C^*-algebras of self-similar groupoids
4. KMS states on self-similar groupoids

1. Self-similar groups



- R. Grigorchuk, *On the Burnside problem on periodic groups*, Funkts. Anal. Prilozen. **14** (1980), 53–54.
- R. Grigorchuk, *Milnor Problem on group growth and theory of invariant means*, Abstracts of the ICM, 1982.
- V. Nekrashevych, Self-Similar Groups, Math. Surveys and Monographs, vol. 117, Amer. Math. Soc., Providence, 2005.

Self-similar groups

- Suppose X is a finite set of cardinality $|X|$;
  - let X^n denote the set of words of length n in X with $X^0 = \varnothing$,
  - let $X^* = \bigcup_{n \geq 0} X^n$.

Definition

Suppose G is a group acting faithfully on X^*. We say (G, X) is a *self-similar group* if, for all $g \in G$ and $x \in X$, there exist $h \in G$ such that

$$g \cdot (xw) = (g \cdot x)(h \cdot w) \quad \text{for all finite words } w \in X^*. \qquad (1)$$

Faithfulness of the action implies the group element h is uniquely defined by $g \in G$ and $x \in X$. So we define $g|_x := h$ and call it *the restriction of g to x*.

Then (1) becomes

$$g \cdot (xw) = (g \cdot x)(g|_x \cdot w) \quad \text{for all finite words } w \in X^*.$$

Self-similar groups

We may replace the letter x by an initial word $v \in X^k$:
For $g \in G$ and $v \in X^k$, define $g|_v \in G$ by

$$g|_v = (g|_{v_1})|_{v_2} \cdots |_{v_k}.$$

Then the self-similar relation becomes

$$g \cdot (vw) = (g \cdot v)(g|_v \cdot w) \quad \text{for all } w \in X^*.$$

Lemma

Suppose (G, X) is a self-similar group. Restrictions satisfy

$$g|_{vw} = (g|_v)|_w, \quad gh|_v = g|_{h\cdot v}\ h|_v, \quad g|_v^{-1} = g^{-1}|_{g\cdot v}$$

for all $g, h \in G$ and $v, w \in X^*$.

Example: the odometer

- Suppose $X = \{0, 1\}$ and $\mathrm{Aut}\, X^*$ is the automorphism group.
- Define an automorphism in $\mathrm{Aut}\, X^*$ recursively by

$$a \cdot 0w = 1w \qquad\qquad a \cdot 1w = 0(a \cdot w)$$

  for every finite word $w \in X^*$

- The self-similar group generated by a is the integers $\mathbb{Z} := \{a^n : \ n \in \mathbb{Z}\}$, and $(\mathbb{Z}, X)$ is commonly called the odometer because the self-similar action is "adding one with carryover, in binary."

Example: the Grigorchuk group

- Suppose $X = \{x, y\}$ and $\operatorname{Aut} X^*$ is the automorphism group.
- The Grigorchuk group is generated by four automorphisms $a, b, c, d \in \operatorname{Aut} X^*$ defined recursively by

$$
\begin{aligned}
a \cdot xw &= yw & a \cdot yw &= xw \\
b \cdot xw &= x(a \cdot w) & b \cdot yw &= y(c \cdot w) \\
c \cdot xw &= x(a \cdot w) & c \cdot yw &= y(d \cdot w) \\
d \cdot xw &= xw & d \cdot yw &= y(b \cdot w).
\end{aligned}
$$

Proposition

The generators a, b, c, d of G all have order two, and satisfy $cd = b = dc$, $db = c = bd$ and $bc = d = cb$. The self-similar action (G, X) is contracting with nucleus $\mathcal{N} = \{e, a, b, c, d\}$.

Properties of the Grigorchuk group

Theorem (Grigorchuk 1980)

The Grigorchuk group is a finitely generated infinite 2-torsion group.

Theorem (Grigorchuk 1984)

The Grigorchuk group has intermediate growth.

(Solved a Milnor problem from 1968)

Example: the basilica group

- Suppose $X = \{x, y\}$ and $\operatorname{Aut} X^*$ is the automorphism group.
- Two automorphisms a and b in $\operatorname{Aut} X^*$ are recursively defined by

$$\begin{aligned} a \cdot xw &= y(b \cdot w) \qquad & a \cdot yw &= xw \\ b \cdot xw &= x(a \cdot w) \qquad & b \cdot yw &= yw \end{aligned}$$

  for $w \in X^*$.
- The *basilica group* B is the subgroup of $\operatorname{Aut} X^*$ generated by $\{a, b\}$. The pair (B, X) is then a self-similar action.
- The nucleus is $\mathcal{N} = \{e, a, b, a^{-1}, b^{-1}, ba^{-1}, ab^{-1}\}$.

Properties of the basilica group

Theorem (Grigorchuk and Żuk 2003)

The basilica group

- *is torsion free,*
- *has exponential growth,*
- *has no free non-abelian subgroups,*
- *is not elementary amenable.*

Theorem (Bartholdi and Virág 2005)

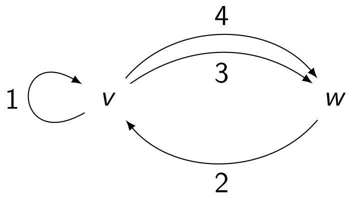
The basilica group is amenable.

2. Self-similar groupoids

- E. Bédos, S. Kaliszewski and J. Quigg, *On Exel-Pardo algebras*, preprint, arXiv:1512.07302.
- R. Exel and E. Pardo, *Self-similar graphs: a unified treatment of Katsura and Nekrashevych C^*-algebras*, to appear in Advances in Math., ArXiv:1409.1107.
- M. Laca, I. Raeburn, J. Ramagge, and M. Whittaker *Equilibrium states on operator algebras associated to self-similar actions of groupoids on graphs*, preprint, ArXiv 1610.00343.

Directed graphs

- Let $E = (E^0, E^1, r, s)$ be a finite directed graph with vertex set E^0, edge set E^1, and range and source maps from E^1 to E^0.



- Given a graph E, the set of paths of length k is

$$E^k := \{\mu = \mu_1\mu_2\cdots\mu_k : \ \mu_i \in E^1, s(\mu_i) = r(\mu_{i+1})\},$$

and let

$$E^* = \bigcup_{k=0}^{\infty} E^k$$

denote the collection of finite paths. A path of length zero is defined to be a vertex.

Partial isomorphisms on graphs

- Suppose $E = (E^0, E^1, r, s)$ is a directed graph. A *partial isomorphism* of the path space E^* consists of two vertices $v, w \in E^0$ and a bijection $g : vE^* \to wE^*$ such that
  - $g(vE^k) = wE^k$ for all $k \in \mathbb{N}$ and
  - $g(\mu\nu) \in g(\mu)E^*$ for all $\mu\nu \in E^*$.

- For each $v \in E^0$ we let $\mathrm{id}_v : vE^* \to vE^*$ denote the partial isomorphism $\mathrm{id}_v(\mu) = \mu$ for all $\mu \in vE^*$.

- We write g for the triple $(g, s(g) := v, r(g) := w)$, and we denote the set of all partial isomorphisms on E by $\mathcal{P}(E^*)$.

Groupoids

- A *groupoid* G with unit space X consists of
  - a set G and a subset $X \subseteq G$,
  - maps $r, s : G \to X$,
  - a set $G^{(2)} = G\ {}_s\!\times_r G := \{(g,h) \in G \times G : s(g) = r(h)\}$ together with a partially defined product $(g,h) \in G^{(2)} \mapsto gh \in G$, and
  - an inverse operation $g \in G \mapsto g^{-1} \in G$

  with some properties.

Proposition

Suppose E is a directed graph. The set $\mathcal{P}(E^*)$ of partial isomorphisms on E^* is a groupoid with unit space E^0. For $g : vE^* \to wE^*$ in $\mathcal{P}(E^*)$ we define

- *$r(g) = w$ and $s(g) = v$,*
- *if $s(g) = r(h)$, the product $gh : s(h)E^* \to r(g)E^*$ is composition, and*
- *$g^{-1} : r(g)E^* \to s(g)E^*$ is the inverse of g.*

Groupoid actions

Suppose that E is a directed graph and G is a groupoid with unit space E^0.

- An *action* of G on the path space E^* is a (unit-preserving) groupoid homomorphism $\phi : G \to \mathcal{P}(E^*)$.
- The action is *faithful* if ϕ is one-to-one.
- If the homomorphism is fixed, we usually write $g \cdot \mu$ for $\phi_g(\mu)$.
  - This applies in particular when G arises as a subgroupoid of $\mathcal{P}(E^*)$, which is how we will define examples.

Self-similar groupoids

Definition

Suppose E is a directed graph and G is a groupoid with unit space E^0 acting faithfully on E^*. Then (G, E) is a *self-similar groupoid* if, for every $g \in G$ and $e \in s(g)E^1$, there exists $h \in G$ satisfying

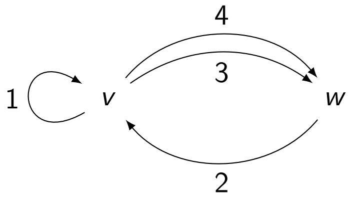
$$g \cdot (e\mu) = (g \cdot e)(h \cdot \mu) \qquad \text{for all } \mu \in s(e)E^*. \tag{2}$$

- Since the action is faithful, there is then exactly one such $h \in G$, and we write $g|_e := h$.
- Now, for $g \in G$ and $\mu \in s(g)E^*$, the analogous definitions to the self-similar group case give us the formula:

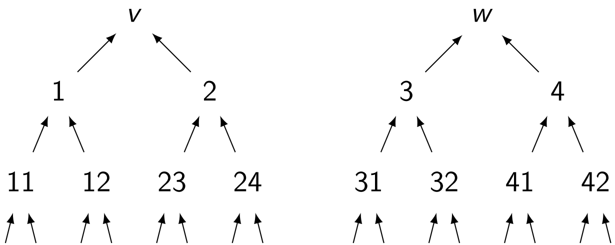
$$g \cdot (\mu\nu) = (g \cdot \mu)(g|_\mu \cdot \nu) \quad \text{for all } \nu \in s(\mu)E^*.$$

Example 1

- Let E be the graph

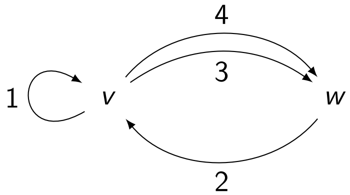


- The path space E^* is



Example 1

- Let E be the graph



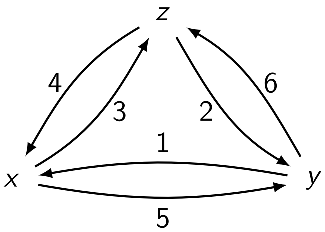
- Define partial isomorphisms $a, b \in \mathcal{P}(E^*)$ recursively by

$$\begin{aligned} a \cdot 1\mu &= 4\mu & b \cdot 3\mu &= 1\mu \\ a \cdot 2\nu &= 3(b \cdot \nu) & b \cdot 4\mu &= 2(a \cdot \mu). \end{aligned} \tag{3}$$

- Let G be the subgroupoid of $\mathcal{P}(E^*)$ generated by A. Then (G, E) is a self-similar groupoid.

Example 2

- Let E be the graph



- Define partial isomorphisms $a, b, c, d, f, g \in \mathcal{P}(E^*)$ recursively by

$$\begin{array}{lll} a \cdot 1\mu = 1(b \cdot \mu) & b \cdot 2\nu = 2\nu & c \cdot 3\lambda = 3(a \cdot \lambda) \\ a \cdot 4\nu = 4(c \cdot \nu) & b \cdot 5\lambda = 5(d \cdot \lambda) & c \cdot 6\mu = 6(b \cdot \mu) \\ d \cdot 1\mu = 4(f \cdot \mu) & f \cdot 2\nu = 6(f^{-1} \cdot \nu) & g \cdot 3\lambda = 5\lambda \\ d \cdot 4\nu = 1(f^{-1} \cdot \nu) & f \cdot 5\lambda = 3\lambda & g \cdot 6\mu = 2(f \cdot \mu) \end{array}$$

- Let G be the subgroupoid of $\mathcal{P}(E^*)$ generated by A. Then (G, E) is a contracting self-similar groupoid

3. C^*-algebras of self-similar groupoids

- R. Exel and E. Pardo, *Self-similar graphs: a unified treatment of Katsura and Nekrashevych C^*-algebras*, to appear in Advances in Math., ArXiv:1409.1107.
- M. Laca, I. Raeburn, J. Ramagge, and M. Whittaker *Equilibrium states on operator algebras associated to self-similar actions of groupoids on graphs*, preprint, ArXiv 1610.00343.
- V. Nekrashevych, *C^*-algebras and self-similar groups*, J. Reine Angew. Math. **630** (2009), 59–123.

C^*-algebras of self-similar groupoids

Proposition

Let E be a finite graph without sources and (G, E) a self-similar groupoid action. There is a Toeplitz algebra $\mathcal{T}(G, E)$ defined by families $\{p_v : v \in E^0\}$, $\{s_e : e \in E^1\}$ and $\{u_g : g \in G\}$ such that

1. *u is a unitary representation of G with $u_v = p_v$ for $v \in E^0$;*
2. *(p, s) is a Toeplitz-Cuntz-Krieger family in $\mathcal{T}(G, E)$, and $\sum_{v \in E^0} p_v$ is an identity for $\mathcal{T}(M)$;*
3. *if $g \in G$ and $e \in E^1$ with $s(g) = r(e)$, then*
$$u_g s_e = s_{g \cdot e} u_{g|_e}$$
4. *if $g \in G$ and $v \in E^0$ with $s(g) = v$, then*
$$u_g p_v = p_{g \cdot v} u_g.$$

C^*-algebras of self-similar groupoids

Proposition

Let (p, s, u) be the universal representation of the Toeplitz algebra $\mathcal{T}(G, E)$. Then

$$\mathcal{T}(G,E) = \overline{\text{span}}\{s_\mu u_g s_\nu^* : \mu, \nu \in E^*, g \in G \text{ and } s(\mu) = g \cdot s(\nu)\}.$$

Proposition

Let (p, s, u) be the universal representation of the Toeplitz algebra $\mathcal{T}(G, E)$. Then the Cuntz-Pimsner algebra $\mathcal{O}(G, E)$ is the quotient of $\mathcal{T}(G, E)$ by the ideal generated by

$$\Big\{p_v - \sum_{\{e \in vE^1\}} s_e s_e^* : v \in E^0\Big\}.$$

The gauge action

- There are natural $\mathbb{R}$-automorphic dynamics on $\mathcal{T}(G,E)$ and $\mathcal{O}(G,E)$ defined by

$$\sigma_t(p_v) = p_v, \quad \sigma_t(s_e) = e^{it}s_e \quad \text{and} \quad \sigma_t(u_g) = u_g$$

- We are interested in (KMS) equilibrium states of the dynamical systems $(\mathcal{T}(G,X),\sigma)$ and of $(\mathcal{O}(G,X),\sigma)$.

4. KMS states on self-similar groupoids

- Z. Afsar, N. Brownlowe, N.S. Larsen, N. Stammeier, *Equilibrium states on right LCM semigroup C^*-algebras*, preprint, ArXiv 1611.01052.
- M. Laca, I. Raeburn, J. Ramagge, and M. Whittaker *Equilibrium states on the Cuntz-Pimsner algebras of self-similar actions*, J. Func. Anal. **266** (2014), 6619–6661.
- M. Laca, I. Raeburn, J. Ramagge, and M. Whittaker *Equilibrium states on operator algebras associated to self-similar actions of groupoids on graphs*, preprint, ArXiv 1610.00343.

The KMS condition

- Suppose $\sigma : \mathbb{R} \to \mathrm{Aut}(A)$ is a strongly continuous action, then there is a dense *-subalgebra of σ-analytic elements: $t \mapsto \sigma_t(a)$ extends to an entire function $z \mapsto \sigma_z(a)$.

- Definition

  The state φ of A satisfies the KMS condition at inverse temperature $\beta \in (0, \infty)$ if

$$\varphi(ab) = \varphi(b\,\sigma_{i\beta}(a))$$

  whenever a and b are analytic for σ.

  Note: it suffices to verify the above for analytic elements that span a dense subalgebra, In our case, the spanning set

$$\{s_\mu u_g s_\nu^* : \ \mu, \nu \in E^*, g \in G \text{ and } s(\mu) = g \cdot s(\nu)\}.$$

KMS states on the Toeplitz algebra

Theorem

Suppose E is a strongly connected finite graph with no sources. Let B be the vertex matrix of E with spectral radius $\rho(B)$.

1. *If $\beta \in [0, \log \rho(B))$, there are no KMS_β states on $(\mathcal{T}(G, E), \sigma)$;*
2. *If $\beta \in (\log \rho(B), \infty)$, there is a homeomorphism between the normalised traces on the groupoid C^*-algebra $C^*(G)$ and the KMS_β states on $(\mathcal{T}(G, E), \sigma)$;*
3. *If $\beta = \log \rho(B)$, the $KMS_{\log \rho(B)}$ states of $(\mathcal{T}(G, E), \sigma)$ arise from KMS states of $(\mathcal{O}(G, E), \sigma)$; and there is at least one such state.*
4. *If the set $\{g|_\mu : \mu \in E^*\}$ is finite for every $g \in G$, then this is the only KMS state of $(\mathcal{O}(G, E), \sigma)$.*

The unique KMS state

- Suppose that E is a finite graph with no sources, that E is strongly connected, and that (G, E) is a self-similar groupoid action such that the set $\{g|_\mu : \mu \in E^*\}$ is finite for every $g \in G$.
- In this situation the vertex matrix B is irreducible, and has a unique unimodular Perron-Frobenius eigenvector $x \in (0, \infty)^{E^0}$.
- For $g \in G$, $v \in E^0$ and $k \geq 0$, define

$$F_g^k(v) := \{\mu \in s(g)E^k v : g \cdot \mu = \mu \text{ and } g|_\mu = \mathrm{id}_v\}, \text{ and}$$
$$c_{g,k} := \rho(B)^{-k} \sum_{v \in E^0} |F_g^k(v)| x_v.$$

- Then for each $g \in G \setminus E^0$, the sequence $\{c_{g,k} : k \in \mathbb{N}\}$ is increasing and converges with limit c_g in $[0, x_{s(g)}]$.

The unique KMS state

Theorem

In the situation from the last slide, the unique $KMS_{\log \rho(B)}$ *state of* $(\mathcal{O}(G,E),\sigma)$ *is given by*

$$\psi(s_\kappa u_g s_\lambda^*) = \begin{cases} \rho(B)^{-|\kappa|} c_g & \text{if } \kappa = \lambda \text{ and } s(g) = r(g) = s(\kappa) \\ 0 & \text{otherwise.} \end{cases}$$

- So we need to compute the values of c_g. This is achieved by evaluating the limit

$$c_g = \lim_{k\to\infty} \rho(B)^{-k} \sum_{v\in E^0} |F_g^k(v)| x_v$$

The Grigorchuk group

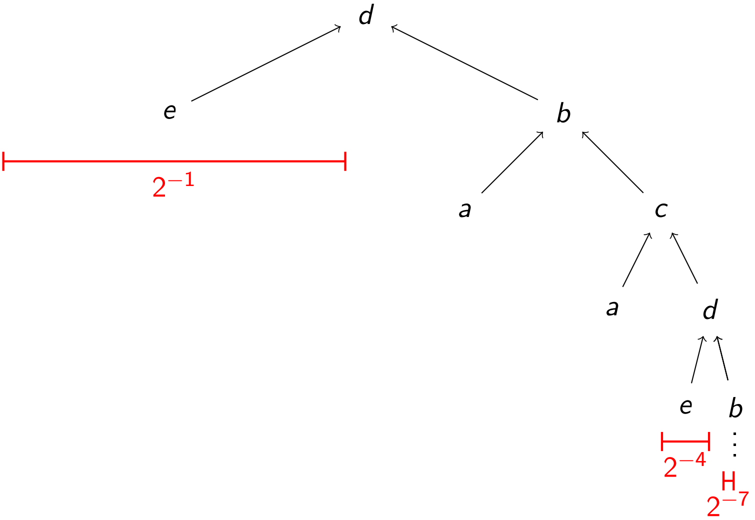
In the case of a self-similar group the graph is a bouquet of loops with a single vertex v. Thus $B = [|X|]$ has spectral radius $\rho(B) = |X|$ with unimodular Perron-Frobenius eigenvector $x_v = 1$.

Proposition

Let (G, X) be the self-similar action of the Grigorchuk group. Then $(\mathcal{O}(G, X), \sigma)$ has a unique $KMS_{\log 2}$ state ψ which is given on generators by

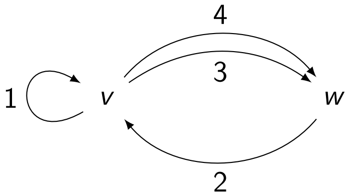
$$\psi_{\log 2}(u_g) = \begin{cases} 1 & \text{for } g = e \\ 0 & \text{for } g = a \\ 1/7 & \text{for } g = b \\ 2/7 & \text{for } g = c \\ 4/7 & \text{for } g = d. \end{cases}$$

Computation of c_d for the Grigorchuk group



$$c_d = \frac{1}{2}\sum_{n=0}^{\infty}\left(\frac{1}{2}\right)^{3n} = \frac{1}{2}\left(\frac{1}{1-\frac{1}{8}}\right) = \frac{4}{7}.$$

Example 1



Recall, (G, E) be the self-similar groupoid defined by:

$$a \cdot 1\mu = 4\mu \qquad b \cdot 3\mu = 1\mu \tag{4}$$
$$a \cdot 2\nu = 3(b \cdot \nu) \qquad b \cdot 4\mu = 2(a \cdot \mu).$$

Proposition

The Cuntz-Pimsner algebra $(\mathcal{O}(G, E), \sigma)$ has a unique $KMS_{\log 2}$ state ψ which is given on generators by

$$\psi(u_g) = \begin{cases} 0 & \text{for } g \in \{a, b, a^{-1}, b^{-1}\}, \\ 1/2 & \text{for } g \in \{v, w\}. \end{cases}$$