The Torelli Group and Representations of Mapping Class Groups

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ABSTRACT

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Let $\mathcal{M}_{g,b,n}$ denote the mapping class group of an orientable surface of genus g with b boundary components and n fixed points. We prove that certain obstructions to the existence of a faithful linear representation do not exist in $\mathcal{M}_{g,b,n}$ for any g, b, and n. We also make explicit the relationship of three known representations of $\mathcal{M}_{g,1,0}$ to each other. In particular, we show how each records the action of mapping class groups on homology and on the winding number of curves on the surface. The action on homology is given by the well known symplectic representation of the mapping class group $\rho : \mathcal{M}_{g,b,n} \to \mathrm{Sp}(2g, \mathbf{Z})$. The kernel of ρ, denoted $\mathcal{I}_{g,b,n}$, is known as the Torelli group. We generalize a construction of Dennis Johnson to find relations amongst Johnson's finite set of generators of $\mathcal{I}_{g,1,0}$ and $\mathcal{I}_{g,0,0}$ and give an alternate technique which yields commutativity relations in these Torelli groups.

Contents

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To my sister Suzanne

1 Introduction

The goal of this thesis is to study certain representations and subgroups of mapping class groups of surfaces. Our investigation has three components. First, we consider the question of whether mapping class groups admit faithful linear representations. We then describe connections between three known representations of certain mapping class groups. Finally, we construct relations among a certain generating set of the Torelli subgroup of the mapping class group.

Let $S_{g,b,n}$ denote an orientable surface of genus g with b boundary components and n punctures. The *mapping class group* of $S_{g,b,n}$, denoted $\mathcal{M}_{g,b,n}$, is defined as the group of all isotopy classes of orientation-preserving homeomorphisms of $S_{g,b,n}$ to itself. In Section 2, which consists of work conducted jointly with Hessam Hamidi-Tehrani, we investigate a natural question which arises in the study of representations of mapping class groups, namely, whether $\mathcal{M}_{g,b,n}$ is *linear*, i.e., admits a faithful representation into $\mathrm{GL}_n(\mathcal{K})$ for some field $\mathcal{K}$. Mapping class groups are closely related to lattices, which are of course linear, and also to braid groups, which were recently shown to be linear [2],[30]. Much work has been done to try to generalize the methods used to demonstrate linearity of braid groups to mapping class groups, but with very limited success. On the other hand, mapping class groups are also closely related to automorphism groups of free groups of rank n, denoted $\mathrm{Aut}(F_n)$. Formanek and Procesi have demonstrated that $\mathrm{Aut}(F_n)$ is not linear if $n \geq 3$ [14]. Hence Hamidi-Tehrani and I took the opposite approach to the linearity question for mapping class groups. Formanek and Procesi's technique is to construct nonlinear groups of a special form, which we call *FP-groups*. They build these groups out of two elements of $\mathrm{Aut}(F_n)$ which act in a particular way on three elements of F_n. We call the group generated by two such maps a *poison group*. Hamidi-Tehrani and I hoped to find analogous poison subgroups in $\mathcal{M}_{g,0,1}$, which acts naturally on $\pi_1(S_{g,0,1})$ and then to mimic Formanek and Procesi's construction of FP-groups in $\mathcal{M}_{g,0,1}$. We instead prove the following surprising result [7].

Theorem *No poison subgroups embed in* $\mathcal{M}_{g,0,1}$.

We further prove a much more general result.

Theorem *No FP-groups of any kind embed in* $\mathcal{M}_{g,b,n}$ *for any* g, b, n.

In other words, not only does the particular construction of Formanek and Procesi fail in the case of mapping class groups, but a more general obstruction to linearity does not exist in any mapping class groups. This gives very strong evidence that mapping class groups may in fact be linear.

We next turn our attention to known representations of mapping class groups. In Section 3, we describe three representations of mapping class groups which arise in very different contexts yet each carry much of the same geometric information. Section 3 is largely expository, and seeks to fill what seems to be a gap in the literature by clarifying some connections between the three representations.

The group $\mathcal{M}_{g,b,n}$ acts naturally on $H = H_1(S_{g,b,n})$, giving rise to what is known as the symplectic representation of the mapping class group.

$$\rho : \mathcal{M}_{g,b,n} \to \mathrm{Sp}(2g, \mathbf{Z}).$$

The kernel of this representation is known as the *Torelli group*, denoted $\mathcal{I}_{g,b,n}$. In [39], Morita constructs representations of $\mathcal{M}_{g,1,0}$ using an extension of Johnson's "torsion" homomorphism $\tau : \mathcal{I}_{g,1,0} \to \Lambda^3 H$ described in [20]. The map τ enables us to determine the action of mapping class groups on the winding number of curves on a surface relative to some non-vanishing vector field. Morita's representation also contains all the information of the symplectic representation ρ.

Trapp [46] (and independently Sipe [44]) gives a linear form of Morita's representation interpreted explicitly in terms of the action of $\mathcal{M}_{g,1,0}$ on winding numbers and on homology. Perron also linearizes Morita's representation, instead building a representation $\psi : \mathcal{M}_{g,1,0} \to \mathrm{GL}_{4g}(\mathbf{Z}[d_1, \ldots, d_{2g}])$ by extending a representation of a certain Artin group [41]. We present a method for extracting the same winding number and homology information directly from Perron's representation.

The Torelli group plays a prominent role in the study of representations of mapping class groups. Hence we focus on this fascinating and poorly understood subgroup of $\mathcal{M}_{g,b,n}$.

Surprisingly little is known about the structure of the Torelli group, but the first serious progress in this direction was made by Dennis Johnson, who wrote a wonderful series of papers on the Torelli group ([20], [21], [22], [24], [25], and [26], all summarized nicely in [23]). One of Johnson's most important results is that both $\mathcal{I}_{g,1,0}$ and $\mathcal{I}_{g,0,0}$ are finitely generated for $g \geq 3$ [22]. (Mess later showed that $\mathcal{I}_{2,b,n}$ is infinitely generated [36].) Johnson also discovered an important surjective map $\tau : \mathcal{I}_{g,1,0} \to \Lambda^3 H$, giving the first nice abelian quotient of the Torelli group.

One important question which remains open, however, is the question of whether the Torelli group admits a finite presentation. It is known that $\mathcal{M}_{g,b,n}$ is finitely presentable (see [16], also [48], [49], [31]). As one approach to this question, we ask what relations can be found amongst Johnson's generators, which remain the only known finite set of generators of the Torelli group. This is no easy task; the order of Johnson's generating set for the Torelli group is exponential in the genus g. As a starting point, we have a technique developed by Johnson which he used to find two families of relations amongst various elements of the Torelli group, including some which are not in his generating set. Johnson uses so-called "lantern relations" in the full mapping class group to obtain these relations in the Torelli group.

We first show how to generalize one of Johnson's families of generators so as to relate only elements from his generating set in a way that yields on the order of g^3 relations for genus $g \geq 4$, using his same technique. The relation is as follows.

Generalized B-Relation *In $\mathcal{I}_{g,1,0}$, for $g \geq 4$ and for $2 \leq l < k \leq g-1$, we have*

$$[W_k^{-1} * (P_g P_l^{-1})][W_2^{-1} * P_k] = [W_g^{-1} * (P_k P_l^{-1})][W_2^{-1} * P_g].$$

In the above relation, the W_i are a certain type of Johnson generator and the P_j are products of two Johnson generators, which also happen to be commutators in the full mapping class group. We obtain from the construction of the relation the following corollary, which Johnson has already proved for genus 3 [22]).

Corollary 1.1 *There are $g-2$ extraneous generators of $\mathcal{I}_{g,1,0}$ in Johnson's set.*

We then give an alternate construction, which also arises from lantern relations but avoids some of the difficulties of Johnson's original technique. This second method yields a new kind of relation, in fact, a commutativity relation.

General Commutator Relation *In $\mathcal{I}_{g,1,0}$, for $g \geq 4$, we have the following relation:*

$$[(B_1^{-1} A_1 B_3), (A_2 B_2^{-1})] = 1.$$

Each curve A_i or B_j is a type of Johnson generator which will be described in detail in Section 4. Taken together, the A_i and B_j satisfy a certain intersection pattern. There are on the order of g^5 of these commutativity relations for $g \geq 4$. We remark that each relation given here, both B-relations and commutator relations, actually represent many more conjugate relations. For example, the simplest case of the generalized B-relation actually yields 33 distinct relations amongst Johnson generators (see Appendix B).

We will also briefly discuss a certain symmetry satisfied by the vast majority of the pairs of Dehn twist curves appearing in the Johnson generating set and some potential applications to the linearity question for the Torelli group. We then present a list of questions, including many raised by this investigation, intended to outline a plan for future study of the Torelli group.

It is worth elaborating at this point on the earlier claim regarding the importance of the role played by the Torelli group in the study of representations of $\mathcal{M}_{g,b,n}$. It comes as no surprise that the Torelli group plays a key role in any representation containing symplectic information. For example, the map τ (to be precise, the contraction of τ) turns out to be important in the Trapp, Morita, and Perron representations of the full mapping class group, as discussed in Section 3.

More surprising is the fact that the Torelli group appears in the study of other representations which arise in ostensibly very different contexts. For example, Kasahara recently showed that Johnson's homomorphism factors through the Jones representation of $\mathcal{M}_2$ restricted to $\mathcal{I}_2$ [27]. In addition, the representations of $\mathcal{M}_g$ arising from topological quantum field theories (TQFTs) also connect with Johnson's work in an interesting way. The TQFT representations of Reshetikhin-Turaev are indexed by a integer parameter r. Wright has calculated these representations explicitly for $r = 4$ and found that the restriction of the representation in this case to $\mathcal{I}_g$ is precisely the sum of the Birman-Craggs homomorphisms from $\mathcal{I}_g$ to $\mathbf{Z}/2\mathbf{Z}$ [51]. Johnson shows in [21] that the sum of the Birman-Craggs homomorphisms is related to his map τ, though neither factors through the other, giving a possibly interesting connection to the representations discussed above.

2 On the Linearity Problem for Mapping Class Groups

In this section we seek to provide some insight into the question of whether mapping class groups are linear. Mapping class groups are often compared with both arithmetic and automorphism groups, and in many ways the three groups are similar and support analogous theories. (There is a nice discussion of this by Karen Vogtmann in [47]. Another good survey of this subject was recently given by Martin Bridson in a series of lectures at Columbia University.) The property of linearity, however, is an area in which these groups differ. Lattices, of course, are linear, but Formanek and Procesi showed in [14] that $\mathrm{Aut}(F_n)$ is not linear for $n \geq 3$ (it follows that $\mathrm{Out}(F_n)$ is also not linear for $n \geq 4$), leading one to ask on which side mapping class groups should fall.

The work in this section was conducted jointly with Hessam Hamidi-Tehrani. We present it here as it appeared in [7], with reference numbers of sections and theorems appropriately altered.

2.1 Introduction

The question of whether mapping class groups are linear has been around for some time. The recent work of Bigelow [2] and also Krammer [30] in determining that the braid group is linear has renewed interest in the subject, due to the close relationship between mapping class groups and braid groups. Let $S_{g,b,n}$ denote a surface of genus g with b boundary components and n fixed points. Let $\mathcal{M}_{g,b,n}$ denote the mapping class group of $S_{g,b,n}$. We assume throughout that maps fix boundary components pointwise. Bigelow and Budney [3] and independently Korkmaz [29] recently determined that $\mathcal{M}_{2,0,0}$ is linear. Korkmaz also showed in [29] that mapping class groups contain very large linear subgroups, namely, the hyperelliptic subgroups. However, the question of linearity remains open for mapping class groups of surfaces of genus 3 or greater.

Let F_n denote the free group of rank n. It is well known that $\mathrm{Out}(F_2)$ and $\mathrm{Aut}(F_2)$ are linear. The former fact is due to Nielsen [40], and the latter follows by [12] from the linearity of the 4-string braid group B_4, which is due to Krammer [30].

On the other hand, Formanek and Procesi demonstrated in [14] that $\mathrm{Aut}(F_n)$ is not a linear group for $n \geq 3$. A simple corollary of this result is that $\mathrm{Out}(F_n)$ is not linear for $n \geq 4$. The well-known fact due to Nielsen [33] that $\mathcal{M}_{g,0,0}$ is isomorphic to $\mathrm{Out}(\pi_1(S_{g,0,0}))$ suggests that it may be possible to apply the methods of Formanek and Procesi to mapping class groups, though it may not be immediately clear how to do so.

Formanek and Procesi define a class of nonlinear groups, which we will generalize slightly and refer to as *Formanek and Procesi groups*, or *FP-groups* for short. We will show that the existence of FP-subgroups of $\mathcal{M}_{g,0,1}$ would imply that $\mathcal{M}_{g+k,0,0}$ is not linear for $k \geq 1$. We will also focus our attention on a special kind of automorphism group, which we call a *poison group*. We will describe the particular method of Formanek and Procesi for constructing FP-groups from poison subgroups.

This work originated in an attempt to use the methods of Formanek and Procesi to show that $\mathcal{M}_{g,0,0}$ is not linear for $g \geq 3$. We prove instead that the essential building blocks of the Formanek and Procesi method do not exist in mapping class groups, first in a special case.

Theorem A *Poison subgroups cannot be embedded in* $\mathcal{M}_{g,0,1}$.

Thus the particular technique of Formanek and Procesi fails to show that certain mapping class groups are not linear. We then generalize this result as follows.

Theorem B *FP-groups do not embed in* $\mathcal{M}_{g,b,n}$ *for any* g, b, *and* n.

Our paper is organized as follows. In Section 2.2, we give an overview of the methods of Formanek and Procesi for constructing a nonlinear subgroup of $\mathrm{Aut}(F_n)$ from a poison subgroup. In Section 2.3, we establish connections between certain mapping class groups and the automorphism group of a closed surface. In Section 2.4 we prove Theorem A. In Section 2.5 we prove Theorem B using very different techniques from those used in Section 2.4. Though Theorem A is a special case of Theorem B, we include a separate proof of Theorem A both for the sake of highlighting the particular construction of Formanek and Procesi and also because the methods used are interesting in their own right. The reader should note, however, that Sections 2.3, 2.4, and 2.5 are completely independent of one another. For example, the reader interested only in Theorem B could read Sections 2.1, 2.2, and 2.5 without any loss of continuity.

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2.2 The Method of Formanek and Procesi

Let G be any group, and let $\mathcal{H}(G)$ denote the following HNN-extension of $G \times G$:

$$\mathcal{H}(G) = \langle G \times G, t \mid t(g,g)t^{-1} = (1,g), g \in G \rangle.$$

In other words, conjugation by t in the HNN-extension carries the diagonal subgroup $G \times G$ onto its second factor. Formanek and Procesi show in the following theorem that such groups exhibit special behavior under a linear representation.

Theorem 2.1 (Formanek and Procesi, [14]) *Let G be a group. Then the image of the subgroup $G \times \{1\}$ under any linear representation of $\mathcal{H}(G)$ is nilpotent-by-abelian-by-finite.*

Corollary 2.2 *Let G be a group, and K a normal subgroup of $\mathcal{H}(G)$ such that the image of $G \times \{1\}$ in $\mathcal{H}(G)/K$ is not nilpotent-by-abelian-by-finite. Then $\mathcal{H}(G)/K$ is not linear.*

Proof. Let $\rho : \mathcal{H}(G)/K \to GL_N(k)$ be a linear representation where k is a field. Let $\pi : \mathcal{H}(G) \to \mathcal{H}(G)/K$ be the natural projection map. Then $\rho \circ \pi$ is a linear representation of $\mathcal{H}(G)$ and hence by Theorem 2.1, $\rho(\pi(G \times \{1\}))$ is nilpotent-by-abelian-by-finite. Thus ρ is not faithful. $\square$

We will call a group of the type described in Corollary 2.2 a *Formanek and Procesi group*, or *FP-group* for short. We now describe the particular construction of Formanek and Procesi in demonstrating the nonlinearity of $\mathrm{Aut}(F_n)$ for $n \geq 3$.

Let G be any group. Let x_1, x_2, x_3 be elements of G such that $\langle x_1, x_2, x_3 \rangle \cong F_3$. Let $\phi_1, \phi_2 \in \mathrm{Aut}(G)$ be two maps such that

1. $\phi_i(x_j) = x_j, \quad i, j = 1, 2, \quad$ and
2. $\phi_i(x_3) = x_3 x_i, \quad i = 1, 2.$

We will call the subgroup $\langle \phi_1, \phi_2 \rangle$ a *poison subgroup* of $\mathrm{Aut}(G)$. We can define poison subgroups of the mapping class group $\mathcal{M}_{g,0,1}$ analogously, since in this case the mapping class group acts on $\pi_1(S_{g,0,1})$. Notice that the second condition implies that $\langle \phi_1, \phi_2 \rangle \cong F_2$. Thus poison groups, being isomorphic to the linear group F_2, are not themselves a kind of FP-group. However, as the following lemma shows, their existence in an automorphism group $\mathrm{Aut}(G)$ implies that $\mathrm{Aut}(G)$ is not linear (hence the name "poison groups", though it suggests a bias towards linearity).

Lemma 2.3 *Let G be any group. If Aut(G) contains a poison subgroup, then it contains an FP-subgroup isomorphic to $\mathcal{H}(F_2)$.*

Proof. Let $\langle \phi_1, \phi_2 \rangle$ be a poison subgroup in $\mathrm{Aut}(G)$. Following Formanek and Procesi's argument in [14], let $\alpha_i \in \mathrm{Aut}(G)$ denote conjugation by x_i. Consider the group

$$H = \langle \phi_1, \phi_2, \alpha_1, \alpha_2, \alpha_3 \rangle.$$

First, note that $\langle \alpha_1, \alpha_2, \alpha_3 \rangle$ is a normal subgroup of H since both ϕ_1 and ϕ_2 preserve the subgroup $\langle x_1, x_2, x_3 \rangle$. Now let $w(a, b)$ denote any non-trivial reduced word in the free group on the letters a and b. By definition of a poison subgroup, we know that $w(\phi_1, \phi_2)(x_i) = x_i$ for i = 1,2. This tells us that if $w(\phi_1, \phi_2)$ is in $\langle \alpha_1, \alpha_2, \alpha_3 \rangle$, then $w(\phi_1, \phi_2)$ must induce conjugation by an element in $\langle x_1, x_2, x_3 \rangle \cong F_3$, which commutes with x_1 and x_2. But the only such element is the identity. Hence $w(\phi_1, \phi_2)$ must be the identity map. But we know this is not the case since

$$w(\phi_1, \phi_2)(x_3) = x_3 w(x_1, x_2). \tag{1}$$

This tells us that the images of ϕ_1 and ϕ_2 mod $\langle \alpha_1, \alpha_2, \alpha_3 \rangle$ will generate a free group. Clearly, the images of ϕ_1 and ϕ_2 also generate the quotient of H by $\langle \alpha_1, \alpha_2, \alpha_3 \rangle$, and so we have a split exact sequence

$$1 \to \langle \alpha_1, \alpha_2, \alpha_3 \rangle \to H \to \langle \phi_1, \phi_2 \rangle \to 1. \tag{2}$$

Thus the only relations we have in a presentation for H are given by conjugation, as follows:

$$H = \langle \phi_1, \phi_2, \alpha_1, \alpha_2, \alpha_3 \ \mid \ \phi_i \alpha_j \phi_i^{-1} = \alpha_j, \ \ \phi_i \alpha_3 \phi_i^{-1} = \alpha_3 \alpha_i, \ \ i, j = 1, 2 \rangle. \tag{3}$$

Rewriting the second set of relations, we obtain $\alpha_3(\alpha_i \phi_i)\alpha_3^{-1} = \phi_i, \quad i = 1, 2$. Since $\langle \phi_1, \phi_2 \rangle \cong \langle \alpha_1, \alpha_2 \rangle \cong F_2$, we have that $H \cong \mathcal{H}(F_2)$, with α_3 playing the role of the element t. Since F_2 is not nilpotent-by-abelian-by-finite, $\mathcal{H}(F_2)$ is an FP-group. $\square$

2.3 The Connection with Mapping Class Groups

Our motivation for the work in this paper is the following observation, the proof of which we defer to the end of the section.

Claim 2.4 *If a poison subgroup exists in* $\mathcal{M}_{g,0,1}$ *for* $g \geq 2$, *then the groups* $\mathcal{M}_{g+k,0,0}$ *are not linear for* $k \geq 1$.

We have been abusing terminology a bit by talking about poison subgroups in $\mathcal{M}_{g,0,1}$ and also in the context of automorphism groups. The distinction between the two contexts is unnecessary for our purposes, as the following lemma shows, since these mapping class groups are isomorphic to automorphism groups.

Lemma 2.5 $\mathcal{M}_{g,0,1} \cong Aut(\pi_1(S_{g,0,0}))$, *for* $g \geq 2$.

Proof. We begin with the exact sequence

$$1 \to \mathrm{Inn}(\pi_1(S_{g,0,0})) \to \mathrm{Aut}(\pi_1(S_{g,0,0})) \to \mathrm{Out}(\pi_1(S_{g,0,0})) \to 1.$$

By the well-known theorem of Nielsen [33], we have that $\mathrm{Out}(\pi_1(S_{g,0,0})) \cong \mathcal{M}_{g,0,0}$. In addition, since $\pi_1(S_{g,0,0})$ is centerless, we can replace $\mathrm{Inn}(\pi_1(S_{g,0,0}))$ with $\pi_1(S_{g,0,0})$ (see, for example, [8]) to obtain

$$1 \to \pi_1(S_{g,0,0}) \to \mathrm{Aut}(\pi_1(S_{g,0,0})) \to \mathcal{M}_{g,0,0} \to 1. \tag{4}$$

By [4], we also have the following exact sequence:

$$1 \to \pi_1(S_{g,0,0}) \to \mathcal{M}_{g,0,1} \to \mathcal{M}_{g,0,0} \to 1. \tag{5}$$

Every short exact sequence $1 \to N \to E \to G \to 1$ induces a homomorphism $G \to \mathrm{Out}(N)$, defined as follows. Let $g \in G$, and let e_g be a lift of $g \in E$. Now, E acts on N by conjugation, hence we can think of e_g as an element of $\mathrm{Aut}(N)$. However, since N is not necessarily abelian, this map is only well defined up to conjugation by an element of N. Thus we get a map $G \to \mathrm{Out}(N)$. According to Corollary 6.8

of [8], given any short exact sequence as above, with N centerless, there is a unique "middle group" E corresponding to any given homomorphism $G \to \mathrm{Out}(N)$.

In Sequence 4 above, it is clear that the map induced is the Nielsen isomorphism between $\mathcal{M}_{g,0,0}$ and $\mathrm{Out}(\pi_1(S_{g,0,0}))$. In Sequence 5, as discussed in [4], the image of a generator a of $\pi_1(S_{g,0,0})$ is the so-called "spin map" associated to each curve, which induces conjugation by that curve, but can be more easily understood as a product of opposite Dehn twists about the boundary of an annular neighborhood of the curve a. In other words, if α and β are the two boundary curves, then the spin map associated to the curve a can be written as $T_\alpha T_\beta^{-1}$, where T_γ denotes the Dehn twist about the curve γ. Let $\phi \in \mathcal{M}_{g,0,0}$, and let $\tilde{\phi}$ denote a lift of ϕ in $\mathcal{M}_{g,0,1}$. Then $\tilde{\phi} T_\alpha T_\beta^{-1} \tilde{\phi}^{-1} = T_{\tilde{\phi}(\alpha)} T_{\tilde{\phi}(\beta)}^{-1}$, which is precisely the spin map associated to $\tilde{\phi}(a)$. Thus, we are simply looking at the action of $\tilde{\phi}$ on $\pi_1(S_{g,0,0})$, but since ϕ does not necessarily fix the basepoint, ϕ is getting mapped to the class of $\tilde{\phi}$ in Aut, modulo inner automorphisms. In other words, the induced map from $\mathcal{M}_{g,0,0} \to \mathrm{Out}(\pi_1(S_{g,0,0}))$ is also the Nielsen isomorphism. Now since $\pi_1(S_{g,0,0})$ has a trivial center, we apply Corollary 6.8 of [8], and the lemma is proved. $\square$

Remark 2.6 The isomorphism given in Lemma 2.5 has received some attention in the literature, though perhaps not as much as it deserves. The map itself is the obvious one, namely, any homeomorphism of a surface with one fixed point induces a natural automorphism of the fundamental group of the closed surface with the fixed point taken as base point. From the geometric point of view, it is not immediately clear that this map from $\mathcal{M}_{g,0,1}$ to $\mathrm{Aut}(\pi_1(S_{g,0,0}))$ should be a surjection, i.e., it is not necessarily obvious that all elements of $\mathrm{Aut}(\pi_1(S_{g,0,0}))$ should be topologically induced.

Lemma 2.7 *If $Aut(\pi_1(S_{g,0,0}))$ is not linear, then $\mathcal{M}_{g,1,0}$ is not linear.*

Before proving the lemma, we make a few observations. From Chapter 4, Section 1 of [4] and Lemma 2.5 we have the short exact sequence

$$1 \to \mathbf{Z} \to \mathcal{M}_{g,1,0} \to \mathrm{Aut}(\pi_1(S_{g,0,0})) \to 1. \tag{6}$$

We note that $\mathbf{Z}$ is actually the center of $\mathcal{M}_{g,1,0}$, generated by a Dehn twist about the boundary curve. Now $\mathrm{Aut}(\pi_1(S_{g,0,0}))$ is the quotient of $\mathcal{M}_{g,1,0}$ by $\mathbf{Z}$. In general, the quotient of a linear group is not necessarily linear, but the extra information we have about the kernel in this case will allow us to draw the desired conclusion. The following two theorems are proved in [50]. Note that the term "closed" refers to the Zariski topology.

Theorem 2.8 *Let G be a linear group and H a closed normal subgroup of G. Then G/H is also linear.*

Theorem 2.9 *The centralizer of any subset of a linear group is closed.*

Proof of Lemma 2.7 Since $\mathbf{Z}$ is the center of $\mathcal{M}_{g,1,0}$, it is normal and also closed by the above. Thus we can apply Theorem 2.8 to the surjection given in Sequence 6, and Lemma 2.7 follows directly. $\square$

We are now ready to prove the claim.

Proof of Claim 2.4 Suppose that $\mathcal{M}_{g,0,1}$ contains a poison subgroup. Then by the isomorphism of Lemma 2.5, $\mathrm{Aut}(\pi_1(S_{g,0,0}))$ also contains a poison subgroup. Then $\mathrm{Aut}(\pi_1(S_{g,0,0}))$ is not linear by Lemma 2.3. Now by Lemma 2.7, $\mathcal{M}_{g,1,0}$ is also not linear. The claim follows from the fact that $\mathcal{M}_{g,1,0}$ is a subgroup of $\mathcal{M}_{g+k,0,0}$, for $k \geq 1$. Although this fact is well-known, for the sake of completeness we include a proof as follows. Consider $S_{g,1,0}$ as a subsurface of $S_{g+k,0,0}$. Let h be the homomorphism from $\mathcal{M}_{g,1,0}$ to $\mathcal{M}_{g+k,0,0}$ defined by extension to the identity on $S_{g+k,0,0} \setminus S_{g,1,0}$. Let $f \in \ker(h)$ such that $f \neq id$. The mapping class $h(f)$ of $S_{g+k,0,0}$ keeps the subsurface $S_{g,1,0}$ invariant up to isotopy. According to Section 7.5 in [19], $h(f)$ induces a well defined mapping class in $\pi_0(\mathrm{Diff}(S_{g,1,0}))$ (the group of homeomorphisms of $S_{g,1,0}$ up to isotopy not necessarily fixing $\partial S_{g,1,0}$). But since $h(f) = id$ and by the definition of h, this implies that f induces the identity in $\pi_0(\mathrm{Diff}(S_{g,1,0}))$, which implies that f could only be a non-trivial power of a Dehn twist in the $\partial S_{g,1,0}$. Then by definition, $h(f)$ will also be a non-trivial power of a Dehn twist, which is a contradiction. $\square$

Remark 2.10 We have defined poison subgroups in the context of $\mathcal{M}_{g,0,1}$ and also in the context of automorphism groups, but the definition also makes sense in the context of any group action on another group. Thus one could use this as a general approach to the linearity question for any such group.

2.4 Poison Subgroups Cannot Be Embedded in $\mathcal{M}_{g,0,1}$

Our strategy for proving this result will be to decompose the surface $S = S_{g,0,0}$ into subsurfaces in a particular way. We then use the machinery of graphs of groups (described in detail in [1]) to analyze the action of the generators of a poison subgroup of $\mathcal{M}_{g,0,1}$ on the elements $x_1, x_2, x_3 \in \pi_1(S)$. After completion of the proof of Theorem A, we discovered that similar methods involving graphs of groups and normal forms were used by Levitt and Vogtmann in [32] to give an algorithm for the Whitehead problem for surface groups. There is a major difference, however, in that we are not given the curves x_1, x_2, and x_3, and hence we cannot apply their algorithm directly, nor would our proof be significantly shortened by direct reference to their results. Thus we have kept the proof of Theorem A in its original form for the sake of self-containment. We have, however, found it useful to adopt their methods for the decomposition of the surface S.

Throughout this section assume that $g \geq 2$, since Theorem A is clear when $g \leq 1$. Fix a point $* \in S$, and identify $S_{g,0,1}$ with $(S, *)$. We use the point $*$ as the base point for the fundamental group of S. Let $\langle \phi_1, \phi_2 \rangle$ be a poison subgroup in $\mathcal{M}_{g,0,1}$. Then there are elements $x_1, x_2, x_3 \in \pi_1(S, *)$ such that $\langle x_1, x_2, x_3 \rangle \cong F_3$ and

1. $\phi_i(x_j) = x_j \quad i, j = 1, 2$, and
2. $\phi_i(x_3) = x_3 x_i \quad i = 1, 2$.

In what follows, we will choose appropriate representatives for ϕ_i and x_j (denoted by the same names by abuse of notation) such that, among other things, a power of ϕ_i fixes a regular neighborhood of x_j pointwise. To this end our main tool will be the following result of Hass and Scott [15]. For $y_1, y_2 \in \pi_1(S, *)$, let

$$\mathrm{Stab}(y_1, y_2) = \{\phi \in \mathcal{M}_{g,0,1} \mid \phi(y_i) = y_i,\ i = 1, 2\}.$$

Lemma 2.11 *Let y_1, y_2 be distinct elements of $\pi_1(S, *)$, which are not proper powers. Then there exists a representative of y_i (denoted by $\tilde{y}_i$) and a subsurface $\mathcal{A}$ formed by a regular neighborhood N of $\tilde{y}_1 \cup \tilde{y}_2$ together with all disk components of $S \setminus N$, such that, for any $\phi \in$ Stab(y_1, y_2), ϕ has a representative homeomorphism $\tilde{\phi}$ such that $\tilde{\phi}(\mathcal{A}) = \mathcal{A}$.*

This lemma follows from Theorem 2.1 in [15] together with the discussion in the beginning of page 32 in the same paper. For further details see Section 2.1 in [32].

Remark 2.12 Notice that, in Lemma 2.11, if $\phi \in \mathrm{Stab}(y_1, y_2)$, the map ϕ induces a unique mapping class in $\pi_0(\mathrm{Diff}(\mathcal{A}, *))$ (see Section 7.5 in [19]).

Since it is possible that x_1 and x_2 are proper powers, we need the following well-known lemma, adapted from [32].

Lemma 2.13 *Given a nontrivial element $x \in \pi_1(S, *)$, there exists a unique $y \in \pi_1(S, *)$ and a unique $t \geq 1$ such that y is not a proper power and $x = y^t$.*

Proof. A proof is given in [32] (Lemma 2.3). Though we will not give details, we note that it is also possible to prove this lemma by elementary hyperbolic geometry, using the discrete action of $\pi_1(S, *)$ on the upper half plane by hyperbolic isometries. $\square$

Corollary 2.14 *Let $z_1, z_2 \in \pi_1(S, *)$ be such that $z_1^N = z_2^N$ for some $N \geq 1$. Then $z_1 = z_2$.*

Proof. Using Lemma 2.13 let $y_i^{t_i} = z_i$ such that y_i is not a proper power and $t_i \geq 1$, for $i = 1, 2$. Let $x = y_1^{t_1 N} = y_2^{t_2 N}$. By the uniqueness guaranteed by Lemma 2.13, we have $y_1 = y_2$ and $t_1 N = t_2 N$. Hence $z_1 = z_2$, as desired. $\square$

Using Lemma 2.13, we can choose elements y_i which are not proper powers and $t_i \geq 1$ such that $x_i = y_i^{t_i}$ for $i = 1, 2$. Then we know that $\phi_i(y_j^{t_j}) = y_j^{t_j}$, which implies that $\phi_i(y_j) = y_j$, by Corollary 2.14. Notice that y_1 and y_2 are distinct since $\langle x_1, x_2 \rangle \cong F_2$. We choose $\tilde{y}_i$ and $\mathcal{A}$ according to Lemma 2.11. Let $\pi_0(\mathrm{Diff}(S, \mathcal{A}))$ be the subgroup of $\mathcal{M}_{g,0,1}$ consisting of mapping classes which have a representative keeping $\mathcal{A}$ fixed pointwise. We now adapt Lemma 3.1 of [32] to our purposes, and repeat their argument nearly verbatim.

Lemma 2.15 *The subgroup $\pi_0(\mathit{Diff}(S, \mathcal{A}))$ has finite index in $\mathit{Stab}(y_1, y_2)$.*

Proof. First note that $\mathcal{A}$ is not an annulus, since x_1 and x_2 generate a free group. Using Lemma 2.11 (and noting Remark 2.12), we can define a map ρ from $\mathrm{Stab}(y_1, y_2)$ to $\pi_0(\mathrm{Diff}(\mathcal{A}, *))$. Now we claim that the image of ρ is finite. To see this, let k be any positive integer. Let T_k denote the set of homotopy classes of simple closed curves in $\mathcal{A}$ whose intersection number with y_1 and y_2 is at most k. Then T_k is finite, since $\mathcal{A} \setminus (\tilde{y}_1 \cup \tilde{y}_2)$ is composed entirely of disks and annuli. Any map $\phi \in \mathrm{Stab}(y_1, y_2)$ will preserve the intersection number of a curve with y_1 and y_2, and hence $\mathrm{Stab}(y_1, y_2)$ acts on the set T_k. Now choose a finite set W of simple closed curves in $\mathcal{A}$ whose image completely determines an element of $\pi_0(\mathrm{Diff}(\mathcal{A}, *))$. Let k be bigger than the intersection number of any element in W with y_1 and y_2. Thus the class of ϕ restricted to $\mathcal{A}$ in $\pi_0(\mathrm{Diff}(\mathcal{A},*))$ is completely determined by the action of ϕ on T_k. But the set of permutations of T_k is finite, and hence the image of $\mathrm{Stab}(y_1, y_2)$ under ρ is finite.

Now let $\iota : \pi_0(\mathrm{Diff}(\mathcal{A},*)) \to \mathrm{Out}(\pi_1(\mathcal{A},*))$ be the natural homomorphism. The image of $\iota \circ \rho$ is also finite by the above argument. Now any element $\phi \in \ker(\iota \circ \rho)$ induces an inner automorphism on $\pi_1(\mathcal{A},*)$, i.e., $\phi(z) = czc^{-1}$. The element c has to commute with both y_1 and y_2, which implies that c has to be a power of both y_1 and y_2 since the centralizer of an element in a surface group is cyclic (this is an exercise in elementary hyperbolic geometry), and y_1 and y_2 are not proper powers. But this implies that $c = 1$ since x_1 and x_2 generate a free group. Hence ϕ induces the identity on $\pi_1(\mathcal{A},*)$. Picking a set of simple generators for $\pi_1(\mathcal{A},*)$, one can use an isotopy of the surface to make sure that ϕ keeps them fixed pointwise, by [13]. Then one can further isotope ϕ to make sure ϕ keeps $\mathcal{A}$ invariant pointwise by Alexander's lemma [43]. Hence $\ker(\iota \circ \rho)$ is contained in $\pi_0(\mathrm{Diff}(S,\mathcal{A}))$, which proves the lemma. □

Proposition 2.16 *There exists an integer M such that ϕ_i^M fixes $\mathcal{A}$ pointwise (up to isotopy).*

Proof. We know $\phi_i \in \mathrm{Stab}(y_1, y_2)$ for $i = 1, 2$. Hence by Lemma 2.15, there is an integer $M_i \geq 0$ such that $\phi_i^{M_i} \in \pi_0(\mathrm{Diff}(S,\mathcal{A}))$. Letting $M = \mathrm{LCM}(M_1, M_2)$, we have $\phi_i^M \in \pi_0(\mathrm{Diff}(S,\mathcal{A}))$ for $i = 1, 2$. □

From this point on, we assume that we are working with the particular representative of ϕ_i^M which fixes $\mathcal{A}$ pointwise.

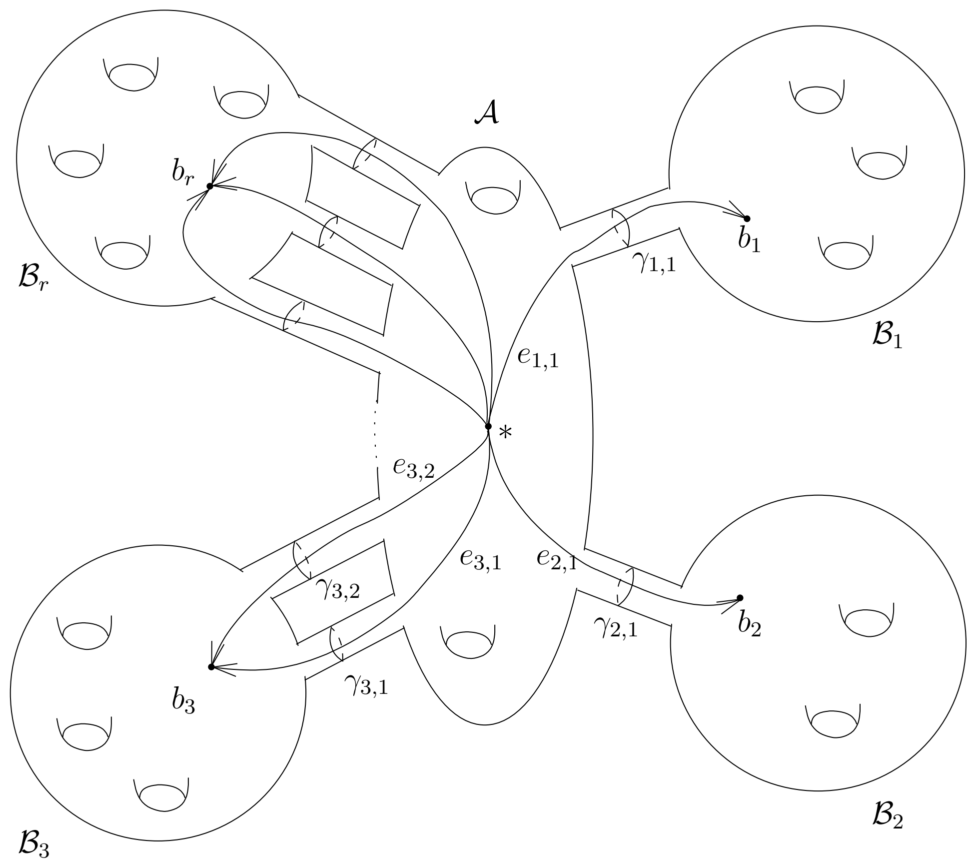


Figure 1: The decomposition of the surface S.

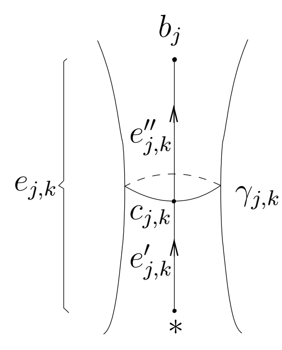


Figure 2: The subarcs of $e_{j,k}$.

Let $\mathcal{B}_1, \cdots, \mathcal{B}_r$ be the respective closures of each component of $S \setminus \mathcal{A}$. Each component is $\mathcal{B}_j$ attached to $\mathcal{A}$ along one or more circles. Hence $\mathcal{A} \cap \mathcal{B}_j$ consists of $n_j \geq 1$ circles, which we denote by $\gamma_{j,1}, \cdots, \gamma_{j,n_j}$.

In what follows we will use this decomposition of S into the subsurfaces $\mathcal{A}, \mathcal{B}_j$ to construct a graph of groups $\mathcal{G}$ whose fundamental group will give a decomposition of $\pi_1(S, *)$. To that end, we introduce some notation.

For an oriented arc e let *start*(e) and *end*(e) be the starting and ending points of the arc e, respectively. Also, let $\bar{e}$ be the same arc with the opposite orientation. In the following discussion, let the pair of indices j, k be such that $1 \leq j \leq r$, and $1 \leq k \leq n_j$.

Choose base points $b_j \in \mathcal{B}_j$. Notice that ϕ_i^M fixes each $\mathcal{B}_j$ setwise. Hence we further isotope ϕ_i^M so that it fixes b_j, for $i = 1, 2$. See Figure 1.

Choose oriented arcs $e_{j,k}$ connecting $*$ to b_j for $1 \leq j \leq r$ and $1 \leq k \leq n_j$. Choose each arc $e_{i,j}$ such that it intersects $\gamma_{j,k}$ exactly once, and does not intersect any other γ's. Moreover, we make the choices in such a way that if $(j,k) \neq (j',k')$, then $e_{j,k}$ and $e_{j',k'}$ do not intersect except possibly at the endpoints. Let $c_{j,k}$ be the point of intersection of $e_{j,k}$ with $\gamma_{j,k}$. Also, let $e'_{j,k}$ be the subarc of $e_{j,k}$ connecting $*$ to $c_{j,k}$, and let $e''_{j,k}$ be the subarc from $c_{j,k}$ to b_j. See Figure 2.

Let G be the graph embedded in S with vertices $*, b_1, \cdots, b_r$ and geometric edges $e_{j,k}$ as above. As a technical point, the arcs with the opposite orientation $\bar{e}_{j,k}$ are also considered edges of the graph G but not drawn separately.

We use the graph G to construct a graph of groups. To each vertex of G we assign the fundamental group of the subsurface in which it is located, namely, to $*$ we assign $A = \pi_1(\mathcal{A}, *)$, to b_j we assign $B_j = \pi_1(\mathcal{B}_j, b_j)$. To each edge $e_{j,k}$ we assign

$\Gamma_{e_{j,k}} = \pi_1(\gamma_{j,k}, c_{j,k}) \cong \mathbf{Z}$. Also, let $\Gamma_{\bar{e}_{j,k}} = \Gamma_{e_{j,k}}$. We also have natural injections of the edge groups into the adjoining vertex groups as follows: for any $e_{j,k}$, since *start*$(e_{j,k}) = *$, the vertex group for *start*$(e_{j,k})$ is A. We have $\alpha_{e_{j,k}} : \Gamma_{e_{j,k}} \to A$ defined by $\alpha_{e_{j,k}}(x) = e'_{j,k} x \bar{e}'_{j,k}$. Corresponding to *end*$(e_{j,k})$, we have $\bar{\alpha}_{e_{j,k}} : \Gamma_{e_{j,k}} \to B_j$ which is defined by $\bar{\alpha}_{e_{j,k}}(x) = \bar{e}''_{j,k} x e''_{j,k}$. For the edges $\bar{e}_{j,k}$ set $\alpha_{\bar{e}_{j,k}} = \bar{\alpha}_{e_{j,k}}$ and $\bar{\alpha}_{\bar{e}_{j,k}} = \alpha_{e_{j,k}}$.

Let $\mathcal{G}$ be the graph of groups constructed by the above data. By the generalized Van Kampen theorem, $\pi_1(S, *)$ is isomorphic to the fundamental group of the graph of groups $\pi_1(\mathcal{G}, *)$.

To understand the elements of $\pi_1(\mathcal{G}, *)$, we quote some definitions from [1]. A *loop* based at $*$ in $\mathcal{G}$ is a sequence

$$t = (g_0, \epsilon_1, g_1, \cdots, \epsilon_n, g_n)$$

where ϵ_i are edges of G and $(\epsilon_1, \cdots, \epsilon_n)$ is a loop in G with *start*$(\epsilon_1) = *$ and *end*$(\epsilon_n) = *$. Also, g_0 and g_n are in A, and for $0 < i < n$, each g_i is in the group assigned to *end*$(\epsilon_i) =$ *start*(ϵ_{i+1}). A loop t in $\mathcal{G}$ is *reduced* if either $n = 0$ and $g_0 \neq 1$, or $n > 0$ and whenever $\epsilon_{i+1} = \bar{\epsilon}_i$, we have $g_i \notin \alpha_{\bar{\epsilon}_i}(\Gamma_{\epsilon_i})$. Geometrically, one can think of t as a loop in S, with g_i being loops in respective subsurfaces, and ϵ_i as arcs connecting these loops. From this point of view, a reduced loop on S does not "travel" to a component $\mathcal{B}_j$ unnecessarily.

By [1], any non-trivial element of $\pi_1(\mathcal{G}, *)$ can be written as $|t| = g_0 \epsilon_1 g_1 \cdots \epsilon_n g_n$, where t is a reduced loop as above.

Remark 2.17 The reduced loop representing 1 is the empty sequence.

Remark 2.18 A non-reduced loop can be made into a reduced loop which represents the same element in $\pi_1(\mathcal{G}, *)$ by the process of *combing*. Namely, if a loop t of length

$n > 1$ is not reduced, it has a subsequence of the form $(g_{i-1}, \epsilon_i, \alpha_{\bar{\epsilon}_i}(h_i), \bar{\epsilon}_i)$. One can replace this subsequence with $(g_{i-1}\alpha_{\epsilon_i}(h_i))$. This process reduces the length, so after finitely many steps one arrives at a reduced loop.

The following theorem is proved in [1].

Theorem 2.19 *Let $t = (g_0, \epsilon_1, g_1, \cdots, \epsilon_n, g_n)$ and $t' = (g'_0, \epsilon'_1, g'_1, \cdots, \epsilon'_m, g'_m)$ be two reduced loops such that $|t| = |t'|$ in $\pi_1(\mathcal{G})$. Then $n = m$, $\epsilon_i = \epsilon'_i$ for $1 \leq i \leq n$, and there exist $h_i \in \Gamma_{\epsilon_i}$ such that*

1. $g'_0 = g_0\ \alpha_{\epsilon_i}(h_1)^{-1}$,
2. $g'_i = \alpha_{\bar{\epsilon}_i}(h_i)\ g_i\ \alpha_{\epsilon_{i+1}}(h_{i+1})^{-1}$,
3. $g'_n = \alpha_{\bar{\epsilon}_n}(h_n)\ g_n$.

Notice that in the above theorem the elements of the form $\alpha_\epsilon(h)$ come from the circles $\gamma_{j,k}$.

Proof of Theorem A. Suppose $\langle \phi_1, \phi_2 \rangle \leq \mathcal{M}_{g,0,1}$ is a poison subgroup with respect to $x_1, x_2, x_3 \in \pi_1(S_{g,0,0}, *)$. We construct the graph of groups $\mathcal{G}$ as above, with $\pi_1(\mathcal{G}, *) \cong \pi_1(S_{g,0,0}, *)$. In the following we will identify these two groups.

By Proposition 2.16, we can choose an integer M such that ϕ_i^M fixes $\mathcal{A}$ pointwise. Since ϕ_i^M also sends each $\mathcal{B}_j$ to itself fixing the base points, we can see that $\phi_i^M(e_{j,k}) = e_{j,k}p_{j,k}$ where $p_{j,k} \in B_j$. Similarly $\phi_i^M(\bar{e}_{j,k}) = {p_{j,k}}^{-1}\bar{e}_{j,k}$.

We will now simplify notation a bit by letting ϕ stand for ϕ_1^M. Let $x_3 = |t|$ where t is the reduced loop $t = (g_0, \epsilon_1, g_1, \cdots, \epsilon_{2n}, g_{2n})$. Notice that since the graph G is "star-shaped", the length of the loop must be even. Therefore

$$\phi(x_3) = |(g_0, \epsilon_1, p_1\phi(g_1){p_2}^{-1}, \epsilon_2, g_2, \epsilon_3, p_3\phi(g_3){p_4}^{-1}, \epsilon_4, \cdots, p_{2n-1}\phi(g_{2n-1}){p_{2n}}^{-1}, \epsilon_{2n}, g_{2n})|$$

(each p_i is in the group which makes this a well-defined path). Now by the condition $\phi_1(x_3) = x_3 x_1$, which implies that $\phi(x_3) = x_3 x_1^M$, we get the equality

$$|(g_0, \epsilon_1, p_1\phi(g_1){p_2}^{-1}, \epsilon_2, g_2, \epsilon_3, p_3\phi(g_3){p_4}^{-1}, \epsilon_4, \cdots, p_{2n-1}\phi(g_{2n-1}){p_{2n}}^{-1}, \epsilon_{2n}, g_{2n})| =$$

$$|(g_0, \epsilon_1, g_1, \cdots, \epsilon_{2n}, g_{2n} x_1^M)|.$$

Let t' and t'' be the paths appearing on the left and right hand sides of the above equation respectively. Since the path t is reduced, so is t''. If t' is not reduced, by Remark 2.18 we can comb it to a reduced path t'_{red}. By the equality and Theorem 2.19, t'_{red} must have the same length as t'', which means t' was reduced in the first place. Using Theorem 2.19 again, there is an $h_1 \in \Gamma_{\epsilon_{2n}}$ such that $g_{2n} x_1^M = \alpha_{\bar{\epsilon}_{2n}}(h_1)\ g_{2n}$, i.e., $x_1^M = {g_{2n}}^{-1} \alpha_{\bar{\epsilon}_{2n}}(h_1)\ g_{2n}$. Similarly, using ϕ_2 in place of ϕ_1, there exists an $h_2 \in \Gamma_{\epsilon_{2n}}$ such that $x_2^M = {g_{2n}}^{-1} \alpha_{\bar{\epsilon}_{2n}}(h_2) g_{2n}$. But $\Gamma_{\epsilon_{2n}} \cong \mathbf{Z}$, therefore h_1, h_2 commute, which implies x_1^M, x_2^M commute. This is a contradiction, since $\langle x_1, x_2 \rangle \cong F_2$. $\square$

2.5 FP-Groups Do Not Embed in Mapping Class Groups

We begin by showing how to narrow our search for an FP-subgroup in a mapping class group.

Lemma 2.20 *Suppose that $\mathcal{M}_{g,b,n}$ contains an FP-subgroup. Then it contains an FP-subgroup H which is isomorphic to a quotient of $\mathcal{H}(F_2)$. Moreover, the image of $F_2 \times \{1\}$ in H is isomorphic to F_2.*

Proof. Suppose $\mathcal{M}_{g,b,n}$ contains an FP-subgroup. Hence there is a group G and a homomorphism $\rho : \mathcal{H}(G) \to \mathcal{M}_{g,b,n}$ such that $\rho(G \times \{1\})$ is not nilpotent-by-abelian-by-finite. Here $\rho(\mathcal{H}(G)) \cong \mathcal{H}(G)/\ker(\rho)$ is the FP-subgroup of $\mathcal{M}_{g,b,n}$. By Tits' alternative for mapping class groups ([19] or [35]), $\rho(G \times \{1\})$ is either abelian-by-finite or contains a subgroup isomorphic to F_2. By assumption, the latter holds. Let $x_1, x_2 \in G$ such that $\langle \rho(x_1, 1), \rho(x_2, 1) \rangle \cong F_2$. Then it is easily seen that for $G_1 = \langle x_1, x_2 \rangle$, $\rho(\mathcal{H}(G_1))$ is an FP-subgroup of $\mathcal{M}_{g,b,n}$ and $G_1 \cong F_2$. $\square$

We now recall the following definition from [19]. A mapping class f is called *pure* if there exists a set (possibly empty) $\mathcal{C} = \{c_1, \cdots, c_k\}$ of non-parallel, non-trivial, non-intersecting simple closed curves on the surface such that:

1. The mapping class f fixes each curve in $\mathcal{C}$ up to isotopy.

2. The mapping class f keeps each component of $S \setminus \mathcal{C}$ invariant up to isotopy.

3. The restriction of f to each component of $S \setminus \mathcal{C}$ is either the identity or pseudo-Anosov. (Recall that the restriction of f to a surface U is pseudo-Anosov if and only if for any non-trivial simple closed curve c in U not isotopic to ∂U and for any $N > 0$, $f^N(c)$ is not isotopic to c.)

For an integer m, let $H_1(S, \mathbf{Z}/m\mathbf{Z})$ be the first homology group of S with coefficients in $\mathbf{Z}/m\mathbf{Z}$. We have an action of $\mathcal{M}_{g,b,n}$ on $H_1(S, \mathbf{Z}/m\mathbf{Z})$, which defines a natural homomorphism $\mathcal{M}_{g,b,n} \to \mathrm{Aut}(H_1(S, \mathbf{Z}/m\mathbf{Z}))$. The following theorem is due to Ivanov ([19], 1.8).

Theorem 2.21 *For any integer $m \geq 3$, the group*

$$\Gamma_m = \ker(\mathcal{M}_{g,b,n} \to \mathrm{Aut}(H_1(S, \mathbf{Z}/m\mathbf{Z})))$$

is a normal subgroup of finite index in $\mathcal{M}_{g,b,n}$ consisting only of pure elements.

In the following discussion we will only need one such subgroup, so we set $m = 3$ for simplicity. Any value $m \geq 3$ would work as well.

The reader should note that in the following theorem, the generators ϕ_i, α_j, and t do not have precisely the same meaning as in Section 2.2.

Theorem 2.22 *Assume $\mathcal{M}_{g,b,n}$ contains an FP-subgroup. Then there exists an FP-subgroup of the form $H = \langle \phi_1, \phi_2, \alpha_1, \alpha_2, t \rangle$ such that ϕ_1, ϕ_2, α_1 and α_2 are in Γ_3 (in particular they are pure), and*

1. *$\langle \phi_1, \phi_2 \rangle \cong F_2$,*
2. *α_i commutes with ϕ_j,*
3. *$t(\phi_i \alpha_i)t^{-1} = \alpha_i$.*

Proof. Let H be an FP-subgroup of the form $\rho(\mathcal{H}(F_2))$ as in Lemma 2.20, where $F_2 = \langle x_1, x_2 \rangle$. Let $\alpha_i = \rho(1, x_i)$ and $\phi_i = \rho(x_i, 1)$. By abuse of notation, we denote $\rho(t)$ by t. Then $H = \langle \phi_1, \phi_2, \alpha_1, \alpha_2, t \rangle$ is an FP-subgroup satisfying (1) - (3) above, by definition of an FP-subgroup and Lemma 2.20. Using Theorem 2.21, Γ_3 is a normal subgroup of $\mathcal{M}_{g,b,n}$ of finite index. Let $N = [\mathcal{M}_{g,b,n} : \Gamma_3]$. Then $\alpha_i^N, \phi_i^N \in \Gamma_3$ are pure, and $\langle \phi_1^N, \phi_2^N \rangle \cong F_2$. Replacing each of α_i, ϕ_j with their Nth powers and keeping the same t, we get an FP-subgroup satisfying the conditions of the theorem. $\square$

In the rest of this paper we assume that α_i, ϕ_j and t are maps as given in Theorem 2.22.

We can now exploit the machinery of pure mapping classes as developed in [19]. For a pure mapping class f, one can always find a representative homeomorphism (which we will also denote by f) which fixes each curve in $\mathcal{C}$ and each component setwise. Moreover, the mapping class f induces well-defined mapping classes on components of $S \setminus \mathcal{C}$ (see Section 7.5 in [19]). As an important technical point, for a component T of $S \setminus \mathcal{C}$, in order to get a well-defined mapping class $f|_T$ in the mapping class group of T, one should allow the isotopies in T to move the points in the components of ∂T which are created as a result of cutting S open. Otherwise, an ambiguity results from combining $f|_T$ with a Dehn twist in a component of ∂T. In other words, when the surface is cut open along $\mathcal{C}$, all the new boundary components which appear will be dealt with essentially as punctures. The same remark holds when considering the mapping class group of a connected subsurface of S. In what follows, the phrase "up to isotopy" will usually be dropped, but should be understood in any discussion of topological equivalence.

In the above discussion, the collection $\mathcal{C}$ corresponding to a pure mapping class f may not be canonical, but in fact one can always choose a canonical collection of isotopy classes of disjoint simple closed curves, denoted by $\sigma(f)$, which we will define shortly. For two 1-submanifolds $\mathcal{C}_1$ and $\mathcal{C}_2$ of S, let

$$i(\mathcal{C}_1, \mathcal{C}_2) = \min\{|\mathcal{C}_1' \cap \mathcal{C}_2'| \mid \mathcal{C}_i' \text{ is isotopic to } \mathcal{C}_i\}.$$

In other words, $i(\mathcal{C}_1, \mathcal{C}_2)$ is the geometric intersection number of $\mathcal{C}_1$ and $\mathcal{C}_2$. We then define $\sigma(f)$ by saying $c \in \sigma(f)$ if the two following conditions hold:

1. $f(c) = c$.

2. For any simple closed curve γ, if $i(\gamma, c) \neq 0$, then $f(\gamma) \neq \gamma$.

The collection $\sigma(f)$ is called the *essential reduction system* for f. It is proved in [19] (see Chapter 7) that $\sigma(f)$ is a finite collection of disjoint simple closed curves, and f restricted to each component of $S \setminus \sigma(f)$ is either the identity or pseudo-Anosov.

If $f \in \mathcal{M}_{g,b,n}$ is not pure, then as discussed above there is some $N > 0$ such that f^N is pure. Thus we can extend the definition of essential reduction systems by defining $\sigma(f)$ to be equal to $\sigma(f^N)$. The notion of an essential reduction system was originally defined in [6] for a mapping class, and was generalized in [19] to an arbitrary subgroup of $\mathcal{M}_{g,b,n}$. Note that $\sigma(f)$ is a topological invariant of the mapping class f. We use this notion to define an invariant for a pair of mapping classes in $\mathcal{M}_{g,b,n}$.

Definition 2.23 For two mapping classes $f, h \in \mathcal{M}_{g,b,n}$, we let

$$i(f, h) = i(\sigma(f), \sigma(h)).$$

Notice that this is invariant under simultaneous conjugacy:

Proposition 2.24 *For $t, f, h \in \mathcal{M}_{g,b,n}$, $i(tft^{-1}, tht^{-1}) = i(f, h)$.*

Proof. First notice that $\sigma(tft^{-1}) = t(\sigma(f))$, for $f, t \in \mathcal{M}_{g,b,n}$ (again see [19], Chapter 7). Then we have that

$$\begin{aligned} i(tft^{-1}, tht^{-1}) &= i(\sigma(tft^{-1}), \sigma(tht^{-1})) \\ &= i(t(\sigma(f)), t(\sigma(h))) \\ &= i(\sigma(f), \sigma(h)) \\ &= i(f, h). \end{aligned}$$

□

The invariant $i(f, h)$ for $f, h \in \mathcal{M}_{g,b,n}$ will be crucial in the proof of Theorem B. We recall the following lemma, proved in [19].

Lemma 2.25 (Ivanov) *Let f be a pure mapping class. If X is a subsurface or a simple closed curve on the surface such that $f^N(X) = X$ for some $N \geq 1$, then $f(X) = X$.*

The following definition is also inspired by [19].

Definition 2.26 Let $f \in \mathcal{M}_{g,b,n}$, and let T be the isotopy class of a connected subsurface of S. We say f keeps T *precisely invariant* if $f(T) = T$ and if $f(c) \neq c$ for each curve c such that $i(c, \partial T) \neq 0$.

In particular we note that a pure mapping class $f \in \mathcal{M}_{g,b,n}$ keeps all components of $S \setminus \sigma(f)$ precisely invariant, by the basic property of $\sigma(f)$. Similarly, f keeps each regular neighborhood of $c \in \sigma(f)$ precisely invariant. We now develop a series of lemmas to prove Theorem B.

Lemma 2.27 *Let f, α be pure mapping classes in $\mathcal{M}_{g,b,n}$ such that $\alpha f = f\alpha$. Let T be a component of $S \setminus \sigma(f)$. Then we have*

(i) $\alpha(T) = T$, *up to isotopy.*

(ii) $\alpha(c) = c$ *for each* $c \in \sigma(f)$.

(iii) $i(f, \alpha) = 0$*; i.e.,* $\sigma(f)$ *and* $\sigma(\alpha)$ *can be isotoped off each other.*

Proof. For any integer N, α^N commutes with f. This implies that $f(\alpha^N(T)) = \alpha^N(f(T)) = \alpha^N(T)$. Suppose a simple closed curve c intersects $\partial\alpha^N(T)$ non-trivially. Then $\alpha^{-N}(c)$ intersects ∂T non-trivially, and so $f(\alpha^{-N}(c)) \neq \alpha^{-N}(c)$, by assumption. Applying α^N to both sides, we get $f(c) \neq c$. Hence f keeps $\alpha^N(T)$ precisely invariant. By the basic property of the essential reduction system, either $f|_T = id$ or $f|_T$ is pseudo-Anosov.

Case 1. Assume $f|_T = id$. Since $f|_{\alpha^N(T)} = (\alpha^N|_T)f|_T(\alpha^N|_T)^{-1}$, we have $f|_{\alpha^N(T)} = id$ for all N. Notice that $i(\partial\alpha^N(T), \partial T) = 0$, since f keeps $\alpha^N(T)$ precisely invariant for all N. Moreover, we claim that no component c of $\partial\alpha^N(T)$ can be isotopic to a simple closed curve in T which is not isotopic to a component of ∂T. Otherwise, one can find a simple closed curve γ in T such that $i(c, \gamma) \neq 0$. But $f(\gamma) = \gamma$, which contradicts the fact that f keeps $\alpha^N(T)$ precisely invariant. Similarly one can show that no component of ∂T can be isotopic to a simple closed curve in $\alpha^N(T)$ which is not isotopic to $\partial\alpha^N(T)$. This shows that either $\alpha^N(T) = T$ or $\alpha^N(T)$ can be isotoped off T. This in turn implies that the collection of subsurfaces $\{\alpha^N(T) \mid N \in \mathbf{Z}\}$ is a collection of disjoint homeomorphic subsurfaces up to isotopy, and hence it is a finite collection. This shows that $\alpha^N(T) = T$ for some N, and since α is pure, $\alpha(T) = T$, up to isotopy, by Lemma 2.25.

Case 2. Let $f|_T$ be pseudo-Anosov. Again, since $f|_{\alpha^N(T)} = (\alpha^N|_T)f|_T(\alpha^N|_T)^{-1}$, we have $f|_{\alpha^N(T)}$ is pseudo-Anosov for all N. Also, notice that $i(\partial\alpha^N(T), \partial T) = 0$, since f keeps $\alpha^N(T)$ precisely invariant for all N. Moreover, we claim that no component c of $\partial\alpha^N(T)$ can be isotopic to a simple closed curve in T which is not isotopic to a component of ∂T. Otherwise, since $c \in \partial\alpha^N(T)$ and f is pure and pseudo-Anosov on $\alpha^N(T)$, we have $f(c) = c$. On the other hand, c is in the interior of T and f is pseudo-Anosov on T, hence $f(c) \neq c$, which is a contradiction. Similarly one can show that no component of ∂T can be isotopic to a simple closed curve in $\alpha^N(T)$ which is not isotopic to $\partial\alpha^N(T)$. This shows that either $\alpha^N(T) = T$ or $\alpha^N(T)$ can be isotoped off T. The rest of the argument is exactly as in Case 1. This proves (i).

To prove (ii), let $c \in \sigma(f)$. Let T be component of $S \setminus \sigma(f)$ such that c is a component of ∂T. Then $\alpha(T) = T$, by (i). This implies that α permutes the components of ∂T, which by Lemma 2.25 implies that $\alpha(c) = c$, proving (ii).

To prove (iii), let $c \in \sigma(f)$ and $\gamma \in \sigma(\alpha)$ such that $i(c, \gamma) > 0$. Then by definition of an essential reduction system, $\alpha(c) \neq c$, which contradicts (ii). □

Let $H = \langle \phi_1, \phi_2, \alpha_1, \alpha_2, t \rangle$ be an FP-subgroup of the type described in Theorem 2.22. Notice that by Lemma 2.27(iii), $\sigma(\phi_i) \cup \sigma(\alpha_j)$ is collection of non-intersecting simple closed curves. For $i = 1, 2$, let $C_i = \sigma(\alpha_i) \cap \sigma(\phi_i)$, $A_i = \sigma(\alpha_i) \setminus C_i$ and $D_i = \sigma(\phi_i) \setminus C_i$. Note that each of A_i, C_i or D_i could be empty.

Lemma 2.28 *For $i = 1, 2$, $A_i \cup D_i \subset \sigma(\alpha_i\phi_i)$.*

Proof. Without loss of generality, we prove $A_i \subset \sigma(\alpha_i\phi_i)$. Let $c \in A_i$. Notice that by Lemma 2.27(ii), $\alpha_i(c) = \phi_i(c) = c$. If $c \notin \sigma(\alpha_i\phi_i)$, by definition, there is a subsurface U containing c where U is a component of $S \setminus \sigma(\alpha_i\phi_i)$. Since $\alpha_i\phi_i|_U$ fixes c, it is not pseudo-Anosov and hence is the identity. Similarly since $c \notin \sigma(\phi_i)$, there is a subsurface V containing c where V is a component of $S \setminus \sigma(\phi_i)$ such that $\phi_i|_V = id$. Therefore $\alpha_i|_{U\cap V} = id$. Since c is not isotopic to any component of ∂U or ∂V, and $i(\partial U, \partial V) = 0$, c is not isotopic to any component of $\partial(U \cap V)$. Then one can find a simple closed curve γ in $U \cap V$ such that $i(c, \gamma) > 0$. But $\alpha_i|_{U\cap V} = id$, so $\alpha_i(\gamma) = \gamma$, which contradicts the fact that $c \in \sigma(\alpha_i)$. $\square$

Lemma 2.29 $i(\phi_1, \phi_2) = 0$.

Proof. Recall that $\sigma(\alpha_i) = A_i \cup C_i$ and $\sigma(\phi_i) = C_i \cup D_i$. By definition of essential reduction system and Lemma 2.27(ii), $i(\alpha_i, \phi_j) = 0$ and so

$$i(A_i, C_j) = i(A_i, D_j) = i(C_1, C_2) = i(C_i, D_j) = 0,$$

for $i, j = 1, 2$. Therefore $i(\alpha_1, \alpha_2) = i(A_1, A_2)$. Now by Lemma 2.28,

$$\begin{aligned} i(\alpha_1\phi_1, \alpha_2\phi_2) &\geq i(A_1, A_2) + i(A_1, D_2) + i(D_1, A_2) + i(D_1, D_2) \\ &= i(A_1, A_2) + i(D_1, D_2). \end{aligned}$$

By part (3) of Theorem 2.22 and Proposition 2.24, we have that

$$\begin{aligned} i(A_1, A_2) &= i(\alpha_1, \alpha_2) \\ &= i(t(\phi_1\alpha_1)t^{-1}, t(\phi_2\alpha_2)t^{-1}) \\ &= i(\phi_1\alpha_1, \phi_2\alpha_2) \\ &\geq i(A_1, A_2) + i(D_1, D_2). \end{aligned}$$

Thus $i(D_1, D_2) = 0$. Hence

$$\begin{aligned} i(\phi_1, \phi_2) &= i(\sigma(\phi_1), \sigma(\phi_2)) \\ &= i(C_1 \cup D_1, C_2 \cup D_2) \\ &= i(C_1, C_2) + i(C_1, D_2) + i(D_1, C_2) + i(D_1, D_2) \\ &= 0, \end{aligned}$$

which proves the lemma. □

For a connected subsurface U of S, we define a subgroup $\Gamma_3(U)$ of the mapping class group of U as follows:

$$\Gamma_3(U) = \{f|_U \mid f \in \Gamma_3 \text{ and } f(U) = U\}.$$

Notice that all elements of $\Gamma_3(U)$ are pure. Also notice that if α_i(respectively ϕ_i) keeps U invariant, then by Theorem 2.22 we have $\alpha_i|_U \in \Gamma_3(U)$ (respectively $\phi_i|_U \in \Gamma_3(U)$). The following lemma is proved in [19] (Lemma 8.13).

Lemma 2.30 *Let Γ be a subgroup of the mapping class group of a connected surface U consisting of pure elements. If $f \in \Gamma$ is a pseudo-Anosov element, then its centralizer in Γ is an infinite cyclic group generated by a pseudo-Anosov element.*

Corollary 2.31 *Let Γ be a subgroup of the mapping class group of a connected surface U consisting of pure elements. If $f, h \in \Gamma$ are pseudo-Anosov elements, then either f commutes with h or their respective centralizers in Γ intersect trivially.*

Proof. Let $C_\Gamma(f)$ denote the centralizer of f in Γ. Suppose there is an element $1 \neq \theta \in C_\Gamma(f) \cap C_\Gamma(h)$. Then $f, h \in C_\Gamma(\theta)$, which is cyclic by Lemma 2.30, so f commutes with h. □

We are going to encounter the following particular situation in different contexts, so we declare it a lemma:

Lemma 2.32 *Let U be a component of $S \setminus \sigma(\phi_i)$ for $i = 1$ or $i = 2$ such that $\Gamma_3(U)$ is non-trivial. Assume that $\alpha_i|_U = id$ and $\phi_i(U) = U$ for $i = 1, 2$. Then the respective centralizers of $\phi_1|_U$ and $\phi_2|_U$ in $\Gamma_3(U)$ intersect non-trivially.*

Proof. Without loss of generality, let U be a component of $S \setminus \sigma(\phi_1)$. Assume on the contrary that the centralizers of $\phi_1|_U$ and $\phi_2|_U$ in the mapping class group of U have only the identity map in common. This in particular implies that $\phi_i|_U \neq id$ for $i = 1, 2$. The map $\phi_1|_U$ is pseudo-Anosov, since U is a component of $S \setminus \sigma(\phi_1)$. Consider the subsurface $t(U)$. By part (3) of Theorem 2.22, we have

$$\alpha_i|_{t(U)} = (t|_U)(\phi_i|_U \alpha_i|_U)(t|_U)^{-1} = (t|_U)(\phi_i|_U)(t|_U)^{-1}. \tag{7}$$

This implies that $\alpha_i|_{t(U)} \neq id$ keeps $t(U)$ invariant, since it is conjugate to $\phi_i|_U$, for $i = 1, 2$. Moreover, $\alpha_1|_{t(U)}$ is pseudo-Anosov. This in particular implies that $t(U)$ is a component of $S \setminus \sigma(\alpha_1)$, and $t(U)$ can be isotoped off U, since $\alpha_1|_U = id$. Moreover, by assumption and by (7), the centralizers of $\alpha_1|_{t(U)}$ and $\alpha_2|_{t(U)}$ intersect trivially in $\Gamma_3(t(U))$. By Lemma 2.27(i), ϕ_i keeps $t(U)$ invariant for $i = 1, 2$, since ϕ_i commutes with α_1. Again, since $\phi_i|_{t(U)}$ commutes with $\alpha_j|_{t(U)}$ and by the assumption about the centralizers, we have $\phi_i|_{t(U)} = id$, for $i, j = 1, 2$. Now we can prove the following statements for $N \geq 1$ simultaneously by induction on N:

1. $\alpha_i|_{t^N(U)} \neq id$ keeps $t^N(U)$ invariant, for $i = 1, 2$.

2. $\alpha_1|_{t^N(U)}$ is pseudo-Anosov (hence, ϕ_i keeps $t^N(U)$ invariant for $i = 1, 2$).

3. The respective centralizers of $\alpha_i|_{t^N(U)}$ in $\Gamma_3(t^N(U))$ intersect trivially, for $i = 1, 2$.

4. $\phi_i|_{t^N(U)} = id$, for $i = 1, 2$.

We have already established all four statements for $N = 1$. The passage from N to $N+1$ follows similarly from the relation

$$\alpha_i|_{t^{N+1}(U)} = (t|_{t^N(U)})(\phi_i|_{t^N(U)}\alpha_i|_{t^N(U)})(t|_{t^N(U)})^{-1} = (t|_{t^N(U)})(\alpha_i|_{t^N(U)})(t|_{t^N(U)})^{-1}.$$

The second statement above shows that $t^N(U)$ can be isotoped off U, since $\alpha_1|_U = id$. Therefore, $t^M(U)$ can be isotoped off $t^N(U)$ for all $M \neq N$. This is clearly a contradiction, since the Euler characteristic of S is finite. $\square$

Lemma 2.33 *For $i = 1, 2$, let U_i be a component of $S \setminus \sigma(\phi_i)$ such that $\phi_i|_{U_i}$is pseudo-Anosov. Then either U_1 and U_2 are disjoint up to isotopy, or U_1 is isotopic to U_2.*

Proof. First we show that if U_1 and U_2 are not disjoint, then either $U_1 \subseteq U_2$ or $U_2 \subseteq U_1$. Suppose $U_1 \nsubseteq U_2$ and $U_2 \nsubseteq U_1$ but U_1 cannot be isotoped off U_2. Throughout the proof, let $j, k \in \{1, 2\}$ be arbitrary such that $j \neq k$. Since $i(\partial U_1, \partial U_2) = 0$, there is some component c_j of ∂U_j such that $c_j \subset U_k$ and c_j is not isotopic to any component of ∂U_k. By Lemma 2.27(i), α_i keeps U_1 and U_2 invariant for $i = 1, 2$. Since $\alpha_i \in \Gamma_3$, we have $\alpha_i|_{U_j} \in \Gamma_3(U_j)$. Since c_j is in the interior of U_k and $\alpha_i(c_j) = c_j$ by Lemma 2.27(ii), this implies that $\alpha_i|_{U_k}$ is not pseudo-Anosov, hence by Lemma 2.30, $\alpha_i|_{U_k} = id$ for $i, k = 1, 2$.

Let $U = U_1 \cup U_2$. At this point we apply a similar argument as in the proof of Lemma 2.32, as follows. By the relation

$$\alpha_i|_{t(U_i)} = (t|_{U_i})(\phi_i|_{U_i}\alpha_i|_{U_i})(t|_{U_i})^{-1} = (t|_{U_i})(\phi_i|_{U_i})(t|_{U_i})^{-1}, \tag{8}$$

we see that $\alpha_i|_{t(U_i)}$ is pseudo-Anosov. This in particular implies that $t(U) = t(U_1) \cup t(U_2)$ can be isotoped off U, since $\alpha_i|_U = id$. Note that $t(U_i)$ is a component of $S \setminus \sigma(\alpha_i)$, so ϕ_j keeps $t(U_i)$ invariant for $i, j = 1, 2$, by Lemma 2.27(i). Since ϕ_i is pure, and $t(c_j)$ is a boundary component of $t(U_j)$, we have $\phi_i(t(c_j)) = t(c_j)$. By the choice of c_j we know that $t(c_j)$ is in the interior of $t(U_k)$. By Lemma 2.30 and the fact that $\phi_i|_{t(U_k)} \in \Gamma_3(U_k)$, we have $\phi_i|_{t(U_k)} = id$, for $i, k = 1, 2$. Now by induction on N we can simultaneously prove the following statements for $N \geq 1$:

1. The map $\alpha_i|_{t^N(U_i)}$ is pseudo-Anosov, for $i = 1, 2$.

2. We have $\phi_i|_{t^N(U_j)} = id$, for $i, j = 1, 2$.

We have already established these two statements for $N = 1$. The passage from N to $N+1$ can be achieved by considering the conjugacy relation

$$\alpha_i|_{t^{N+1}(U_i)} = t|_{t^N(U_i)}\ \phi_i|_{t^N(U_i)}\ \alpha_i|_{t^N(U_i)}\ t|^{-1}_{t^N(U_i)} = t|_{t^N(U_i)}\ \alpha_i|_{t^N(U_i)}\ t|^{-1}_{t^N(U_i)}. \tag{9}$$

This proves statement (1) above. Now use Lemma 2.27(i) to see that ϕ_i keeps $t^{N+1}(U_j)$ invariant. This implies that $\phi_i|_{t^{N+1}(U_j)} \in \Gamma_3(t^{N+1}(U_j))$, and by Lemma 2.30, we have statement (2).

In particular, statement (1) shows that $t^N(U)$ can be isotoped off U for all $N > 1$, which is a contradiction as in Lemma 2.32. This proves that either $U_1 \subseteq U_2$ or $U_2 \subseteq U_1$, or U_1 and U_2 can be isotoped off each other.

Now without loss of generality, suppose that $U_1 \subseteq U_2$, but U_1 is not isotopic to U_2. Then there exists a component c_1 of ∂U_1 such that c_1 is not isotopic to a component of ∂U_2. By Lemma 2.27(i), α_i keeps U_1 and U_2 invariant for $i = 1, 2$. Also, by Lemma 2.27(ii), $\alpha_i(c_1) = c_1$, which implies $\alpha_i|_{U_2} = id$, by Lemma 2.30. Again, using (8) we get statement (1) for $N = 1$. Hence $t(U_i)$ is a component of $S \setminus \sigma(\alpha_i)$. So ϕ_i keeps U_j invariant. Thus $\phi_i(t(c_1)) = t(c_1)$, which gives $\phi_i(U_2) = id$, by Lemma 2.30. This proves statement (2) for $N = 1$. The passage from N to $N + 1$ follows by using equation (9) above. Then again we have that U_2 can be isotoped off $t^N(U_2)$ for all $N > 1$, which is a contradiction. This proves that U_1 is isotopic to U_2. $\square$

Lemma 2.34 *Let U be a component of both $S \setminus \sigma(\phi_1)$ and $S \setminus \sigma(\phi_2)$ such that $\phi_i|_U$ is pseudo-Anosov for $i = 1, 2$. Then $\phi_1|_U$ commutes with $\phi_2|_U$.*

Proof. If $\phi_1|_U$ and $\phi_2|_U$ do not commute, then their centralizers in $\Gamma_3(U)$ have trivial intersection by Corollary 2.31. This implies that $\alpha_i|_U = id$, which contradicts Lemma 2.32. $\square$

We are finally ready to prove Theorem B.

Proof of Theorem B. Let U be a component of $S \setminus \sigma(\phi_1)$ such that $\phi_1|_U$ is pseudo-Anosov. We first prove that $\phi_2|_U$ is either pseudo-Anosov or the identity. Suppose $\phi_2|_U$ is neither pseudo-Anosov nor the identity (in particular, U is not a component of $S \setminus \sigma(\phi_2)$). Let $V_1, V_2, \cdots, V_s$ be components of $S \setminus \sigma(\phi_2)$, which cover U up to isotopy. We can assume that the cover is minimal in the sense that none of the V_k can be isotoped off U. By Lemma 2.33, $\phi_2|_{V_k} = id$ for all $1 \leq k \leq s$. (This does not mean that $\phi_2|_U = id$, since ϕ_2 may involve Dehn twists about boundary components of V_k.) By Lemma 2.29, $i(\partial U, \partial V_k) = 0$ for all $1 \leq k \leq s$, which shows that ϕ_2 keeps U invariant. Moreover, $\phi_2|_U$ is a non-trivial composition of Dehn twists about disjoint simple closed curves. Using Lemma 2.27(i), α_i keeps U invariant. Since $\alpha_i|_U, \phi_j|_U \in \Gamma_3(U)$ and $\alpha_i|_U$ commutes with $\phi_2|_U$, using Lemma 2.30 we see that $\alpha_i|_U$ cannot be pseudo-Anosov. Moreover, $\alpha_i|_U$ commutes with $\phi_1|_U$ so $\alpha_i|_U = id$. Now by Lemma 2.32, we get that the centralizers of $\phi_1|_U$ and $\phi_2|_U$ must intersect non-trivially. Lemma 2.30 then implies that $\phi_2|_U$ is either pseudo-Anosov or the identity, which is a contradiction.

We have proved that for a component U of $S \setminus \sigma(\phi_1)$ where $\phi_1|_U$ is pseudo-Anosov, $\phi_2|_U$ is either pseudo-Anosov or the identity. In the case that $\phi_2|_U$ is pseudo-Anosov, $\phi_1|_U$ and $\phi_2|_U$ commute by Lemma 2.34. Similarly, for a component V of $S \setminus \sigma(\phi_2)$ where $\phi_2|_V$ is pseudo-Anosov, $\phi_1|_V$ is either a commuting pseudo-Anosov or the identity.

Let S_1 be the subsurface of S which is the union of subsurfaces T such that either $\phi_1|_T$ or $\phi_2|_T$ is pseudo-Anosov. We have proved that ϕ_1 and ϕ_2 both keep S_1 invariant, and $\phi_1|_{S_1}$ commutes with $\phi_2|_{S_1}$.

On $S_2 = S \setminus S_1$ both ϕ_1 and ϕ_2 are compositions of Dehn twists about disjoint curves, by Lemma 2.29. Hence $\phi_1|_{S_2}$ and $\phi_2|_{S_2}$ commute. We conclude that ϕ_1 and ϕ_2 commute, contradicting part (1) of Theorem 2.22. This shows that FP-groups do not embed in $\mathcal{M}_{g,b,n}$, as desired. $\square$

3 Winding Number and Representations of Mapping Class Groups

First Morita, then Trapp, and more recently Perron, all construct representations of the mapping class group, each using very different techniques. These three separate approaches, however, yield closely connected representations of $\mathcal{M}_{g,1,0}$. The three authors are probably aware of this fact (Trapp and Perron each credit Morita), yet the precise connections amongst the three representations have not been made explicit. Our goal in this section is to clarify these connections and hence to fill in a perceived gap in the literature by presenting all three together as different interpretations of what is essentially one representation.

Throughout this section we are only interested in $\mathcal{M}_{g,b,n}$ in the case where $n = 0$, and so we let $\mathcal{M}_{g,b}$ denote $\mathcal{M}_{g,b,0}$, and likewise for any corresponding surfaces and subgroups. Also, whenever we have need to make reference to a homology basis, we will consider the standard symplectic basis used in both [20] and [46]. For reasons which will become clear later, we now denote the symplectic representation of $\mathcal{M}_{g,1}$ by $\rho_2 : \mathcal{M}_{g,1} \to Sp(2g, \mathbf{Z})$. Recall that ρ_2 records the action of the mapping class group on homology, and its kernel is the Torelli group, $\mathcal{I}_{g,1}$.

After presenting some necessary background, we describe Morita's representation ρ_3, which generalizes both the symplectic representation ρ_2 as well as Johnson's crossed homomorphism $\tau : \mathcal{I}_{g,1} \to \Lambda^3 H_1(S_{g,1}, \mathbf{Z})$. We focus on a crossed homomorphism induced by Morita's representation, which will serve as an important link between Morita's representation and that of Trapp and of Perron.

3.1 Group Cohomology Background

The cohomology of groups is usually defined using the language of cochain complexes. It is more useful for our purposes to understand group cohomology in terms of crossed homomorphisms, and therefore we follow Brown's treatment of this subject in [8].

Let E be a group. We say that E is an *extension of G by A*, if we have a short exact sequence

$$0 \to A \to E \to G \to 1$$

(though, as Brown notes, some authors, in particular Bernard Perron in [41], would reverse the terminology and refer to this as an extension of A by G). For our purposes, we will only need to consider the case where the kernel A is an abelian group (hence the use of 0 on the left). An extension is *split* if we have a section $s : G \to E$, or equivalently, if $E \cong A \rtimes G$. (Recall that the semi-direct product $A \rtimes G$ is equal to the set $A \times G$ together with multiplication given by $(a, g) \cdot (b, h) = (a + gb, gh)$.)

There is a unique split extension corresponding to any given action of G on A. However, there are possibly many different sections which induce the split extension. Let $s : G \to A \rtimes G$ be a section of a split extension. Then the induced map onto the first factor $s : G \to A$ is necessarily a *crossed homomorphism*. A crossed homomorphism (sometimes called a *derivation*) is a function $d : G \to A$ such that $d(gh) = d(g) + g\, d(h)$. Let $\mathrm{Der}(G, A)$ denote the set of all crossed homomorphisms $d : G \to A$. The set of possible sections of the given split extension is in 1-1 correspondence with the set $\mathrm{Der}(G, A)$ [8].

We now introduce an equivalence relation on crossed homomorphisms, and hence on sections, which will clarify our use of the word "unique" in the previous paragraph. Let $a \in A$, and define a crossed homomorphism $d_a : G \to A$ by $d_a(g) = ga - a$.

Then we will say that $d_1 \sim d_2$ if $d_2 - d_1 = d_a$ for some $a \in A$. Such a function d_a is called a *coboundary* or a *principal derivation.* Denote by $P(G, A)$ the set of all such coboundaries. We define the *first cohomology of* G *with coefficients in* A, denoted $H^1(G, A)$, as $\mathrm{Der}(G, A)/P(G, A)$. The corresponding notion of equivalence for sections is simply conjugation, i.e., if $a \in A$, and $i(a)$ represents the inclusion of a in $A \rtimes G$, then two sections s_1 and s_2 are equivalent if there exists an $a \in A$ which satisfies $s_1(g) = i(a)s_2(g)i(a)^{-1}$ for all $g \in G$ [8].

There is another interpretation of $H^1(G, A)$ which will also be useful to us in the case of mapping class groups. Thus we now follow Morita's treatment of group cohomology in [37] in the special case where $G = \mathcal{M}_{g,1}$ and $A = H^1(S_{g,1}; \mathbf{Z})$ (we will usually drop the $\mathbf{Z}$ from this notation, but it is to be understood). We know that $\mathcal{M}_{g,1}$ acts on $H_1(S_{g,1})$ via the symplectic representation ρ_2, and we will denote this by $\phi_*(x) = \rho_2(\phi)(x)$. If we employ the usual identification of $H^1(S_{g,1})$ with $\mathrm{Hom}(H_1(S_{g,1}), \mathbf{Z})$, we can describe the action of $\mathcal{M}_{g,1}$ on $H^1(S_{g,1})$ by $\phi u(x) = u(\phi_*^{-1}(x)) = u(\rho_2(\phi^{-1})(x))$ for $\phi \in \mathcal{M}_{g,1}, u \in H^1(S_{g,1})$, and $x \in H_1(S_{g,1})$.

Let $F(\mathcal{M}_{g,1} \times H_1(S_{g,1}), \mathbf{Z})$ denote the set of all functions $f : \mathcal{M}_{g,1} \times H_1(S_{g,1}) \to \mathbf{Z}$ such that:

1. $f(\phi, x + y) = f(\phi, x) + f(\phi, y)$

2. $f(\phi\psi, x) = f(\phi, \psi_*(x)) + f(\psi, x)$

Let $d \in \mathrm{Der}(\mathcal{M}_{g,1}; H^1(S_{g,1}))$, and let $f_d : \mathcal{M}_{g,1} \times H_1(S_{g,1}) \to \mathbf{Z}$ be given by $f_d(\phi, x) = d(\phi^{-1})(x)$. We have a bijection given as follows:

$$\begin{aligned} \mathrm{Der}(\mathcal{M}_{g,1}; H^1(S_{g,1})) \quad &\longleftrightarrow \quad F(\mathcal{M}_{g,1} \times H_1(S_{g,1}), \mathbf{Z}) \\ d \quad &\longleftrightarrow \quad f_d \end{aligned} \tag{10}$$

In view of this fact, Morita also refers to elements of $F(\mathcal{M}_{g,1} \times H_1(S_{g,1}), \mathbf{Z})$ as crossed homomorphisms. Let $a \in H^1(S_{g,1}; \mathbf{Z})$. Then the bijection carries the coboundary $d_a \in \mathrm{Der}(\mathcal{M}_{g,1}; H^1(S_{g,1}; \mathbf{Z}))$ to a map $d_a : \mathcal{M}_{g,1} \times H_1(S_{g,1}), \mathbf{Z})$ defined by $d_a(\phi, x) = a(\phi_*(x) - x)$. Here we follow Morita in abusing notation by retaining the name d_a, and we will also refer to such maps as coboundaries. Thus another way to interpret $H^1(\mathcal{M}_{g,1}; H^1(S_{g,1}))$ is as the set $F(\mathcal{M}_{g,1} \times H_1(S_{g,1}), \mathbf{Z})$ modulo coboundaries.

3.2 Morita's Representation ρ_3

Let π denote the fundamental group of the surface $S_{g,1}$, and let π' denote its commutator subgroup. Also, we will use H^1 as a shorthand notation for the first integral cohomology group of $S_{g,1}$, and H_1 to denote the first integral homology group. By Poincaré duality we have $H^1 \cong H_1$, so we will often simply use H when it is unnecessary to be more specific. Johnson constructs homomorphism $\tau : \mathcal{I}_{g,1} \to \Lambda^3 H$ in [20] based on the action of $\mathcal{M}_{g,1}$ on the quotient $\pi/[\pi, \pi']$. Morita generalizes Johnson's approach to the extent of finding a sequence of representations of $\mathcal{M}_{g,1}$ based on its action on the lower central series of the fundamental group of the surface $S_{g,1}$ in [39]. We denote the lower central series by Γ_j. Thus $\Gamma_1 = \pi_1(S_{g,1})$, and $\Gamma_{j+1} = [\Gamma_1, \Gamma_j]$ for $j \geq 1$. The quotient group $N_j = \Gamma_1/\Gamma_j$ is known as the *j-th nilpotent quotient* of Γ_1. Note that N_2 is just H, the first integral homology of $S_{g,1}$. Now if we fix a base point on the boundary of our surface, $\mathcal{M}_{g,1}$ acts naturally on the fundamental group Γ_1. This action induces an action on each nilpotent quotient N_j, which then yields a sequence of representations $\rho_j : \mathcal{M}_{g,1} \to \mathrm{Aut}\ N_3$. Since $N_2 = H$, ρ_2 is just the symplectic representation of the mapping class group, and $\mathrm{Im}\ \rho_2 = \mathrm{Sp}(H) \cong \mathrm{Sp}(2g, \mathbf{Z})$.

We would like to develop a similarly useful understanding of $\mathrm{Im}\ \rho_3$. "Useful" here means that we can embed the image in a semi-direct product, each factor of which is a piece which we can interpret geometrically and connect to our other representations. Morita outlines this process very carefully in [39], and we follow his exposition here. Though we do not give every detail, we aim to describe Morita's construction well enough that the reader will be able to compare his methods to those of Trapp and Perron. We will incorporate most of Morita's notation and terminology so that the interested reader will have an easier time reading the more detailed version in [39].

Let $I_j = \rho_j(\mathcal{I}_{g,1})$, and let $I(N_j)$ denote the subgroup of Aut N_j whose elements act trivially on the first homology of N_j. Morita first shows that I_j is an extension of $\mathrm{Sp}(H)$ by $I(N_j)$. Unfortunately, this extension is not split. Morita determines that Aut N_j is an extension of $\mathrm{GL}(H)$ by $I(N_j)$ (each of which contains an embedded copy of its respective counterpart above), but again this extension is not split.

Morita turns to the *Mal'cev completion* of a nilpotent group (see [34] for the definition) and considers a new representation $\rho_j \otimes \mathbf{Q} : \mathcal{M}_{g,1} \to \text{Aut } (N_j \otimes \mathbf{Q})$. The case $k = 3$ is special because there is an explicit product description for the Mal'cev completion $N_3 \otimes \mathbf{Q}$. Namely, $N_3 \otimes \mathbf{Q} \cong \Lambda^2 H_{\mathbf{Q}} \times H_{\mathbf{Q}}$, where $H_{\mathbf{Q}} = H \otimes \mathbf{Q}$, with multiplication defined by $(\xi, u)(\eta, v) = (\xi + \eta + \frac{1}{2} u \wedge v, u + v)$. We skip over the details, but this product description ultimately enables Morita to give a split extension.

$$0 \to \mathrm{Hom}(H_{\mathbf{Q}}, \Lambda^2 H_{\mathbf{Q}}) \to \text{Aut } (N_3 \otimes \mathbf{Q}) \to \mathrm{GL}(H_{\mathbf{Q}}) \to 1$$

where the action of $\mathrm{GL}(H_{\mathbf{Q}})$ on $\mathrm{Hom}(H_{\mathbf{Q}}, \Lambda^2 H_{\mathbf{Q}})$ is given by $(Af)(u) = Af(A^{-1}u)$ for $A \in \mathrm{GL}(H_{\mathbf{Q}}), f \in \mathrm{Hom}(H_{\mathbf{Q}}, \Lambda^2 H_{\mathbf{Q}})$, and $u \in H_{\mathbf{Q}}$.

For any split extension, the projection map from the semi-direct product onto its first factor is necessarily a crossed homomorphism (see [8] or [17]). Let q : Aut $(N_3 \otimes \mathbf{Q}) \to \mathrm{Hom}(H_{\mathbf{Q}}, \Lambda^2 H_{\mathbf{Q}})$ be the crossed homomorphism associated to the split extension Aut $(N_3 \otimes \mathbf{Q}) \cong \mathrm{Hom}(H_{\mathbf{Q}}, \Lambda^2 H_{\mathbf{Q}}) \rtimes \mathrm{GL}(H_{\mathbf{Q}})$ given above. For simplicity, let $r = \rho_3 \otimes \mathbf{Q}$. If we compose q with the representation $r : \mathcal{M}_{g,1} \to \mathrm{Aut}\ (N_3 \otimes \mathbf{Q})$, we obtain another crossed homomorphism $\tilde{k} : \mathcal{M}_{g,1} \to \mathrm{Aut}\ (N_3 \otimes \mathbf{Q})$, which Morita calls *the crossed homomorphism associated to* r. Since $\mathrm{Hom}(H_{\mathbf{Q}}, \Lambda^2 H_{\mathbf{Q}})$ becomes an $\mathcal{M}_{g,1}$-module via the symplectic representation ρ_2, we can express the fact that $\tilde{k}$ is a crossed homomorphism as follows:

$$\tilde{k}(\phi\psi) = \tilde{k}(\phi) + \rho_2(\phi)\tilde{k}(\psi).$$

In addition, we can use the semi-direct product structure of Aut $(N_3 \otimes \mathbf{Q})$ to write $r(\phi) = (\tilde{k}(\phi), \rho_2(\phi))$ for any $\phi \in \mathcal{M}_{g,1}$.

We now return our attention to the group Aut N_3 and the representation ρ_3. Again, building on special properties of the Mal'cev completion of the nilpotent quotients N_j in the case $j = 3$, Morita is able to show that there exists an embedding $i : N_3 \to N_3 \otimes \mathbf{Q}$ which induces an injection $i_* : \mathrm{Aut}\ N_3 \to \mathrm{Aut}\ (N_3 \otimes \mathbf{Q})$ such that $r = i_* \circ \rho_3$. Thus Im $r \cong$ Im ρ_3, and we have a copy of Im ρ_3 embedded in Aut $(N_3 \otimes \mathbf{Q}) \cong \mathrm{Hom}(H_{\mathbf{Q}}, \Lambda^2 H_{\mathbf{Q}}) \rtimes \mathrm{GL}(H_{\mathbf{Q}})$.

Using the respective natural embeddings of $\Lambda^2 H$ and H into $\Lambda^2 H_{\mathbf{Q}}$ and $H_{\mathbf{Q}}$, Morita shows that Im $\rho_3 \subseteq \mathrm{Hom}(H, \frac{1}{2}\Lambda^2 H) \rtimes \mathrm{Sp}(H)$ [39]. We will follow Morita's abuse of notation and also use the symbol $\tilde{k}$ to denote the crossed homomorphism associated to ρ_3.

We must now examine the dependence of the above construction on the injection i. A different embedding of N_3 in $N_3 \otimes \mathbf{Q}$ may change our embedding of Imρ_3 in Aut $(N_3 \otimes \mathbf{Q})$. Suppose that $i' : N_3 \to N_3 \otimes \mathbf{Q}$ were another such injection. Then Morita shows in [39] that there exists an element $f \in \mathrm{Hom}(H_{\mathbf{Q}}, \Lambda^2 H_{\mathbf{Q}}) \subset$ Aut $(N_3 \otimes \mathbf{Q})$ such that $i' = f \circ i$. Each i' induces a representation $r' = i'_* \circ \rho_3$ with an associated crossed homomorphism $\tilde{k}' : \mathcal{M}_{g,1} \to \mathrm{Hom}(H_{\mathbf{Q}}, \Lambda^2 H_{\mathbf{Q}})$. Then the following equation holds for all $\phi \in \mathcal{M}_{g,1}$:

$$\tilde{k}'(\phi) = \tilde{k}(\phi) + f - \rho_2(\phi) f.$$

But according to our previous notation from Section 3.1, $f - \rho_2(\phi)f$ is just the coboundary $d_{(-f)}$. If we denote this coboundary by δ_f, we can write simply that $\tilde{k}' = \tilde{k} + \delta_f$. We shall call $\tilde{k}'$ *the crossed homomorphism associated to the map* f. In other words, any two crossed homomorphisms arising from such a construction will differ by a coboundary and hence are identical as elements of cohomology.

Remark 3.1 There are some technical details which are being glossed over here. By referring to i' as "another such injection", we mean that N_3 is an extension of H by $\Lambda^2 H$, $N_3 \otimes \mathbf{Q}$ is an extension of $H_{\mathbf{Q}}$ by $\Lambda^2 H_{\mathbf{Q}}$, and the injection $i' : N_3 \to N_3 \otimes \mathbf{Q}$ must make the short exact sequences corresponding to these extensions commute with the natural inclusions of $\Lambda^2 H$ and H into $\Lambda^2 H_{\mathbf{Q}}$ and $H_{\mathbf{Q}}$, respectively.

We can now adjust the image of our crossed homomorphisms within $\mathrm{Hom}(H_{\mathbf{Q}}, \Lambda^2 H_{\mathbf{Q}})$ (and hence the image of ρ_3 within $\mathrm{Hom}(H_{\mathbf{Q}}, \Lambda^2 H_{\mathbf{Q}}) \rtimes \mathrm{Sp}(H)$) simply by choosing different maps f. In [39], Morita explicitly constructs a map $f \in \mathrm{Hom}(H, \frac{1}{2}\Lambda^2 H)$ with the property that $\mathrm{Im}(\tilde{k} + \delta_f) \subset \frac{1}{2}\Lambda^3 H \subset \mathrm{Hom}(H, \frac{1}{2}\Lambda^2 H)$. In yet another abuse of notation, we denote by $\tilde{k}$ the crossed homomorphism associated to this particular map f. We are now in a position to write down the representation ρ_3 in such a way that $\mathrm{Im}\, \rho_3$ is useful to us.

Theorem 3.2 (Morita, [39]) *Let $\tilde{k} : \mathcal{M}_{g,1} \to \frac{1}{2}\Lambda^3 H$ be the crossed homomorphism associated to the map f given above. Then we can embed Im ρ_3 into $\frac{1}{2}\Lambda^3 H \rtimes Sp(H)$*

and the representation ρ_3 can be described explicitly by the formula

$$\rho_3(\phi) = (\tilde{k}(\phi), \rho_2(\phi))$$

for any $\phi \in \mathcal{M}_{g,1}$. Furthermore, the restriction of the crossed homomorphism $\tilde{k}$ to the Torelli group $\mathcal{I}_{g,1}$ is precisely Johnson's homomorphism $\tau : \mathcal{I}_{g,1} \to \frac{1}{2}\Lambda^3 H$.

Note that Johnson defined his map τ with values in $\Lambda^3 H$, so we make the obvious identification. The map τ carries some interesting geometric information about mapping classes in $\mathcal{I}_{g,1}$ [22], and we shall see that $\tilde{k}$ carries the same information for all of $\mathcal{M}_{g,1}$.

3.3 Crossed Homomorphisms $\mathcal{M}_{g,1} \to H$

Using calculations involving projections of $\pi_1(S_{g,1})$ onto F_2, Morita constructs a crossed homomorphism $k : \mathcal{M}_{g,1} \times H_1(S_{g,1}) \to \mathbf{Z}$ [37]. He proceeds to show in [39] that k is the contraction of the crossed homomorphism $\tilde{k} : \mathcal{M}_{g,1} \to \frac{1}{2}\Lambda^3 H$. In other words, if we let $C : \Lambda^3 H \to H$ be the standard contraction map given by

$$C(x \wedge y \wedge z) = 2[(x \cdot y)z + (y \cdot z)x + (z \cdot x)y]$$

where $\cdot$ denotes the intersection pairing of homology classes, then $k = C \circ \tilde{k} : \mathcal{M}_{g,1} \to H$. From either point of view, Morita's constructions of the crossed homomorphism k are purely algebraic. Since we will actually be more concerned with the contraction of $\tilde{k}$, we understand "Morita's representation" to refer to the map ρ_3 composed with contraction of the first factor.

However, we know that $\tilde{k}$ restricts to Johnson's homomorphism τ. We now describe the geometric information carried by τ. Johnson proves in [22] that $C \circ \tau$ is what is known as the *Chillingworth homomorphism* of $\mathcal{I}_{g,1}$, which records how a given element of the Torelli group acts on the winding number of curves on a surface, a concept introduced by Chillingworth in [9]. Let X be a nonsingular vector field on $S_{g,1}$. If γ is an oriented, direct, regular curve on $S_{g,1}$, then its *winding number relative to* X, denoted $\omega_X(\gamma)$, is the number of times the tangent vector to the curve rotates relative to the vector field X (for more details see [9]).

Remark 3.3 Chillingworth's winding number function is well-defined on homotopy classes and for technical reasons must be computed using a representative curve which is both regular and direct. We recall that a curve is *regular* if continuously varying non-zero tangents exist at all points of the curve. A closed curve which self-intersects transversally finitely many times is said to be *direct* if it contains no nullhomotopic loop.

Remark 3.4 We note two interesting facts about winding numbers of curves on surfaces:

1. Formula 1 of [9] shows how the winding number of any homotopy class is determined by the winding numbers of a particular basis of $\pi_1(S_{g,1})$.

2. Let $\gamma_1, \ldots, \gamma_{2g}$ be a basis of $\pi_1(S_{g,1})$. Then Theorem 4.2 of [9] points out that a non-vanishing vector field X can be chosen to satisfy any given assignments $\omega_X(\gamma_i) = a_i, \quad a_i \in \mathbf{Z}$.

Following Trapp ([46]), we define a function $e_X : \mathcal{M}_{g,1} \to H^1$ given by $e_X(f)(\gamma) = \omega_X(f(\gamma)) - \omega_X(\gamma)$. This function measures the change in winding numbers of our homology basis effected by a given element of the mapping class group relative to a fixed vector field X. Johnson first defined such a function on the Torelli group in [20].

Remark 3.5 In general, winding numbers are not well defined on homology classes, but Trapp shows in [46] that $e_X(f)$ is actually well defined on homology classes for all $f \in \mathcal{M}_{g,1}$. (An argument in the restricted case of $\mathcal{I}_{g,1}$ is also given in [20].) Note that we will often abuse notation by referring to both the curve and its homology class as γ. Since $H^1 \cong \mathbf{Z}^{2g}$, we will think of elements in H^1 as row vectors.

The function e_X is not a homomorphism on the entire mapping class group, but it is a *crossed homomorphism* in the sense that it obeys the following composition law (as proved in [46]):

$$e_X(fh) = e_X(f)\rho_2(h) + e_X(h). \tag{11}$$

It is easy to check that Trapp's map $e_X : \mathcal{M}_{g,1} \to H^1$ is a also crossed homomorphism in the sense of Morita. We will view e_X as an element of the set $F(\mathcal{M}_{g,1} \times H_1(S_{g,1}), \mathbf{Z})$. In this case, we need to check the two criteria set forth by Morita. The first is clear. To check the composition law, let $f, h \in \mathcal{M}_{g,1}$, and let $x \in H_1$. Then what we need to show is that

$$\begin{aligned} e_X(fh)(x) &= e_X(f)(h_* x) + e_X(h)(x) \qquad (12) \\ &= e_X(f)\rho_2(h)(x) + e_X(h)(x) \end{aligned}$$

since the action of $\mathcal{M}_{g,1}$ on H_1 is given by the symplectic representation. But this is precisely the criterion described in Equation11.

In any case, it is immediately clear that e_X is a homomorphism when restricted to the Torelli group. Johnson proves this directly in [20]. It is worth noting that we also have from Johnson the fact that e_X is independent of the choice of the vector field X when restricting to $\mathcal{I}_{g,1}$. In particular, in the restricted case we can write $e(f)$ or e_f for $e_X(f)$ and dualize to a homology class $t(f)$ determined by intersection pairing: $\gamma \cdot t(f) = e_f(\gamma)$; Johnson refers to $t(f)$ as the *Chillingworth class* of f. Then we can think of t as a homomorphism from the Torelli group to H, which we will call the *Chillingworth homomorphism.* Thus, from our previous discussion, we can now write $C \circ \tau = t$.

Using the map k described at the beginning of Section 3.3, Morita establishes that crossed homomorphisms of the form $\mathcal{M}_{g,b} \to H$ are essentially unique.

Theorem 3.6 (Morita, [37]) *$H^1(\mathcal{M}_{g,1}; H^1(S_{g,1})) \cong \mathbf{Z}$, with generator k.*

Combining Theorem 3.6 with Theorem 3.2, we have therefore established that the Chillingworth homomorphism t extends essentially uniquely to the full mapping class group (up to coboundary and sign). More precisely, thinking of e_X as the extension of t to $\mathcal{M}_{g,1}$, we have:

Proposition 3.7 *$k = e_X$, as elements of $H^1(\mathcal{M}_{g,1}; H^1(S_{g,1}))$.*

We note that Theorem 6.1 of [38] is the special case of Proposition 3.7 when the crossed homomorphisms are restricted to $\mathcal{I}_{g,1}$, and that the proof utilizes the bijection given in Equation 10.

We have established that Trapp's crossed homomorphism e_X which measures the action of a mapping class on winding numbers is essentially equal to Morita's crossed homomorphism k, a fact first realized by Trapp in [46]. The result here is not new, since both Trapp and Morita realized that the Chillingworth homomorphism extended to the entire mapping class group, and Trapp realized that his crossed homomorphism was "essentially" the same as Morita's. Besides pinning down the connections, our goal is to see the geometric interpretation of each representation. What Morita does not explicitly state is that k carries winding number information not only for the Torelli group, but for the whole mapping class group. Thus it is possible to extract this winding number information directly from Morita's representation.

3.4 Trapp's Representation

Trapp uses his crossed homomorphism e_X to construct a linear representation $T_X : \mathcal{M}_{g,1} \to GL(2g+1, \mathbf{Z})$ given as follows:

$$T_X(f) = \begin{pmatrix} 1 & e_X(f) \\ 0 & \rho(f) \end{pmatrix}$$

where X is a given nonsingular vector field on $S_{g,1}$. We again let $\rho : \mathcal{M}_{g,1} \to \mathrm{Sp}(2g, \mathbf{Z})$ denote the symplectic representation, dropping the index 2 introduced for other purposes in Section 3.2. The 0 in the matrix denotes a column of $2g$ zeros. We note that Trapp's representation can be factored through $\mathcal{M}_{g,0}$, though we do not address it here, and that Patricia Sipe previously discovered this version of the representation in [44].

Remark 3.8 Until now, we have ignored the issue of the choice of the vector field X. Trapp proves that e_X does not depend on X when restricting to $\mathcal{I}_{g,1}$, and moreover, that if X and X' are different vector fields on $S_{g,1}$, then T_X and $T_{X'}$ are conjugate representations. We henceforth denote the above representation simply by T. It follows from Morita's work that outside of $\mathcal{I}_{g,1}$, different choices of X should correspond to varying the crossed homomorphism k by a coboundary.

Trapp's definition of this representation is particularly nice because it interprets Morita's purely algebraic constructions explicitly in terms of geometry, and also because it linearizes Morita's representation. Trapp also proves that $T(f)$ can be understood to measure the acton of Df on the first homology with $\mathbf{Z}$ coefficients of the unit tangent bundle of the surface (Theorem 2.2, [46]).

We claim that $\mathrm{Im}\, T$ is a split extension of $\mathrm{Sp}(2g, \mathbf{Z})$ by $2H \cong H$. If $\pi : \mathrm{Im}\, T \to \mathrm{Sp}(2g, \mathbf{Z})$ is the obvious map which picks out the $2g \times 2g$ lower right-hand block, then we have the following short exact sequence

$$0 \longrightarrow \ker \pi \longrightarrow \mathrm{Im}\, T \longrightarrow \mathrm{Sp}(2g, \mathbf{Z}) \longrightarrow 1.$$

The splitting is given in the obvious way, by sending a matrix $A \in \mathrm{Sp}(2g, \mathbf{Z})$ to the matrix

$$\begin{pmatrix} 1 & 0 \\ 0 & A \end{pmatrix}.$$

Also, Proposition 2.8 of [46] tells us that $\ker \pi \cong 2H$, which establishes our claim. Trapp addresses the issue of splittings only in the case where he factors T through $\mathcal{M}_{g,0}$.

3.5 Perron's Representation

We now describe a completely different approach to the linearization of Morita's representation given by Bernard Perron in [41].

To any graph Γ we will associate a particular Artin group, denoted $A(\Gamma)$. To each vertex of Γ we associate a generator of $A(\Gamma)$, and we shall abuse notation by using the same symbol to denote both. If x and y are two vertices of Γ bounding a common edge, then the corresponding generators "braid", that is, $xyx = yxy$. If no such edge exists, then x and y commute.

Remark 3.9 As noted, we are only concerned here with one specific kind of Artin group, but the reader is probably familiar with a much more general construction. There are different conventions for using Coxeter graphs to present an Artin group. Here we take the definition used in [10]. A Coxeter graph consists of vertices and labelled edges, with edge labels taken from the set $\{3, 4, \ldots, \infty\}$. Then the associated Artin group has a presentation in which generators correspond to vertices, and relations correspond to edges as follows. If s, t are vertices bounding a common edge with label $m_{s,t}$, then we have the relation $ststs \cdots = tstst \cdots$, where the word on each side of the equation has length $m_{s,t}$. If $m_{s,t} = \infty$, there is no relation. If s, t do not bound a common edge, then we have the relation $st = ts$. In our case, then, any edge in our graph would have the label 3, corresponding to the braid relation $sts = tst$. Hence, for simplicity in this special case, we are simply dropping the labels.

Example 3.10 Let A_n be the graph in Figure 3. Then $A(A_n) = B_{n+1}$, where B_k

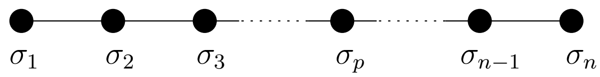


Figure 3: The graph A_n

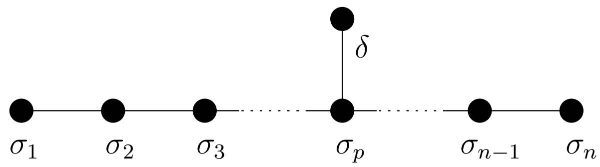


Figure 4: The graph $E_{n+1,p}$

denotes the braid group on k strands.

The graph A_n embeds in the graph $E_{n+1,p}$ shown in Figure 4. This inclusion induces an injection on the corresponding groups: $B_{n+1} \to A(E_{n+1,p})$ [41]. The connection to mapping class groups comes when we let $n = 2g$ and $p = 4$, for Matsumoto has given $\mathcal{M}_{g,1}$ explicitly as a quotient of $A(E_{2g+1,4})$ [34].

This connection between $A(E_{n+1,p})$ and $\mathcal{M}_{g,1}$ is easy to see if we look at Humphries' generating set for $\mathcal{M}_{g,1}$, which consists of Dehn twists about the curves shown in Figure 5. Recall that Dehn twists about disjoint curves will commute and that Dehn twists about curves which intersect once will braid. Then each vertex $\sigma_i \in A(E_{2g+1,4})$ naturally corresponds to the Dehn twist about the curve C_i, and δ corresponds to the second meridian curve B. Hence Perron's strategy is to begin with a representation of the braid group, extend it to $A(E_{n+1,p})$, and then factor it through $\mathcal{M}_{g,1}$.

Perron begins with the well-known (reduced) Burau representation

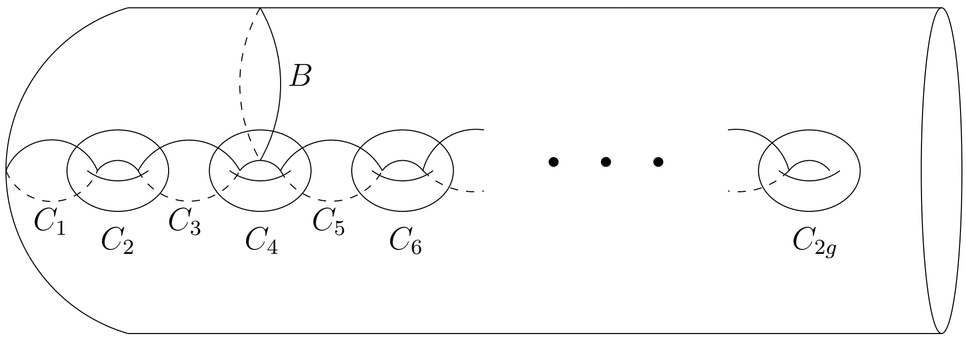


Figure 5: Humphries' generating set for $\pi_1(S_{g,1})$

$B_{n+1} \longrightarrow \mathrm{GL}_n(\mathbf{Z}[t, t^{-1}])$, defined as follows:

$$\sigma_i \mapsto J_i = \begin{pmatrix} I_{i-2} & 0 & 0 \\ & & \\ 0 & \begin{matrix} 1 & 0 & 0 \\ t & -t & 1 \\ 0 & 0 & 1 \end{matrix} & 0 \\ & & \\ 0 & 0 & I_{n-i-1} \end{pmatrix},$$

where I_k stands for the $k \times k$ identity matrix. We note that in the case $i = 1, n$, we have, respectively:

$$\sigma_1 \mapsto J_1 = \begin{pmatrix} \begin{matrix} -t & 1 \\ 0 & 1 \end{matrix} & 0 \\ & \\ 0 & I_{n-2} \end{pmatrix}, \quad \sigma_n \mapsto J_n = \begin{pmatrix} I_{n-2} & 0 \\ & \\ 0 & \begin{matrix} 1 & 0 \\ t & -t \end{matrix} \end{pmatrix}.$$

The problem is that the Burau representation does not extend to $A(E_{n+1,p})$ when $p = 4$. Therefore Perron extends the Burau representation slightly. Let R_i denote an $n \times n$ block of zeros with a t placed in the $(i, i)^{th}$ position. It is easy to check that the map

$$\begin{aligned} B_{n+1} &\rightarrow \mathrm{GL}_{2n}(\mathbf{Z}[t, t^{-1}]) \\ \sigma_i &\mapsto \begin{pmatrix} I_n & 0 \\ R_i & J_i \end{pmatrix} \end{aligned}$$

is a well-defined representation, which Perron refers to as the *Burau bis* representation.

The Burau bis representation extends to $A(E_{n+1,p})$ for all possible values of n and p, in the following way. Let $\vec{b} = \begin{pmatrix} b_1 \\ \cdot \\ \cdot \\ \cdot \\ b_n \end{pmatrix}$, $\vec{d} = \begin{pmatrix} d_1 \\ \cdot \\ \cdot \\ \cdot \\ d_n \end{pmatrix}$, and $\lambda = (\lambda_1, \ldots, \lambda_n)$. We define the following $n \times n$ matrices:

$$\begin{aligned} A &= \begin{pmatrix} \lambda_1\vec{b} & \lambda_2\vec{b} & \cdots & \lambda_n\vec{b} \end{pmatrix} \\ B &= \begin{pmatrix} \vec{0} & \cdots & \vec{0} & \vec{b} & \vec{0} & \cdots & \vec{0} \end{pmatrix} \\ C &= \begin{pmatrix} \lambda_1\vec{d} & \lambda_2\vec{d} & \cdots & \lambda_n\vec{d} \end{pmatrix} \\ D &= \begin{pmatrix} \vec{0} & \cdots & \vec{0} & \vec{d} & \vec{0} & \cdots & \vec{0} \end{pmatrix}, \end{aligned}$$

where $\vec{0}$ denotes a column of n zeros. Let us further assume that the b_i, $i = 1, \ldots, n$ satisfy the following conditions:

$$\begin{aligned} tb_i &= -td_{i-1} + (1+t)d_i - d_{i+1}, \quad i \neq p \\ tb_p &= -td_{p-1} + (1+t)d_p - d_{p+1} + t \\ \sum_{i=1}^{n} \lambda_i b_i &= -(1 + d_p + t), \end{aligned}$$

setting any undefined d_j equal to zero. If we make the assignments

$$\begin{aligned} \sigma_i &\mapsto \begin{pmatrix} I_n & 0 \\ R_i & J_i \end{pmatrix} \\ \delta &\mapsto \begin{pmatrix} I_n + A & B \\ C & I_n + D \end{pmatrix}, \end{aligned}$$

then for each choice of λ, we get a linear representation $\psi_\lambda : A(E_{n+1,p}) \to \mathrm{GL}_{2n}(R)$, where we can take R to be the field of rational fractions in $n+1$ indeterminates $Q(t, d_1, \ldots, d_n)$ (Proposition 2.2 of [41]). Perron then shows that one can obtain a representation of $\mathcal{M}_{g,1}$ by making choices for t and λ.

Proposition 3.11 (Perron, [41]) *The representation ψ_λ factors through $\mathcal{M}_{g,1}$ if and only if we have*

$$\begin{aligned} t &= -1 \\ \lambda_1 &= \lambda_3 = -1 \\ \lambda_i &= 0, \quad i \neq 1, 3. \end{aligned}$$

Thus we get a linear representation $\psi : \mathcal{M}_{g,1} \to \mathrm{GL}_{4g}(\mathbf{Z}[d_1, \ldots, d_n])$

Immediately we can see that ψ is likely to be connected with the symplectic representation $\rho : \mathcal{M}_{g,1} \to \mathrm{Sp}(2g, \mathbf{Z})$, since it is well known that setting $t = -1$ in the Burau representation will give us the associated action on homology for the σ_i. In fact, Perron is able to prove the following:

Theorem 3.12 (Perron, [41]) *The image of ψ is a non-split extension of $Sp(2g, \mathbf{Z})$ by $2H$. However, Im ψ embeds as a finite index subgroup in $H \rtimes Sp(2g, \mathbf{Z})$. Furthermore, ψ restricted to $\mathcal{I}_{g,1}$ is precisely equal to the Chillingworth homomorphism t.*

Thus we have another representation linearizing Morita's representation, this time coming from representations of Artin groups. We therefore could hope to "read off" both the symplectic information as well as the winding number information straight from the matrices in the image of ψ. The symplectic information is easy to extract, and it turns out that the winding number information will take only a bit more work.

Even though Im ψ doesn't split, the embedding Im $\psi < H \rtimes \mathrm{Sp}(2g, \mathbf{Z})$ induces a crossed homomorphism from $\mathcal{M}_{g,1}$ into H. By Theorem 3.6, we know that the crossed homomorphism induced by ψ must be equivalent to k and hence to e_X, up to an integer multiple. Therefore the winding number information must somehow be contained in Morita's representation. The last statement in Theorem 3.12 tells us where to look for it.

We recall that $\mathrm{Im}\ \psi < \mathrm{GL}_{4g}(\mathbf{Z}[d_1, \ldots, d_n])$. Let $\bar{\rho} : \mathrm{Im}\ \psi \to \mathrm{Sp}(2g, \mathbf{Z})$ be the map defined by extracting the lower right-hand $2g \times 2g$ block from the matrix and by setting $d_1 = d_3 = 1$ and $d_i = 0$ for $i \neq 1, 3$. Then Perron shows that the following diagram of short exact sequences commutes, and thus application of $\bar{\rho}$ yields the symplectic data.

$$\begin{tikzcd}
1 \arrow[r] & \mathcal{I}_{g,1} \arrow[r, "i"] \arrow[d, "t"] & \mathcal{M}_{g,1} \arrow[r, "\rho"] \arrow[d, "\psi"] & \mathrm{Sp}(2g, \mathbf{Z}) \arrow[r] \arrow[d, "\mathrm{id}"] & 1 \\
0 \arrow[r] & 2H \arrow[r, "\xi"] & \mathrm{Im}\ \psi \arrow[r, "\bar{\rho}"] & \mathrm{Sp}(2g, \mathbf{Z}) \arrow[r] & 1
\end{tikzcd}$$

If we want to understand how winding number information is contained in the representation ψ, we must therefore understand the injection $\xi : H \to \mathrm{Im}\ \psi$. For an explicit definition of ξ, see [41]. For our purposes, it suffices to know that for $m \in H$, $\xi(m)$ has the form $\begin{pmatrix} N_1 & m\vec{b} \\ N_2 & N_3 \end{pmatrix}$, where the N_i are $2g \times 2g$ blocks, $\vec{b}$ is a column vector of indeterminates dependent upon the d_i as previously defined, and $m\vec{b}$ denotes the $2g \times 2g$ matrix

$$\begin{pmatrix} m_1\vec{b} & \cdots & m_{2g}\vec{b} \end{pmatrix} = \begin{pmatrix} m_1 b_1 & \cdots & m_{2g} b_1 \\ m_1 b_2 & \cdots & m_{2g} b_2 \\ \vdots & & \vdots \\ m_1 b_{2g} & \cdots & m_{2g} b_{2g} \end{pmatrix}.$$

Let us now examine exactly how $\mathrm{Im}\ \psi$ embeds in $H \rtimes \mathrm{Sp}(2g, \mathbf{Z})$. Let δ^* denote

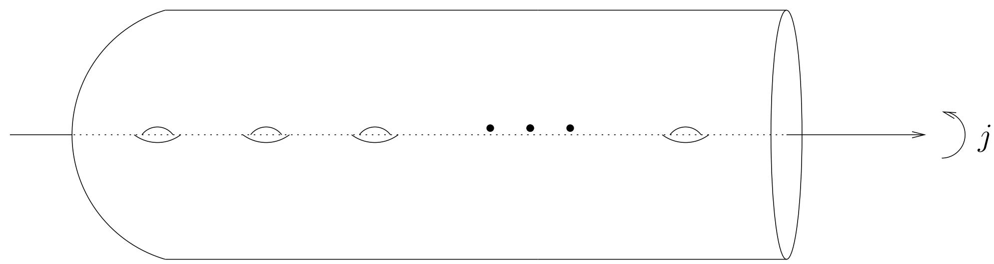


Figure 6: The involution j

the image under ψ of the Dehn twist about the curve $j(B)$, where j is rotation by π about the axis indicated in Figure 6. Let $\langle \mathrm{Im}\ \psi, \delta^* \rangle$ denote the subgroup of $\mathrm{GL}_{4g}(\mathbf{Z}[d_1, \ldots, d_n])$ generated by $\mathrm{Im}\ \psi$ and δ^*. Following Perron, we can extend the map $\bar{\rho} : \mathrm{Im}\ \psi \to \mathrm{Sp}(2g, \mathbf{Z})$ to a map $\bar{\rho}' : \langle \mathrm{Im}\ \psi, \delta^* \rangle \to \mathrm{Sp}(2g, \mathbf{Z})$ by setting

$$\bar{\rho}'(\delta^*) = \begin{pmatrix} \begin{matrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{matrix} & 0 \\ 0 & I_{2g-4} \end{pmatrix}.$$

Then it follows from Proposition 8.10 of [41] that we have another commutative diagram of exact sequences.

$$\begin{tikzcd}
0 \arrow[r] & 2H \arrow[r, "\xi"] \arrow[d, "\mathrm{incl}"] & \mathrm{Im}\ \psi \arrow[r, "\bar{\rho}"] \arrow[d, "\mathrm{incl}"] & \mathrm{Sp}(2g, \mathbf{Z}) \arrow[r] \arrow[d, "\mathrm{id}"] & 1 \\
0 \arrow[r] & H \arrow[r, "\xi"] & \langle \mathrm{Im}\ \psi, \delta^* \rangle \arrow[r, "\bar{\rho}'"] & \mathrm{Sp}(2g, \mathbf{Z}) \arrow[r] & 1
\end{tikzcd}$$

The lower sequence splits, and hence the above gives us the embedding of $\mathrm{Im}\ \psi$ in a semidirect product, as previously mentioned. The corresponding section S :

$\mathrm{Sp}(2g, \mathbf{Z}) \to \langle \mathrm{Im}\ \psi, \delta^* \rangle$ is given by

$$S(A) = \begin{pmatrix} I_g & 0 \\ R_A & A \end{pmatrix}.$$

The $g \times g$ matrix R_A is not relevant to our calculations; an explicit definition can be found in [41]. Any element in $H \rtimes \mathrm{Sp}(2g, \mathbf{Z})$, and hence any element in Im ψ, can be written uniquely as a product $\xi(m)S(A)$ for some $m \in H$ (actually, $m \in 2H$) and some matrix $A \in \mathrm{Sp}(2g, \mathbf{Z})$. Since the crossed homomorphism induced by ψ comes from the first factor, and since the first pair of commuting exact sequences tells us that ξ restricted to $\mathcal{I}_{g,1}$ is precisely the Chillingworth homomorphism t, $\xi(m)$ must carry the winding number information. A simple calculation now tells us how to "see" winding number information in Im ψ. Let $Y \in$ Im ψ. Then

$$Y = \xi(m)S(A) = \begin{pmatrix} N_1 & m\vec{b} \\ N_2 & N_3 \end{pmatrix} \begin{pmatrix} I_g & 0 \\ R_A & A \end{pmatrix}$$

for some $m \in H, A \in \mathrm{Sp}(2g, \mathbf{Z})$. Multiplying this out tells us that any $Y \in$ Im ψ is of the form

$$Y = \begin{pmatrix} N_1 + (m\vec{b})R_A & (m\vec{b})A \\ N_2 + N_3 R_A & N_3 A \end{pmatrix}.$$

Since ξ is an injection, we might as well think of m as carrying the winding number information. Then winding number information is actually embedded in many places in this matrix, as m also appears in all the N_i. But the easiest place to see it is the $2g \times 2g$ upper right-hand block. Since $A \in \mathrm{Sp}(2g, \mathbf{Z})$, A is invertible and $m\vec{b}$ can be calculated. To be more explicit, let $\tilde{\rho}$ denote the process of setting $d_1 = d_3 = 1$ and $d_i = 0$ for $i \neq 1, 3$. Therefore, if we have

$$\psi(f) = \begin{pmatrix} M_1 & M_2 \\ M_3 & M_4 \end{pmatrix},$$

then we can compute

$$m\vec{b} = M_3[\tilde{\rho}(A_4)]^{-1}.$$

By construction of the matrix $m\vec{b}$, we can then "read off" the vector m and the winding number information contained therein.

Remark 3.13 We first note that Perron's representation cannot be equivalent to Trapp's, since Im ψ does not split, but, like Morita's, construction, only embeds in a splitting. We also remind the reader that the vector describing winding number will depend on some choice of vector field X. The reader will note that the upper right-hand block in $\psi(\sigma_i)$ is 0 by definition for all $i = 1, \ldots, 2g$, and hence the winding number vector will be the zero vector for each, a fact which may surprise the reader (and indeed surprised the author). However, a careful application of Theorem 4.2 and Formula 1 in [9] (see Remark 3.4) shows that indeed a vector field X can be found on $S_{g,1}$ satisfying this condition. Moreover, the remaining generator of $\mathcal{M}_{g,1}$ does effect change on winding number, i.e., does not correspond to a zero block, and therefore winding number is still "interesting" on the full mapping class group.

For the sake of completeness, we must also mention certain linear representations of Artin groups found by Squier in [45]. As Trapp points out, his representation T is precisely the case where Squier's parameters satisfy a certain condition which allows Squier's map to factor through mapping class groups. In this context, then, the connection between Trapp's representation and Perron's construction is not so surprising.

In summary, therefore, we have described three representations of $\mathcal{M}_{g,1}$, each arising in a very different context and employing vastly different methods of construction, from Mal'cev completions to actions on tangent bundles to quotients of Artin groups. The amazing fact is that each representation contains essentially the same information. We hope that the above discussion has served the purpose of making explicit the connections between these three representations, and, since the author has a bias towards the geometric interpretation, we hope that it is now clear how the geometric information explicitly given in Trapp's representation is encoded in Morita and Perron's representations. We also emphasize the important role played here by the Torelli group, especially by Johnson's map $\tau : \mathcal{I}_{g,1} \to \Lambda^3 H$, which motivates and lays the groundwork for Morita's original construction. An attempt to understand the Torelli group better will make up the final section of this thesis.

Before closing Section 3, however, we make a few remarks concerning the potential application of representations of mapping class groups to the classification problem for Heegaard splittings of a given closed and orientable 3-manifold M. A little background will be necessary at this point. A *Heegaard splitting* of M is a decomposition of a 3-manifold into two handlebodies of genus g , F_1 and F_2, such that $F_1 \cap F_2 = \partial F_1 = \partial F_2$, and $F_1 \cup F_2 = M$. We can also construct Heegaard splittings by beginning with the two handlebodies and specifying a "gluing" homeomorphism $\phi : \partial F_1 \to \partial F_2$, producing a 3-manifold M. We denote this as follows: $M = F_1 \cup_\phi F_2$. Isotopic homeomorphisms yield homeomorphic 3-manifolds; thus we may simply specify the mapping class of ϕ in $\mathcal{M}_{g,0}$. Two Heegaard splittings of a given 3-manifold M are equivalent if there exists an isotopy of M taking one splitting surface to the other. We note that the gluing map is necessarily an element of $\mathcal{M}_{g,0}$, while the previous discussion focused on representations of $\mathcal{M}_{g,1}$. However, each of the above representations factors through $\mathcal{M}_{g,0}$, though the details were not relevant to our discussion here.

Equivalence classes of Heegaard splittings correspond to double cosets in $\mathcal{M}_{g,0}$ mod $\mathcal{H}_{g,0}$, where $\mathcal{H}_{g,0}$ denotes the *handlebody subgroup*, which is the subgroup of $\mathcal{M}_{g,0}$ consisting of mapping classes which extend to a homeomorphism of a fixed handlebody whose boundary is the surface $S_{g,0}$. Invariants of these double cosets, or of any homomorphic image of these double cosets, will be invariants of Heegaard splittings. For example, Birman found invariants of Heegaard splittings arising from double cosets under the symplectic representation [5].

The process of adding a trivial "handle" to a Heegaard splitting is known as *stabilization*. In other words, we add a 1-handle to each handlebody and glue them by mapping the longitude of one to the meridian of the other. It is known that given any two Heegaard splittings of any 3-manifold M, only a finite number of stabilizations are required before the two splittings become equivalent. In fact, there are no known examples in which more than one stabilization is necessary. The obvious question to ask is then, does it only require one stabilization in general to render two inequivalent splittings equivalent?

Birman's invariant has one crucial shortcoming. It is always trivial on a stabilized Heegaard splitting. A new, better invariant, would almost certainly shed light on the second question raised above. Fascinating examples have been developed recently by Moriah and separately by Menasco which may provide counterexamples, but these examples must wait for an invariant.

In order to calculate invariants of these double cosets under some representation, one must first understand the image of $\mathcal{H}_{g,0}$. Birman's success in using the symplectic representation to develop invariants of Heegaard splittings comes from the fact that $\rho(\mathcal{H}_{g,0})$ has a simple, algebraic description which lends itself to calculations. Unfortunately, in general, the image of $\mathcal{H}_{g,0}$ is quite difficult to characterize, but a thorough study of $\mathcal{I}_{g,0}$ might be of use here. For example, the Trapp/Sipe representation is an excellent candidate for obtaining a "nice" image of $\mathcal{H}_{g,0}$, since, after all, it is a quotient of the symplectic representation together with one extra row of winding number data. One can nearly characterize double cosets mod $\mathcal{H}_{g,0}$ under the Trapp/Sipe representation, but one cannot complete the calculation without knowing $\mathcal{H}_{g,0} \cap \mathcal{I}_{g,0}$. Given the important role frequently played by the Torelli group in other representations, one suspects the similar information would eventually be necessary in calculating double cosets there as well.

4 Relations in the Torelli Group

Obtaining information about the mapping class group of a surface from its known representations is a difficult problem, so we now turn our attention to one of the most important subgroups of the mapping class group, namely, the Torelli group. We continue with the notation of Section 3 by letting $S_{g,b}$ denote $S_{g,b,0}$, with similar notation for the corresponding mapping class groups and Torelli groups.

Powell gave an infinite set of generators of the Torelli group consisting of two kinds of maps [42]. The first type are so-called *bounding pair maps* (BP-maps for short). Let T_γ denote a Dehn twist about a curve γ. A BP-map is a product of Dehn twists $T_\alpha T_\beta^{-1}$, where the curves α and β are each non-separating but together bound a subsurface. The second type are commonly called BSCC-maps, short for *bounding simple closed curves.* It is interesting to note that the subgroup of $\mathcal{I}_{g,1}$ generated by BSCC-maps is precisely the kernel of Johnson's homomorphism τ [24]. However, it is BP-maps which play the key role in generating $\mathcal{I}_{g,1}$. Johnson was able to show in [26] that BP-maps whose curves bound a genus 1 subsurface generate $\mathcal{I}_{g,0}$ and $\mathcal{I}_{g,1}$ for $g \geq 3$ (again, an infinite set), before proving in [22] that a certain finite set of BP-maps of various genus would suffice (thus settling a problem first raised in Kirby's problem list [28]). The question of whether the Torelli group is finitely presentable remains open, however. Thus we approach this problem by identifying relations amongst Johnson's finite generating set.

4.1 Johnson's finite generating set

Johnson's finite generating set for $\mathcal{I}_{g,1}$ is constructed to be aligned closely with Humphries' generating set for $\mathcal{M}_{g,1}$, which consists of Dehn twists about the curves pictured in Figure 5 [18]. Please note that we shall often abuse notation by confusing a curve with the Dehn twist about that curve. Johnson showed in [22] that these $2g+1$ Dehn twists generate $\mathcal{M}_{g,0}$ as well. For simplicity, we shall restrict our attention to the surface $S_{g,1}$, but similar results hold for $S_{g,0}$ in all that follows.

We recall Johnson's definition of a chain on a surface as presented in [22]. A *chain* is an ordered collection of oriented simple closed curves $(c_1, \ldots, c_n)$ on $S_{g,1}$ such that:

1. $c_i \cap c_{i+1}$ transversely at a single point,
2. $c_i \cdot c_{i+1} = 1$, where $a \cdot b$ denotes the algebraic intersection, and
3. $c_i \cap c_j = \emptyset$ if $|i-j| > 1$.

The term *n-chain* will refer to a chain with n oriented curves. Two examples of chains are shown in Figures 7 and 8. The $2g$-chain in Figure 7 is known as a *straight chain* and consists of all the Humphries generators except for the second meridian B. Figure 8 shows a $(2g-3)$-chain known as a *β-chain.* Note that the curve β in Figure 8 is simply the result of applying the Humphries map B to the curve C_4, i.e., $B * C_4 = \beta$. (Throughout this section, $*$ will denote conjugation, i.e., $a * b = aba^{-1}$.)

Johnson's notion of a subchain is much more general than the obvious idea of taking a consecutive subset of a chain. We define the sum of two oriented curves which intersect transversally to be the oriented curve resulting from "smoothing out" points of intersection, as shown in Figure 9. Then if $(c_1, \ldots, c_n)$ is chain, $(c_1, \ldots, c_{i-1}, c_i +$

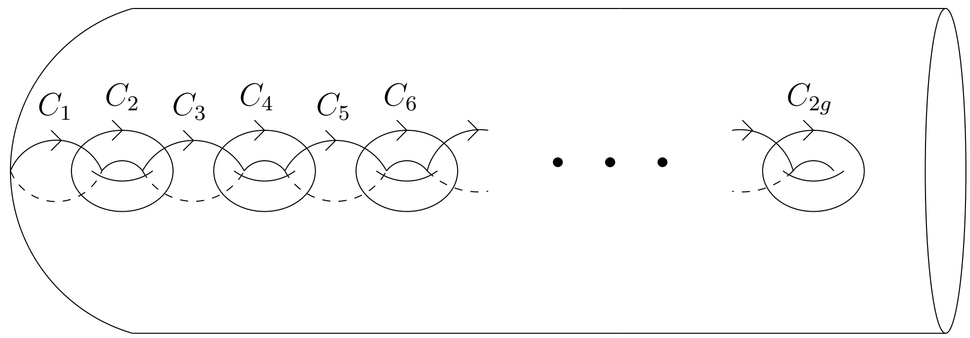


Figure 7: Straight chain on $S_{g,1}$

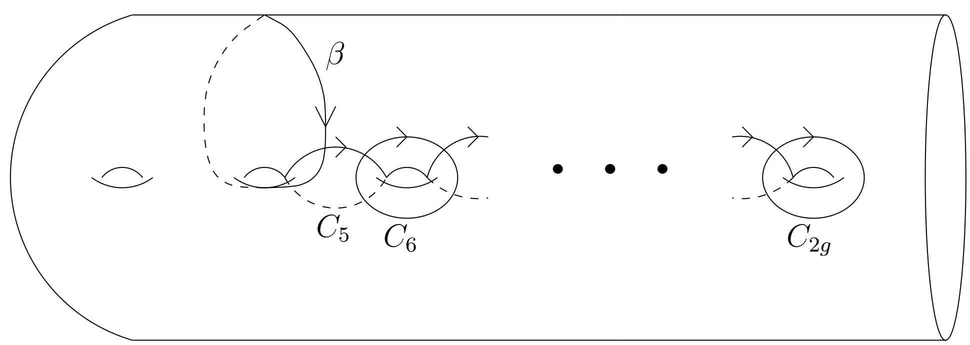


Figure 8: β-chain on $S_{g,1}$

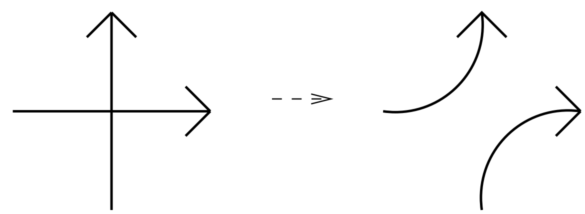


Figure 9: Summing oriented curves by smoothing out intersections

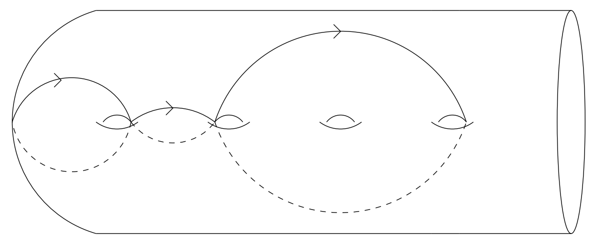


Figure 10: The straight 3-subchain (1349) on $S_{4,1}$

$\cdots + c_j, c_{j+1}, \ldots, c_n)$ is also a chain. Any chain obtained from another chain after a finite number of such summations will be called a *subchain* of the original chain. We will only be concerned with *straight subchains* and *β-subchains*.

For example, $(C_1 + C_2, C_3, C_4 + C_5 + C_6 + C_7 + C_8)$ is a straight 3-subchain map for $g \geq 4$ (see Figure 10). We shall require a shorthand notation for subchains of our straight chains and β-chains. We use i to denote the curve C_i and β for the curve β. A consecutive sequence of numbers $(i, \ldots, j)$ will stand for the consecutive subchain $(C_i, \ldots, C_j)$. A gap in a sequence of numbers $(i_1, \ldots, i_j, i_k, \ldots, i_n)$, where $i_k \neq i_j + 1$, will indicate the subchain $(C_{i_1}, \ldots, C_{i_j} + C_{i_j+1} + \cdots + C_{i_k-1}, C_{i_k}, \ldots, C_{i_n})$. Thus, the straight subchain given above would be denoted (1349). Note that in this example, the curve C_9 is not included in the subchain. In fact, in the case $g = 4$, no such curve exists in the straight chain. In this case the number 9 merely serves as a "cut-off",

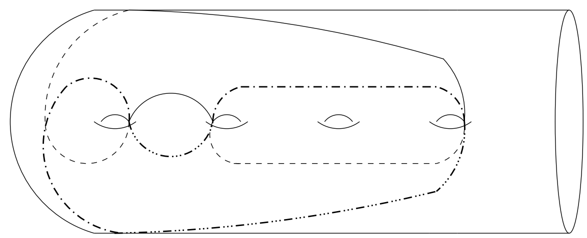


Figure 11: The straight 3-subchain map [1349] on $S_{4,1}$

an indication that the last curve included in the summation is C_8.

In order to obtain elements of the Torelli group from these subchains, we take a regular neighborhood K of an n-chain. If n is odd, K has two boundary components, α and β (choose α to lie to the left of the odd-indexed curves $c_1, c_3, \ldots$ in the chain). By construction, the product $T_\alpha T_\beta^{-1}$ is a BP-map of genus $\frac{n-1}{2}$ and hence an element of $\mathcal{I}_{g,1}$. We call such a map a *chain map* and denote it by replacing the parentheses in the chain notation with brackets. For example, the chain map associated to the 3-subchain given in the proceeding paragraph will be denoted [1349]. The curves which define this BP-map are shown in Figure 11. Note that Figure 11 shows two distinct curves. One curve is solid on the "top" of the surface and dashed when it travels underneath the surface. The second curve is dashed with one dot on the top and dashed with three dots underneath. The reader should take a moment to convince himself or herself that these two curves are disjoint (or at least are isotopic to disjoint curves), though they appear to cross when simultaneously traveling into a hole.

We are now able to state Johnson's result.

Theorem 4.1 (Johnson, [22]) *The odd straight-subchain maps together with the odd β-chain maps generate $\mathcal{I}_{g,1}$ and $\mathcal{I}_{g,0}$ for $g \geq 3$.*

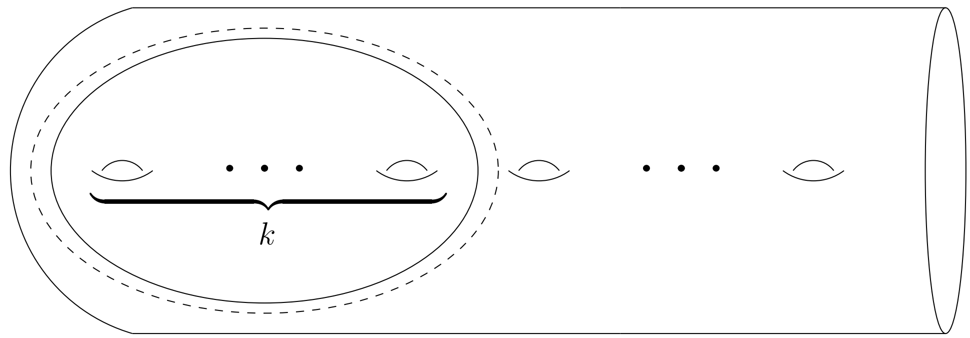


Figure 12: The curves defining W_k.

Remark 4.2 The order of Johnson's generating set is exponential in g. More precisely, there are $9 \cdot 2^{2g-3} - 4g^2 + 4g - 5$ Johnson generators for $\mathcal{I}_{g,1}$ and $9 \cdot 2^{2g-3} - 4g^2 + 2g - 6$ for $\mathcal{I}_{g,0}$. Johnson previously obtained a lower bounds of $\frac{1}{3}(4g^3 + 5g + 3)$ and $\frac{1}{3}(4g^3 - g)$ for the number of generators for $\mathcal{I}_{g,1}$ and $\mathcal{I}_{g,0}$, respectively [21].

There is one particularly important type of Johnson generator which will appear so often that we introduce some special notation for it. We therefore let W_k denote the consecutive straight $(2k-1)$-subchain map $[234\cdots(2k+1)]$. The curves defining W_k are shown in Figure 12. We note that W_g is clearly in the kernel of the natural map $\mathcal{I}_{g,1} \to \mathcal{I}_{g,0}$ obtained by gluing in a disk along the boundary component of $S_{g,1}$.

4.2 Lantern relations in the Torelli group

Johnson discovered relations amongst elements of the Torelli group. The main tool he used to construct such relations is commonly known as the *lantern relation* in the mapping class group. This relation was first discovered by Dehn in [11] and independently described by Johnson in [22]. The relation is carried by the surface

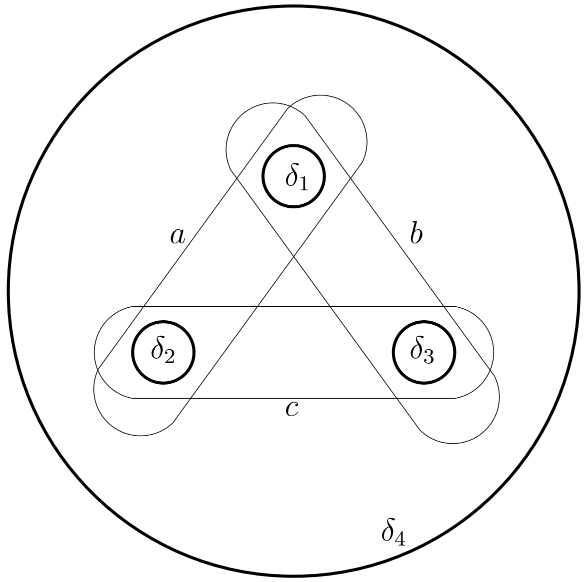


Figure 13: The lantern relation on $S_{0,4}$

$S_{0,4}$, and for the curves $a, b, c, \delta_1, \delta_2, \delta_3$, and δ_4 as shown in Figure 13, we have the following relation amongst the corresponding Dehn twists:

$$T_c T_b T_a = T_{\delta_1} T_{\delta_2} T_{\delta_3} T_{\delta_4}$$

Clearly the left-hand side of the equation can be cyclically permuted. Also note that for $i = 1, \ldots, 4$, T_{δ_i} commutes with T_γ for all other curves γ in the relation since the δ_i are disjoint from all other curves.

4.2.1 Johnson's B-relations

Johnson realized that he could exploit the symmetry of certain chain maps in order to get relations in the Torelli group out of lantern relations in the mapping class group. We give a simple example of his method before stating the families of relations he

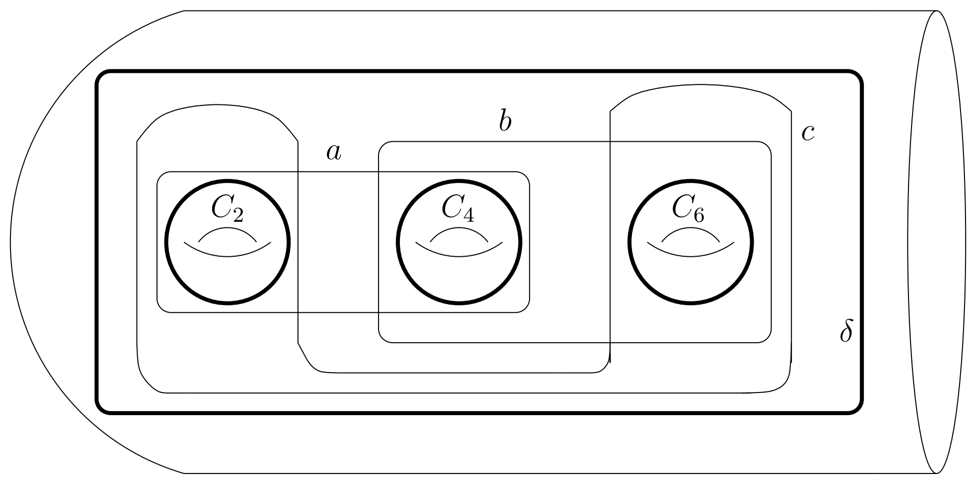


Figure 14: A lantern relation on $S_{3,1}$

found in $\mathcal{I}_{g,1}$.

Referring to Figure 14, we see that the curves in bold, C_2, C_4, C_6 and δ, together bound an $S_{0,4}$. Thus the curves a, b, and c complete the lantern relation, and (continuing our abuse of notation in the case of the C_i), we obtain:

$$T_c T_b T_a = C_2 C_4 C_6 T_\delta. \tag{13}$$

On the "bottom" of this surface, we will have another such relation. Let μ be reflection through the plane of the page, and let γ' denote $\mu(\gamma)$. After inverting each side of the relation, our "reflected" lantern relation becomes:

$$T_{c'}^{-1} T_{b'}^{-1} T_{a'}^{-1} = C_2^{-1} C_4^{-1} C_6^{-1} T_{\delta'}^{-1}. \tag{14}$$

We now combine Equations 13 and 14, noting that Dehn twists about disjoint curves commute, to obtain:

$$T_c T_{c'}^{-1} T_b T_{b'}^{-1} T_a T_{a'}^{-1} = T_\delta T_{\delta'}^{-1}.$$

To simplify notation, we let A_γ denote the product $T_\gamma T_{\gamma'}^{-1}$. Thus we can rewrite our equation as follows:

$$A_c A_b A_a = A_\delta.$$

We observe that three of the four BP-maps in this relation are straight 3-subchain maps (the reader is referred to Appendix A for a complete list of Johnsons's generators for genus 3). Thus we can further rewrite our equations:

$$\begin{aligned} A_c[4567][2345] &= [234567] \\ A_c[4567]W_2 &= W_3 \end{aligned} \tag{15}$$

Now, A_c is certainly a BP-map, and hence we have obtained a relation in the Torelli group. However, A_c is not a Johnson generator, and hence we would like to eliminate it somehow. Johnson's method for dealing with this issue is to conjugate Equation 15 above by B, the Dehn twist corresponding to the second meridian of the Humphries generating set for $\mathcal{M}_{g,1}$. After doing so, we obtain:

$$A_c[\beta 567]B * W_2 = B * W_3.$$

Since B commutes with A_c, we can invert Equation 15 and combine with the above to obtain:

$$W_2^{-1}[4567]^{-1}[\beta 567]B * W_2 = B * W_3.$$

In this way, starting with the $S_{0,4}$ shown in Figure 15, Johnson obtains the following family of relations for $g \geq 3$ [22]:

$$W_2^{-1}[45\cdots(2g+1)]^{-1}[\beta 5\cdots(2g+1)]B * W_2 = [23\cdots(2g+1)]^{-1}B * [23\cdots(2g+1)]. \tag{16}$$

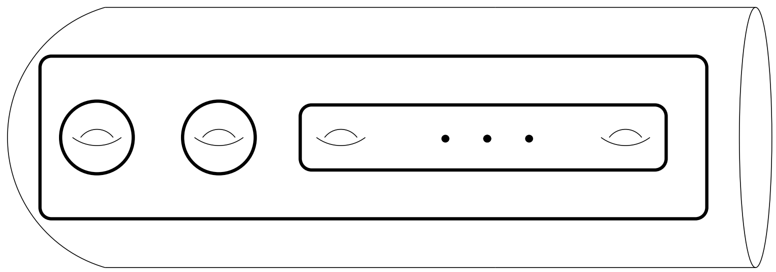


Figure 15: An $S_{0,4}$ on $S_{g,1}$

As it stands, this relation is key to Johnson's proof that his finite set of odd subchain maps generates $\mathcal{I}_{g,1}$. Moreover, we observe that by a lemma of Johnson [22], $B * W_2$ is a product of straight 3-subchains in $\mathcal{I}_{2,1}$. In addition, W_g lies in the kernel of the natural map $\mathcal{I}_{g,1} \to \mathcal{I}_{g,0}$. Therefore if we pass to $\mathcal{I}_{g,0}$, we have successfully obtained a relation amongst Johnson generators:

$$W_2^{-1}[45 \cdots (2g+1)]^{-1}[\beta 5 \cdots (2g+1)]B * W_2 = 1. \tag{17}$$

Though this relation is not explicitly given in terms of Johnson generators, Johnson points out that it allows us to eliminate the β-subchain map $[\beta 5 \cdots (2g+1)]$ in $\mathcal{I}_{g,0}$. In $\mathcal{I}_{3,0}$, this actually eliminates the only β-chain generator, leaving us with 35 straight 3-subchain generators and thus attaining the known lower bound on generators of $\mathcal{I}_{g,0}$ (see Appendix A). Relation 16 also enables us to find a minimal generating set for $\mathcal{I}_{g,1}$, using a different argument again due to Johnson, which we will present in the proof of Corollary 4.4 in the next section.

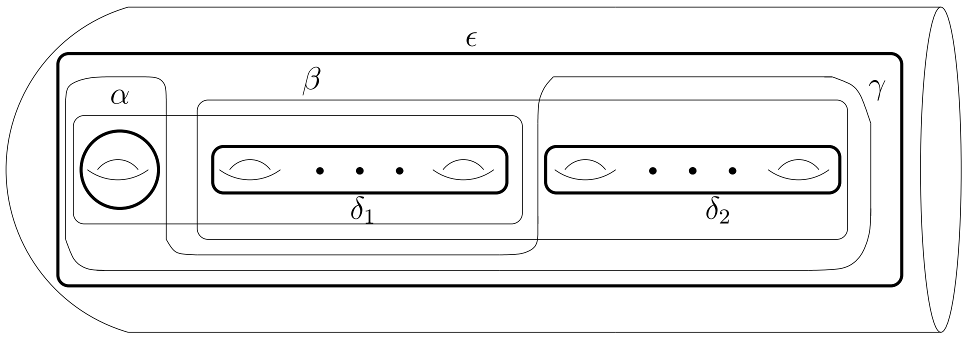


Figure 16: Another $S_{0,4}$ on $S_{g,1}$

4.2.2 Generalized B-relations

Despite its usefulness in realizing a minimal generating set for genus 3, Relation 16 is unsatisfactory on two counts. First, we do not know how to write $B * W_2$ explicitly as a product of Johnson generators, and secondly, we still do not have relations strictly amongst Johnson generators in $\mathcal{I}_{g,1}$. We shall now generalize Johnson's construction in such a way that addresses these two issues. We will do so first by generalizing Johnson's "B-relations", so called because of their dependence on the second meridian B.

The obvious generalization is to let the second "inside" boundary component of the $S_{0,4}$ contain as much genus as we like, as Johnson does with the third "inside" boundary component to obtain his Relation 16. In other words, we begin with the $S_{0,4}$ outlined in bold in Figure 16. To keep the pictures simple, we will not draw the image of the curves under reflection in the plane of the page, but these curves are to be understood when required.

Remark 4.3 Note that for such an $S_{0,4}$ the left-most "inside" boundary component of the $S_{0,4}$ must necessarily be the curve C_2 as in Figure 16, since otherwise our lantern curves will intersect the meridian B and the conjugation trick will not work.

We begin, as above, by writing down the two lantern relations corresponding to the curves in Figure 16 to obtain:

$$A_\gamma A_\beta A_\alpha = A_{\delta_1} A_{\delta_2} A_\epsilon. \tag{18}$$

We assume that the curve δ_1 encloses k' holes and that the curve δ_2 encloses at least one hole. The BP-map A_γ is the only one in the relation which is not a Johnson generator, and γ is disjoint from the curve B. Thus we are in a position to apply the B-trick described in Section 4.2.1, and after conjugating Equation 18 by B, we obtain:

$$A_\gamma (B * A_\beta)(B * A_\alpha) = (B * A_{\delta_1}) A_{\delta_2} (B * A_\epsilon).$$

Inverting Equation 18 and combining with the above yields:

$$A_\alpha^{-1} A_\beta^{-1} (B * A_\beta)(B * A_\alpha) = A_\epsilon^{-1} A_{\delta_1}^{-1} (B * A_{\delta_1})(B * A_\epsilon)$$

since A_{δ_2} commutes with everything.

We can now make the following substitutions:

$$\begin{aligned}
A_\alpha &= W_{k'+1} = [23 \cdots (2k'+3)] \\
A_\beta &= [45 \cdots (2g+1)] \\
A_{\delta_1} &= [45 \cdots (2k'+3)] \\
A_\epsilon &= W_g = [23 \cdots (2g+1)].
\end{aligned}$$

For $1 \leq k' \leq g-2$, we thus have in $\mathcal{I}_{g,1}$:

$$\begin{aligned} & W_{k'+1}^{-1}[45\cdots(2g+1)]^{-1}[\beta 5\cdots(2g+1)](B*W_{k'+1}) \\ = \; & W_g^{-1}[45\cdots(2k'+3)]^{-1}[\beta 5\cdots(2k'+3)](B*W_g). \end{aligned} \tag{19}$$

In $\mathcal{I}_{g,0}$, W_g drops out, leaving a somewhat simpler relation:

$$W_{k'+1}^{-1}[45\cdots(2g+1)]^{-1}[\beta 5\cdots(2g+1)](B*W_{k'+1}) = [45\cdots(2k'+3)]^{-1}[\beta 5\cdots(2k'+3)]$$

To simplify the notation a bit, we let $k = k'+1$. We also let P_n denote the product $[45\cdots(2n+1)]^{-1}[\beta 5\cdots(2n+1)]$. We can also understand P_n as the commutator $[[45\cdots(2n+1)], B^{-1}]$, where $[a,b] = a^{-1}b^{-1}ab$. Note, however, that since $B \notin \mathcal{I}_{g,1}$, P_n lies in the commutator subgroup of the full mapping class group, but not necessarily in the commutator subgroup of $\mathcal{I}_{g,1}$. Our relation in $\mathcal{I}_{g,1}$ now becomes:

$$W_k^{-1}P_g(B*W_k) = W_g^{-1}P_k(B*W_g).$$

We shall refer to this relation as $\mathrm{Rel}(g,k)$, with $2 \leq k \leq g-1$.

Corollary 4.4 *The Johnson generators* $[12\cdots(2q)]$ *can be eliminated for* $3 \leq q \leq g$

Proof. The proof follows Johnson's argument for eliminating the map [123456] from the generating set of $\mathcal{I}_{3,1}$ given in Section 5 of [22]. According to Lemma 5 of [22], the straight $(2q-1)$-chain map $[12\cdots(2q)]$ can be written as a product of $B*W_q$ and other straight $(2q-1)$-chain maps in $\mathcal{I}_{q,1}$. But $\mathrm{Rel}(q,2)$ allows us to write $B*W_q$ in terms of other Johnson generators. Since $\mathcal{I}_{q,1}$ embeds in $\mathcal{I}_{q+1,1}$ (see the proof of Claim 2.4 in Section 2.3), we can therefore eliminate the generator $[12\cdots(2q)]$ for all $3 \geq q \geq g$. $\square$

A useful rewriting of $\mathrm{Rel}(g,k)$ is the following:

$$B * W_g = P_k^{-1} W_g W_k^{-1} P_g (B * W_k).$$

In $\mathrm{Rel}(g,k)$, everything is a Johnson generator or a product of Johnson generators with the exception of $B*W_g$ and $B*W_k$. In order to obtain relations strictly amongst Johnson generators, we again make use of the fact that $\mathcal{I}_{g,1}$ embeds in $\mathcal{I}_{g+1,1}$. If we let $k=2$ in $\mathrm{Rel}(g,k)$, two nice things happen. First of all, the relation simplifies since $P_2 = 1$. Secondly, as previously mentioned, $B * W_2$ is some product of straight 3-subchain maps in $\mathcal{I}_{2,1}$, and so we shall at least obtain some relation amongst Johnson generators, if not an explicit one. We therefore eliminate $B * W_g$ using $\mathrm{Rel}(g,2)$:

$$B * W_g = W_g W_2^{-1} P_g (B * W_2).$$

Similarly, we can eliminate $B * W_k$ using $\mathrm{Rel}(k,2)$:

$$B * W_k = W_k W_2^{-1} P_k (B * W_2).$$

Note that in using $\mathrm{Rel}(k,2)$ we are implicitly assuming that $2 \leq k-1$ and hence that $3 \leq k \leq g-1$. We now substitute $\mathrm{Rel}(k,2)$ and $\mathrm{Rel}(g,2)$ into $\mathrm{Rel}(g,k)$ to obtain:

$$W_k^{-1} P_g W_k W_2^{-1} P_k (B * W_2) = W_g^{-1} P_k W_g W_2^{-1} P_g (B * W_2).$$

This is now a relation amongst Johnson generators only, but even better, $B*W_2$ drops out of the relation to give the following explicit relation amongst Johnson generators:

$$W_k^{-1} P_g W_k W_2^{-1} P_k = W_g^{-1} P_k W_g W_2^{-1} P_g.$$

If we now multiply on the right by the map W_2, we can rewrite this relation in an interesting form.

Generalized B-Relation I *In $\mathcal{I}_{g,1}$, for $g \geq 4$ and for $3 \leq k \leq g-1$, we have*

$$(W_k^{-1} * P_g)(W_2^{-1} * P_k) = (W_g^{-1} * P_k)(W_2^{-1} * P_g).$$

There are precisely $\binom{g-2}{2}$ such relations in $\mathcal{I}_{g,1}$ (all relations in $\mathcal{I}_{g,1}$ are also relations in $\mathcal{I}_{g+1,1}$). We note that the first generalized B-Relation can also be understood as a commutator relation in $\mathcal{M}_{g,1}$:

$$[B^{-1}, [45\cdots(2k+1)]W_g W_k^{-1}[45\cdots(2g+1)]] = 1.$$

We obtained the first generalized B-relation by setting $k = 2$, a value chosen only for the relative simplicity of $\mathrm{Rel}(g,2)$. We now investigate other possible choices. Let us begin again with $\mathrm{Rel}(g,k)$, $2 \leq k \leq g-1$. Now let us choose any l such that $2 \leq l \leq k-1$ (and thus $k \leq 3$ and $g \leq 4$). Then we can use $\mathrm{Rel}(k,l)$:

$$B * W_k = P_l^{-1} W_k W_l^{-1} P_k (B * W_l).$$

and also $\mathrm{Rel}(g,l)$:

$$B * W_g = P_l^{-1} W_g W_l^{-1} P_g (B * W_l).$$

Substituting these back into $\mathrm{Rel}(g,k)$, and rewriting as with the first relation, we can now state the most general possible form of the B-relations.

Generalized B-Relation II *In $\mathcal{I}_{g,1}$, for $g \geq 4$ and for $2 \leq l < k \leq g-1$, we have*

$$[W_k^{-1} * (P_g P_l^{-1})][W_l^{-1} * P_k] = [W_g^{-1} * (P_k P_l^{-1})][W_l^{-1} * P_g].$$

Note that the second generalized B-relation contains the first in the case $l = 1$ (recall $P_2 = 1$). There are $\binom{g-1}{3}$ such relations in $\mathcal{I}_{g,1}$.

We observe that these relations involve only consecutive subchain maps, which represent a rather small portion of the Johnson generating set. Despite this, the relations are of value in that they currently represent the only known relations in the Torelli group other than commutativity arising from sets of pairwise disjoint curves. Moreover, the following lemma of Dennis Johnson allows us to see that we may use a single relation amongst consecutive subchain maps as the basis for obtaining many more relations amongst other non-consecutive generators.

Lemma 4.5 (Johnson, [22])

(i) *C_j commutes with the subchain map $[i_1 i_2 \cdots i_n]$ if and only if both $j, j+1 \in i_1, i_2, \ldots, i_n$ or both $j, j+1 \notin i_1, i_2, \ldots, i_n$.*

(ii) *If $j = i_m$, but $j+1 \neq i_{m+1}$, then*

$$C_j^{-1} * [i_1 \cdots j \;\; i_{m+1} \cdots i_n] = [i_1 \cdots (j+1) \;\; i_{m+1} \cdots i_n].$$

(iii) *If $j+1 = i_m$, but $j \neq i_{m-1}$, then*

$$C_j * [i_1 \cdots i_{m-1} \;\; (j+1) \cdots i_n] = [i_1 \cdots i_{m-1} \;\; j \cdots i_n].$$

The intricate notation required in the statement of the lemma unfortunately obscures a simple idea which is best understood with some examples corresponding to the three statements in the lemma:

(i) C_3 commutes with the map [2345] since both 3 and 4 appear as indices in the map. C_3 also commutes with [6789] since neither 3 nor 4 appear. However, C_3

does not commute with [2357] since 3 appears without 4, nor with [1456] since 4 appears without 3.

(ii) $C_3^{-1} * [2367] = [2467]$.

(iii) $C_5 * [\beta 689] = [\beta 589]$.

It is also worth noting that, for example, $C_5 * [2345]$ is not known explicitly. We know only that it is a product of straight subchain maps (by another lemma in [22]).

Conjugation thus gives us a convenient way to get from one subchain map to another. This is our first clue that relations amongst consecutive subchain maps are not as limited in scope as one might at first expect. In fact, Lemma 4.5 is a powerful tool for creating new relations based on the generalized B-relations. Just how powerful is demonstrated in the calculations in Appendix B, but we give a quick illustration here. For example, in the first generalized B-relation, we could conjugate by C_5^{-1} to turn $W_2 = [2345]$ into [2346], and then subsequently conjugate by C_6^{-1} to replace [2346] with [2347]. We can continue on conjugating by C_j^{-1}, increasing j by one each time, until we reach $j = 2k + 1$. However, we need not stop at this point. We must merely observe that conjugating by C_{2k+1}^{-1} will simply involve W_k and P_k in the morphing of the relation. This is only the tip of the iceberg. As we shall see in Appendix B, we can get a total of 33 relations simply by conjugating the lone generalized B-relation on a genus 4 surface. A word of caution: we must be careful in our application of Lemma 4.5. For example, backing up to where we had the map [2346], we might be tempted to conjugate by C_4^{-1} in order to come up with a relation involving [2356]. However, $C_4^{-1} * [\beta 567]$ is not explicitly known, and so we do not get an explicit relation.

Counting the number of new relations we can obtain in $\mathcal{I}_{g,1}$ from the generalized B-relations in this way is possible, but we will skip over such a tedious computation. It is only important to realize that each generalized B-relation stands for many, many other relations.

4.2.3 Commutator relations

The B-relations are limited by the fact that a BP-map appearing in the initial lantern relation which is not a Johnson generator must satisfy the condition that each of its two defining curves be disjoint from the second meridian B. This is an extremely restrictive condition, and so we next develop a technique whereby we are freed from our dependence on positioning relative to the meridian B.

The basic idea is to start with a slightly different $S_{0,4}$ on the surface and write down the lantern relation. Any curve in the lantern relation which does not become a Johnson generator when paired with its reflection in the plane of the page must somehow be eliminated. For each such curve, we then seek another $S_{0,4}$ in which the curve plays a different role in the lantern relation, hoping to use it to eliminate these "bad" BP-maps and obtain an explicit relation strictly amongst Johnson generators.

We begin with the $S_{0,4}$ on a surface of genus 4 bounded by the curves C_4, C_6, δ, and ϵ as shown in Figure 17. In Figures 18 - 21, the curve ϵ will be understood though it is not explicitly drawn. Figure 18 illustrates the curves for a lantern relation on this $S_{0,4}$. As usual, we reflect through the plane of the page, invert the resulting relation, and combine the two to obtain a relation amongst BP-maps. For curves which are not Johnson generators, we continue the notation of the previous section. For simplicity in the notation of later relations, we will use a different notation system

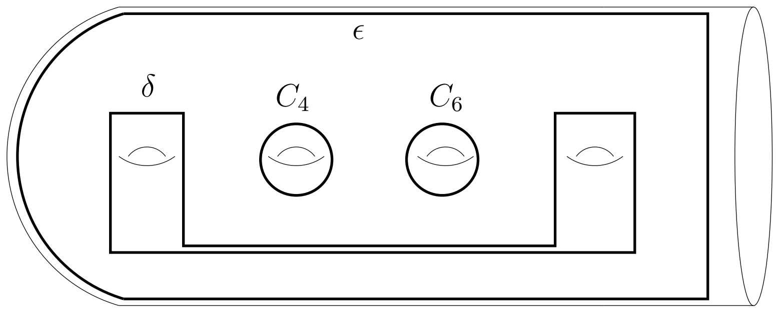


Figure 17: An $S_{0,4}$ on $S_{4,1}$

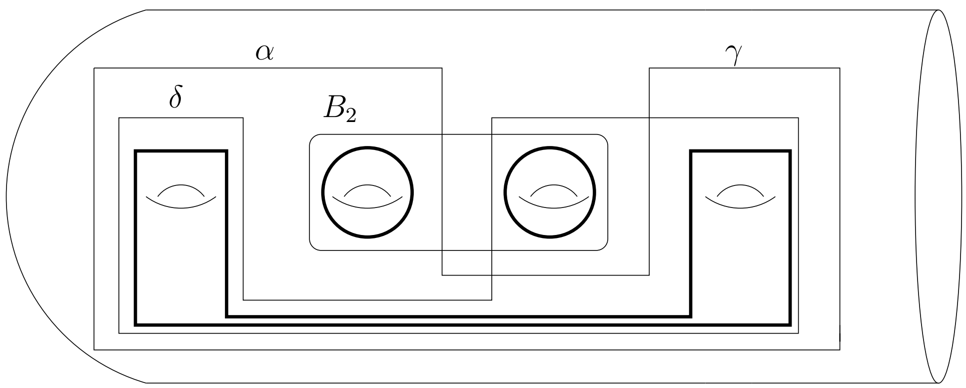


Figure 18: A lantern relation on $S_{4,1}$

for any BP-maps which are Johnson generators, with the exception of the map A_ϵ. For example, in this section, B_2 will be used to denote the BP-map corresponding to the curve B_2 shown in Figure 18 and its reflection through the plane of the page. Therefore we can write down our relation as follows:

$$A_\gamma B_2 A_\alpha = A_\delta A_\epsilon.$$

There are three maps in this relation which are not Johnson generators, namely, A_α, A_γ, and A_δ. We can eliminate A_α with the lantern relation drawn in Figure 19,

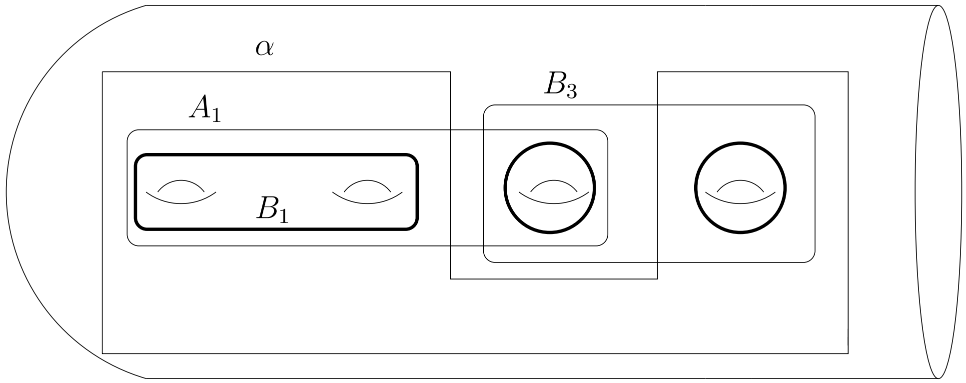


Figure 19: Eliminating A_α

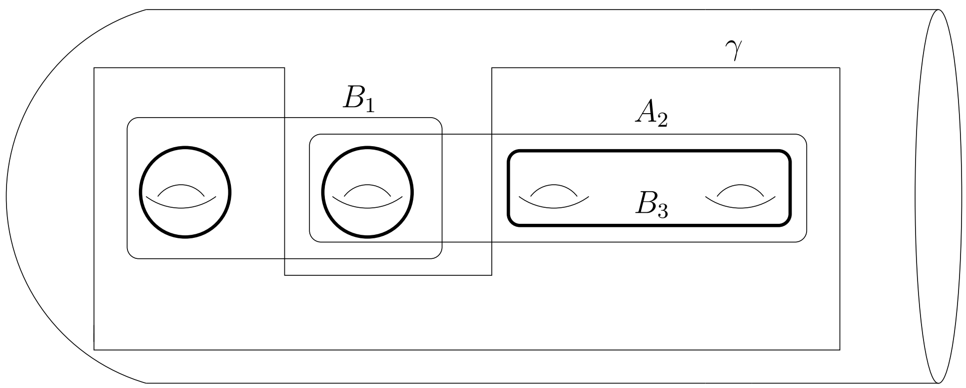


Figure 20: Eliminating A_γ

which yields the following BP-relation:

$$A_1 B_3 A_\alpha = B_1 A_\epsilon$$

or solving for A_α:

$$A_\alpha = B_3^{-1} A_1^{-1} B_1 A_\epsilon.$$

Similarly, we can eliminate A_γ and A_δ by using the lantern relations given in

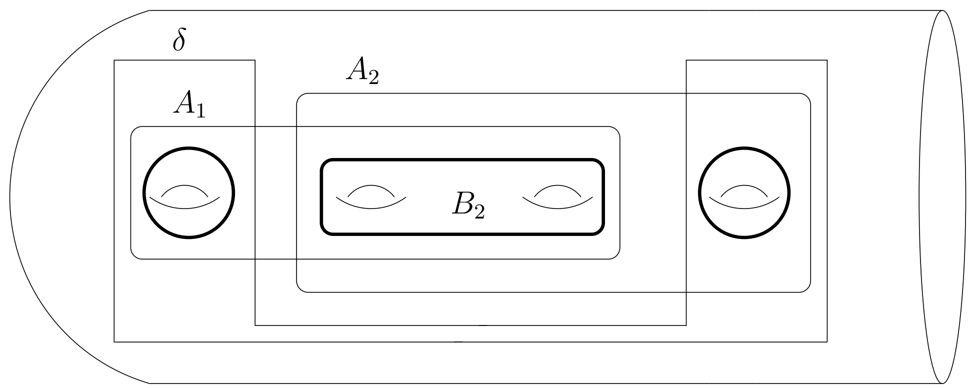


Figure 21: Eliminating A_δ

Figures 20 and 21, respectively. We get the following relations:

$$\begin{aligned} A_\gamma &= B_3 A_\epsilon B_1^{-1} A_2^{-1} \\ A_\delta &= B_2 A_\epsilon A_1^{-1} A_2^{-1}. \end{aligned}$$

Substituting in for A_α, A_γ, and A_δ, we see that A_ϵ drops out of the relation (the curve ϵ is disjoint from all other curves involved in the relation). The result is:

$$B_3 B_1^{-1} A_2^{-1} B_2 B_3^{-1} A_1^{-1} B_1 = B_2 A_1^{-1} A_2^{-1}.$$

This relation looks a bit unwieldy at first, but after some reorganizing using the fact that many pairs of these BP-maps commute, we realize that it is actually a commutativity relation:

$$(B_1^{-1} A_1 B_3)(A_2 B_2^{-1}) = (A_2 B_2^{-1})(B_1^{-1} A_1 B_3).$$

or

$$[(B_1^{-1} A_1 B_3), (A_2 B_2^{-1})] = 1. \tag{20}$$

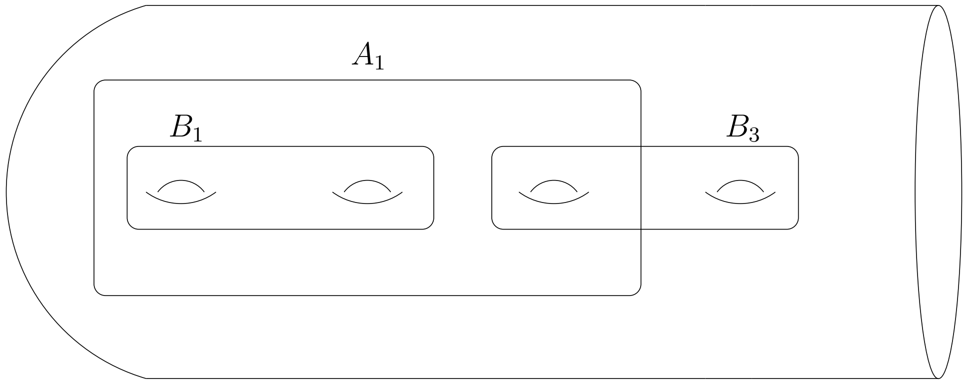


Figure 22: Curves defining the product $B_1^{-1}A_1B_3$

The curves involved within each of the two products in the commutator relation are given in Figures 22 and 23, respectively. It is also interesting to rewrite Relation 20 as a "not quite commuting" relation:

$$(A_2B_2^{-1})(A_1B_1^{-1}) = (A_1B_1^{-1})[B_3 * (A_2B_2^{-1})].$$

It is worth noting that this relation cannot be obtained merely by a straightforward applicatoin of the commutativity relations between the various pairs of Johnson generators.

This single elementary idea gives us a wealth of relations amongst Johnson generators. To begin with, we can simply turn the picture upside down to obtain another commutator relation:

$$[A_1B_2^{-1}, A_2B_1B_3^{-1}] = 1.$$

We can also rewrite the relation using Johnson's notation. We have:

$$\begin{aligned} A_1 &= [234567] = W_3 \\ A_2 &= [456789] \end{aligned}$$

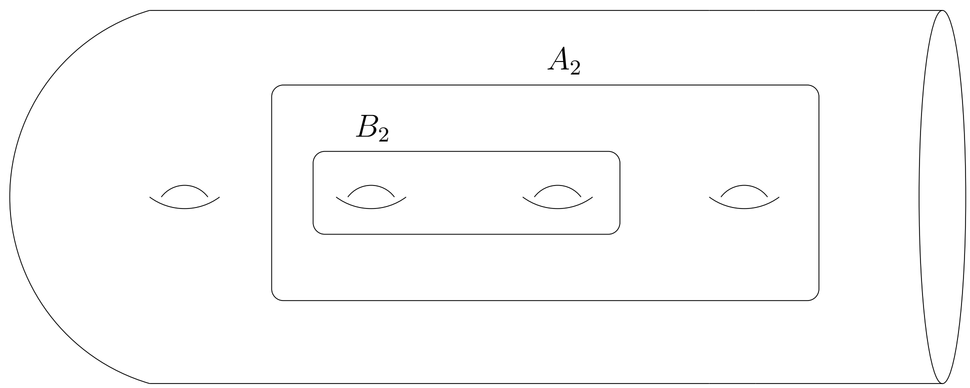


Figure 23: Curves defining the product $A_2B_2^{-1}$

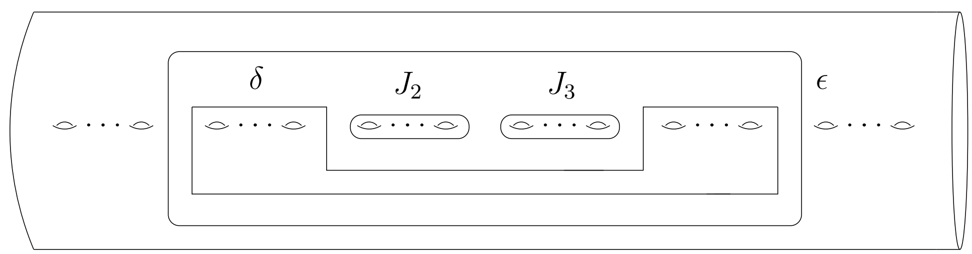


Figure 24: An $S_{0,4}$ on a higher genus surface

$$\begin{aligned} B_1 &= [2345] = W_2 \\ B_2 &= [4567] \\ B_3 &= [6789] \end{aligned}$$

Lemma 4.5 can also be applied to this commutativity relations to obtain new commutativity relations amongst certain non-consecutive straight subchain maps. Furthermore, we can find analogous relations in higher genus by applying the same idea to the $S_{0,4}$ shown in Figure 24.

If we define analogous curves α and γ, and introduce the curves J_1 and J_4 as

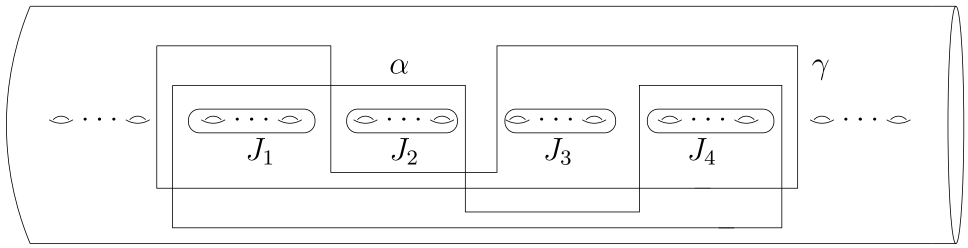


Figure 25: Lantern curves

shown in Figure 25, we can write down the following relation:

$$A_\gamma B_2 A_\alpha = A_\delta A_\epsilon J_2 J_3$$

Now we simply find three new lantern relations analogous to those given in Figures 19, 20, and 21. The only difference is that now we have the curves J_i in our relations. We find that we can make the following substitutions:

$$\begin{aligned} A_\alpha &= B_3^{-1} A_1^{-1} B_1 A_\epsilon J_3 J_4 \\ A_\gamma &= B_3 A_\epsilon B_1^{-1} A_2^{-1} J_1 J_2 \\ A_\delta &= B_2 A_\epsilon A_1^{-1} A_2^{-1} J_1 J_4 \end{aligned}$$

Noting that A_ϵ and the J_i commute with every map in these lantern relations, we see that these factors drop out of the relation, and we are left with the precise analog of our genus 4 commutator relation:

General Commutator Relation *In $\mathcal{I}_{g,1}$, for $g \geq 4$ and for the curves shown in Figure 26, we have*

$$[(B_1^{-1} A_1 B_3), (A_2 B_2^{-1})] = 1.$$

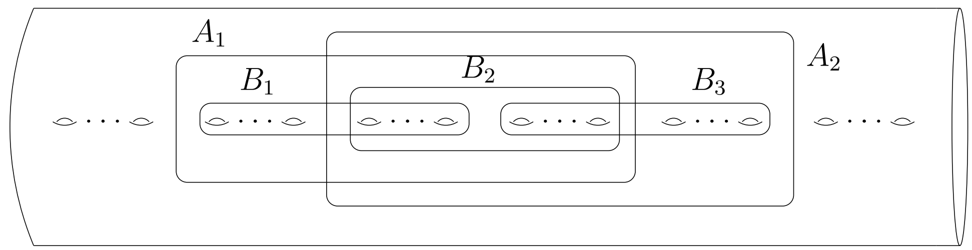


Figure 26: Curves in the generalized commutator relation

We need some notation at this point, so we assume that B_i encloses the m_i^{th} hole to the $(m_{i+2}-1)^{th}$ hole. In other words, B_i "begins" with the m_i^{th} hole and ends before the m_{i+2}^{th} hole. Thus our curves represent the following Johnson generators:

$$\begin{aligned} A_1 &= [(2m_1)(2m_1+1)\cdots(2m_4)] \\ A_2 &= [(2m_2)(2m_2+1)\cdots(2m_5)] \\ B_1 &= [(2m_1)(2m_1+1)\cdots(2m_3)] \\ B_2 &= [(2m_2)(2m_1+1)\cdots(2m_4)] \\ B_3 &= [(2m_3)(2m_1+1)\cdots(2m_5)] \end{aligned}$$

We can of course use Lemma 4.5 on all these relations to obtain morelations.

Remark 4.6 Johnson found other lantern relations amongst elements of $\mathcal{I}_{g,1}$. For example, also in [22], he finds that

$$[1234][1256\cdots 2g](B*[345\cdots 2g]) = [56\cdots 2g][123\cdots 2g].$$

This relation is not useful for our present purpose since $B*[345\cdots 2g]$ is not a Johnson generator. All efforts to use either of our two methods introduced are in vain, however. We have no recursively defined relation with which to take advantage of the embeddings of $\mathcal{I}_{g,1}$ in higher genus groups, as in Section 4.2.2. After some calculations, one also finds that when attempting to apply the technique of letting "bad" curves play different roles in other lantern relations as in Section 4.2.3, one simply runs out of room on the surface. Thus the real trick to finding relations amongst Johnson generators is to find a "good" $S_{0,4}$ in the first place.

4.3 Symmetry of straight chain maps and further questions

There is a great deal of interesting geometry in the Johnson generating set. As one might guess from the symmetry of the straight chain itself, it turns out that each straight subchain map $T_\alpha T_\beta^{-1}$ has the property that rotation through 180 degrees about the axis shown in Figure 27 takes the curve α onto β (see, for instance, the straight 3-subchain given in Figure 11). Let us call this rotation j. The centralizer of j in $\mathcal{M}_{g,b,n}$ is known as the *hyperelliptic subgroup* of $\mathcal{M}_{g,b,n}$ and is known to be linear ([3], [29]). Now, let A denote a Johnson generator $T_\alpha T_\beta^{-1}$. It is not true that A commutes with j; however we do have the property that $jAj^{-1} = A^{-1}$. It also has the immediate consequence that the product jA is itself an involution and also that $A = [T_\alpha^{-1}, j]$. This is an intriguing property in its own right, and, since Johnson shows in [22] that $\mathcal{I}_{3,0}$ is generated by 35 straight 3-subchain maps (given explicitly in Appendix A), it piques one's curiosity regarding the linearity question for the Torelli group in genus 3. One also could hope to generalize any result along these lines to higher genus to a certain extent, since the vast majority of Johnson generators are straight subchain maps.

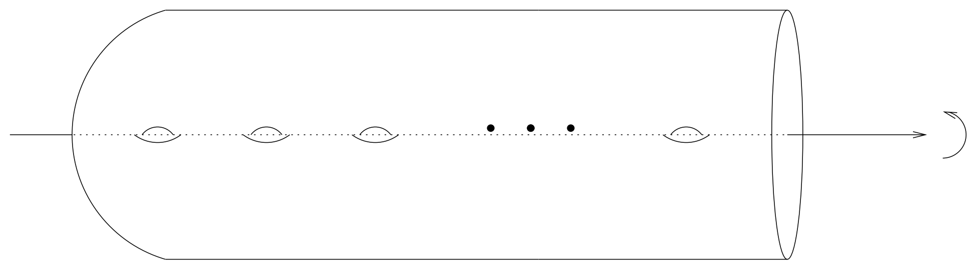


Figure 27: The involution j

Beyond the linearity question, there are many interesting questions which remain unanswered regarding the Torelli group. For example, the question of finite presentation remains open. Is there a more useful generating set than Johnson's, one that would perhaps lend itself better to the finite presentation question? The question of a minimal generating set also remains of interest. What is a minimal generating set for $\mathcal{I}_{g,1}$ or $\mathcal{I}_{g,0}$ when $g \geq 4$? Can Johnson's cubic lower bound on the number of generators of $\mathcal{I}_{g,1}$ and $\mathcal{I}_{g,0}$ be realized? Failing that, can we find a better lower bound? More particularly, can the β-subchain maps be eliminated from Johnson's generating set in higher genus? Do the straight 3-chain maps suffice to generate $\mathcal{I}_{g,0}$ for $g \geq 4$?

This last question warrants a bit of discussion. First of all, Johnson's proof of his finite generating set relies on the fact that his set contains at least one 3-subchain map. Since $\mathcal{I}_{g,0}$ is normal in $\mathcal{M}_{g,0}$, and by Lemma 4.5, we know that if we have one straight 3-subchain map in $\mathcal{I}_{g,0}$, we have them all, and likewise one β 3-subchain map gives us all β 3-subchain maps. Thus if Johnson's method of proof could somehow be adapted to find a smaller generating set from amongst his generators, the 3-subchain maps involved. The size of this set is reasonable, being on the order of g^4, and we have seen that straight 3-chain maps suffice to generate $\mathcal{I}_{3,0}$. This bit of speculation of course leads to another question: is there a generating set better suited than Johnson's to attaining a minimal set?

Furthermore, we might ask what the intersection of the Torelli group is with other interesting subgroups of the mapping class group, such as the handlebody subgroup, or the Crisp-Paris subgroups [10], which are right-angled Artin groups (and hence linear) generated by squares of certain Dehn twists? As discussed in Section 3, the former question arises in the context of classifying Heegaard splittings of 3-manifolds, yet another context in which better knowledge of the Torelli group could have vast implications.

These and many other questions must be answered if enormous potential of the Torelli group is to be realized fully.

A Johnson's generators in genus 3

We present here a complete and explicit list of all Johnson's generators for both $\mathcal{I}_{3,1}$ and $\mathcal{I}_{3,0}$, with a summary of relevant facts from Section 4.

1. In the genus 3 case, Johnson's generators are all straight subchain maps with the exception of a single β-subchain map.

2. We first give the 5-subchain maps (six indices), followed by the 3-subchain maps (four indices).

3. The straight 5-subchain maps listed all lie in the kernel of the natural map $\mathcal{I}_{g,1} \to \mathcal{I}_{g,0}$.

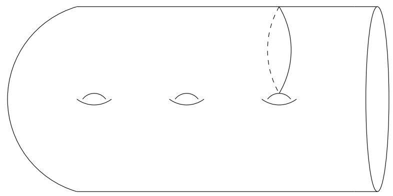
4. Since all straight subchain maps have rotational symmetry, as discussed in Section 4.3, for such maps we only give one of the two curves defining the BP-map. The second can then be obtained by applying the involution j shown in Figure 27.

5. In the case of the β-subchain, both curves are drawn.

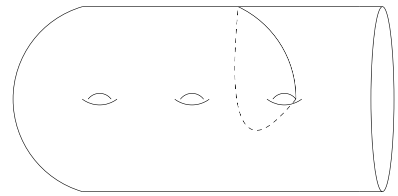
6. The 36 3-subchain maps together with all of the 5-subchain maps but the first ([123456]) represent a minimal generating set for $\mathcal{I}_{3,1}$.

7. The 35 straight 3-subchain maps (which have four indices) shown here represent a minimal generating set for $\mathcal{I}_{3,0}$.

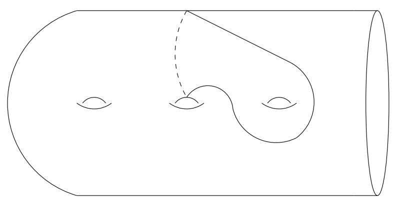
[123456]



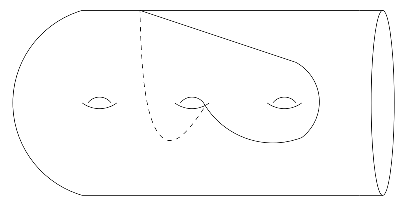
[123457]



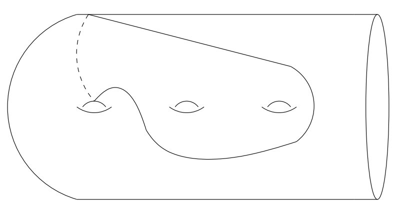
[123467]



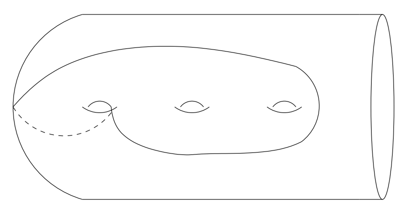
[123567]



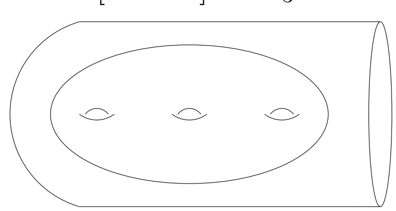
[124567]



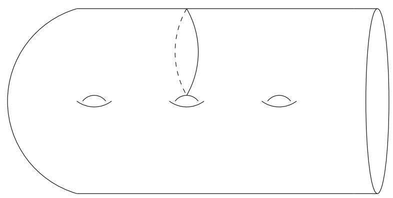
[134567]



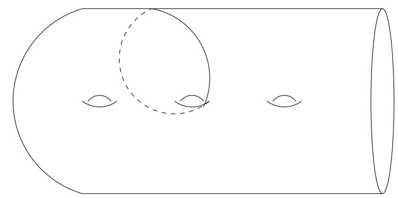
$[234567] = W_3$



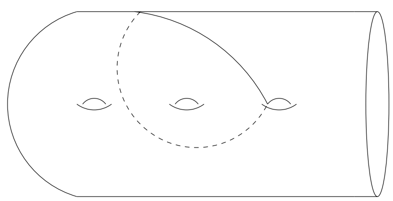
[1234]



[1235]



[1236]



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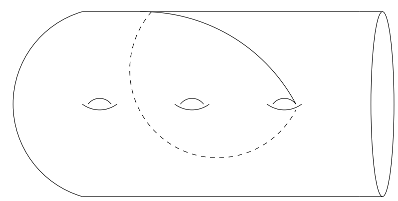

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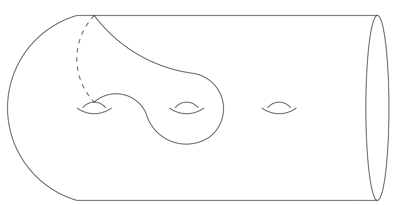

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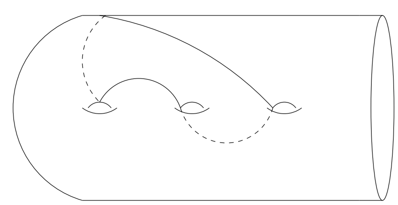

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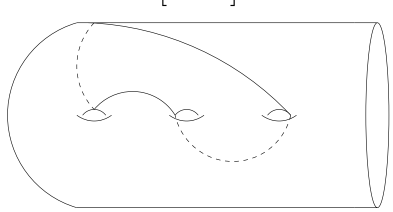
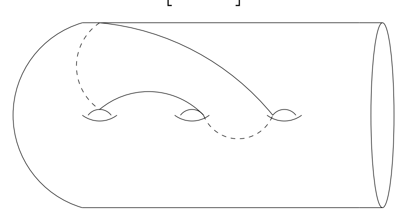
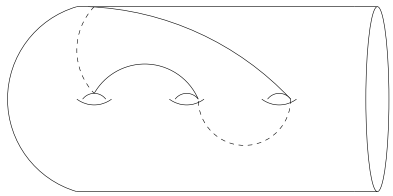


[1256]



[1257]



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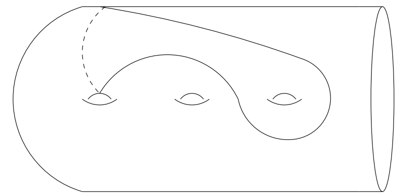

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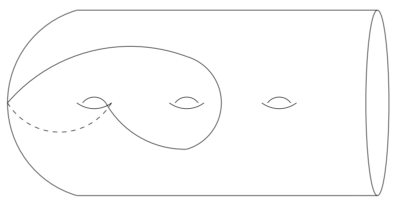

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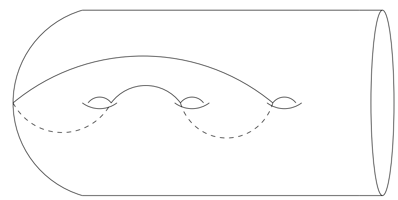

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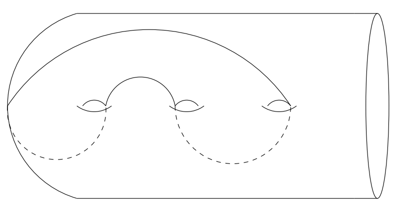

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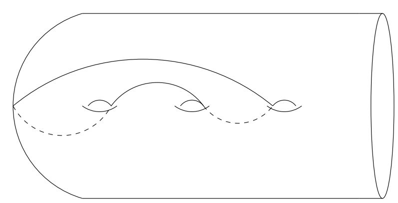

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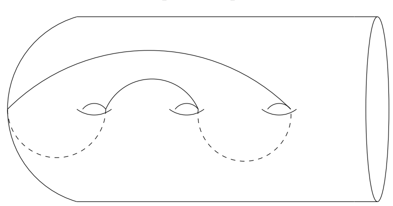

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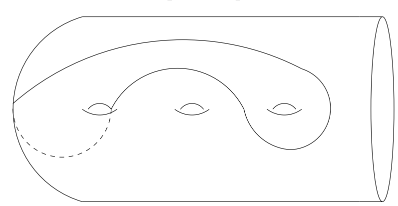

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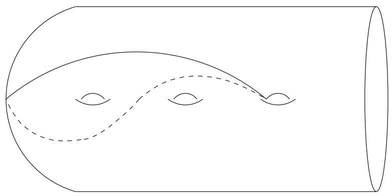

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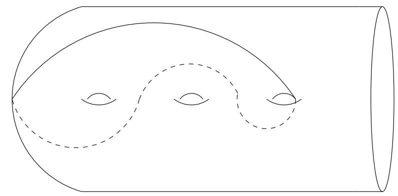

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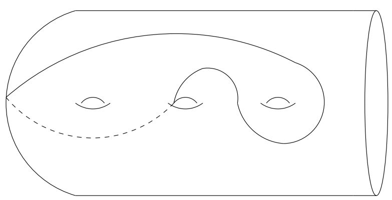

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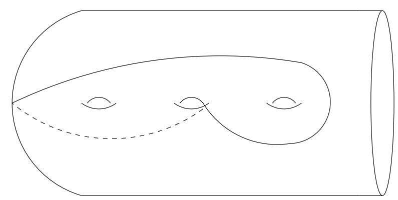

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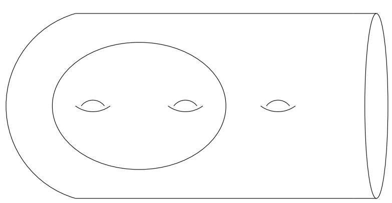
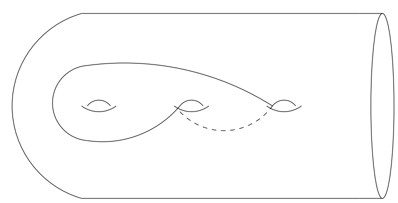


[2346]



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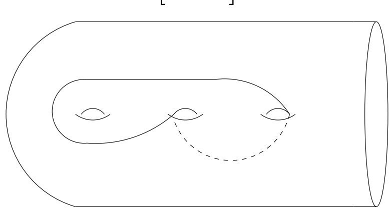
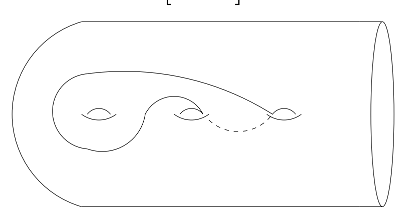


[2356]



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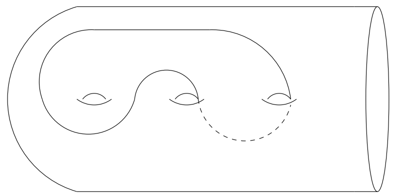
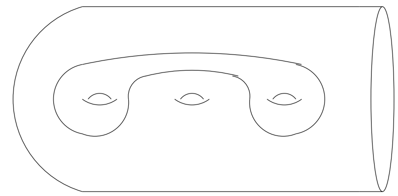
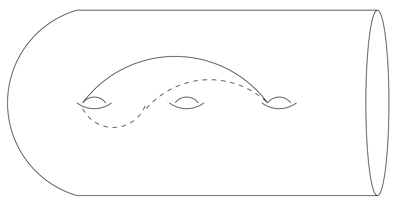


[2367]



[2456]



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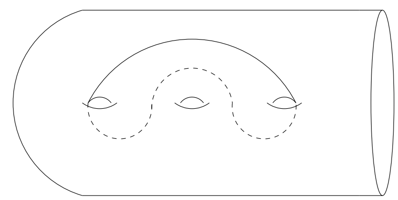

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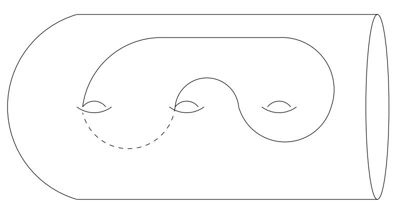

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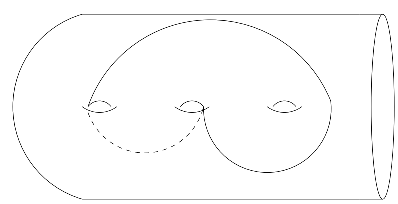

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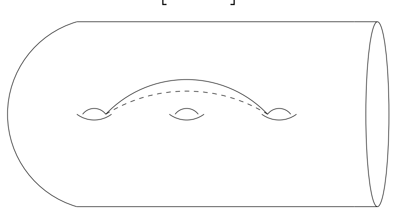

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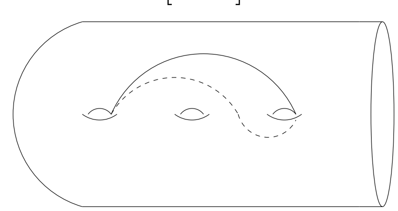

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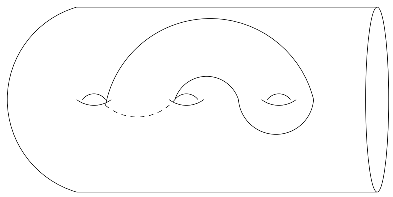

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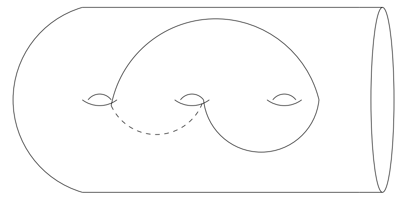
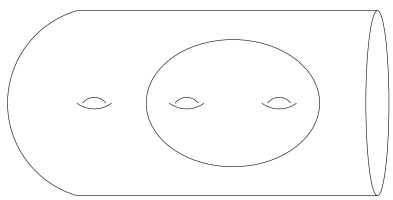
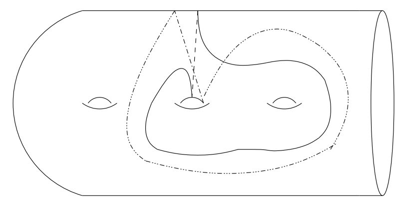


[4567]



[β 567]



B Some calculations of relations in low genus

Before getting started with calculations in genus 4, we note that Johnson's Equation 16 gives us a single relation in $\mathcal{I}_{3,0}$, though it is not explicit since we only know that $B * W_2$ is a product of straight 3-chains in $\mathcal{I}_2, 1$. The relation is as follows:

$$W_2^{-1}[4567]^{-1}[\beta 567]B * W_2 = 1.$$

Since mapping class groups of closed surfaces do not embed in higher genus mapping class groups, this does not give us an explicit form for $B * W_2$. As previously discussed in Section 4.2.1, it does allow us to eliminate the generator $[\beta 567]$ from Johnson's generating set in $\mathcal{I}_{3,0}$. Note that we could rewrite the above relation in the notation of Section 4 as follows:

$$W_2^{-1} P_3 B * W_2 = 1$$

or equivalently,

$$P_3 = [W_2, B^{-1}]$$

We also have the analogous relation in $\mathcal{I}_{g,0}$:

$$P_g = [W_2, B * W_2].$$

Turning now to the surface with one boundary component, we calculate all relations arising from the generalized B-relation in the simplest case, $\mathcal{I}_{4,1}$.

There is only one generalized B-relation in genus 4, with $k = 3$ (and $l = 2$):

$$(W_3^{-1} * P_4)(W_2^{-1} * P_3) = (W_4^{-1} * P_3)(W_2^{-1} * P_4)$$

. We write this out in Johnson's notation so that we can find all conjugates:

$$\begin{aligned} & [234567]^{-1} * ([456789]^{-1}[\beta 56789]) \cdot [2345]^{-1} * ([4567]^{-1}[\beta 567]) \\ = \; & [23456789]^{-1} * ([4567]^{-1}[\beta 567]) \cdot [2345]^{-1} * ([456789]^{-1}[\beta 56789]) \qquad (1) \end{aligned}$$

Conjugate by C_5^{-1} and then by C_6^{-1} to obtain the next two relations:

$$\begin{aligned} & [234567]^{-1} * ([456789]^{-1}[\beta 56789]) \cdot [2346]^{-1} * ([4567]^{-1}[\beta 567]) \\ = \;& [23456789]^{-1} * ([4567]^{-1}[\beta 567]) \cdot [2346]^{-1} * ([456789]^{-1}[\beta 56789]) \end{aligned} \tag{2}$$

$$\begin{aligned} & [234567]^{-1} * ([456789]^{-1}[\beta 56789]) \cdot [2347]^{-1} * ([4567]^{-1}[\beta 567]) \\ = \;& [23456789]^{-1} * ([4567]^{-1}[\beta 567]) \cdot [2347]^{-1} * ([456789]^{-1}[\beta 56789]) \end{aligned} \tag{3}$$

Now conjugate by C_7^{-1}, which involves four more factors, with result:

$$\begin{aligned} & [234568]^{-1} * ([456789]^{-1}[\beta 56789]) \cdot [2348]^{-1} * ([4568]^{-1}[\beta 568]) \\ = \;& [23456789]^{-1} * ([4568]^{-1}[\beta 568]) \cdot [2348]^{-1} * ([456789]^{-1}[\beta 56789]) \end{aligned} \tag{4}$$

Then conjugate by C_8^{-1}:

$$\begin{aligned} & [234569]^{-1} * ([456789]^{-1}[\beta 56789]) \cdot [2349]^{-1} * [4569]^{-1}[\beta 569]) \\ = \;& [23456789]^{-1} * ([4569]^{-1}[\beta 569]) \cdot [2349]^{-1} * ([456789]^{-1}[\beta 56789]) \end{aligned} \tag{5}$$

Note that the genus 4 relations embed in all higher genus, and in that case, we could continue on in this way until running out of genus.

We now work on the "middle" of the indices. Beginning with Relation 4 we have some room to maneuver inside the larger factors. First we conjugate Relation 4 by C_6^{-1} and then by C_5^{-1}:

$$\begin{aligned} & [234578]^{-1} * ([456789]^{-1}[\beta 56789]) \cdot [2348]^{-1} * ([4578]^{-1}[\beta 578]) \\ = \;& [23456789]^{-1} * ([4578]^{-1}[\beta 578]) \cdot [2348]^{-1} * ([456789]^{-1}[\beta 56789]) \end{aligned} \tag{6}$$

$$\begin{aligned}
& [234678]^{-1} * ([456789]^{-1}[\beta 56789]) \cdot [2348]^{-1} * ([4678]^{-1}[\beta 678]) \\
= \;& [23456789]^{-1} * ([4678]^{-1}[\beta 678]) \cdot [2348]^{-1} * ([456789]^{-1}[\beta 56789]) \qquad (7)
\end{aligned}$$

We can create even more relations from Relation 5 by conjugating by $C_6^{-1}, C_7^{-1}, C_5^{-1}$, and C_6^{-1} (in that order) to obtain, the following four relations, respectively:

$$\begin{aligned}
& [234579]^{-1} * ([456789]^{-1}[\beta 56789]) \cdot [2349]^{-1} * ([4579]^{-1}[\beta 579]) \\
= \;& ([23456789]^{-1} * [4579]^{-1}[\beta 579])([2349]^{-1} * [456789]^{-1}[\beta 56789]) \qquad (8)
\end{aligned}$$

$$\begin{aligned}
& [234589]^{-1} * ([456789]^{-1}[\beta 56789]) \cdot [2349]^{-1} * ([4589]^{-1}[\beta 589]) \\
= \;& [23456789]^{-1} * ([4589]^{-1}[\beta 589]) \cdot [2349]^{-1} * ([456789]^{-1}[\beta 56789]) \qquad (9)
\end{aligned}$$

$$\begin{aligned}
& [234689]^{-1} * ([456789]^{-1}[\beta 56789]) \cdot [2349]^{-1} * ([4689]^{-1}[\beta 689]) \\
= \;& [23456789]^{-1} * ([4689]^{-1}[\beta 689]) \cdot [2349]^{-1} * ([456789]^{-1}[\beta 56789]) \qquad (10)
\end{aligned}$$

$$\begin{aligned}
& [234789]^{-1} * ([456789]^{-1}[\beta 56789]) \cdot [2349]^{-1} * ([4789]^{-1}[\beta 789]) \\
= \;& [23456789]^{-1} * ([4789]^{-1}[\beta 789]) \cdot [2349]^{-1} * ([456789]^{-1}[\beta 56789]) \qquad (11)
\end{aligned}$$

At this point, we are stuck on the 4's, i.e., we cannot conjugate by C_4^{-1} to obtain, for example, [2359] in place of [2349], since C_4^{-1} clashes with the β-subchain maps. So we turn our attention to the lower indices. We begin again with the original relation:

$$\begin{aligned}
& [234567]^{-1} * ([456789]^{-1}[\beta 56789]) \cdot [2345]^{-1} * ([4567]^{-1}[\beta 567]) \\
= \;& [23456789]^{-1} * ([4567]^{-1}[\beta 567]) \cdot [2345]^{-1} * ([456789]^{-1}[\beta 56789])
\end{aligned}$$

We now conjugate by C_1 and then by C_2, which involves three factors.

$$\begin{aligned}
& [134567]^{-1} * ([456789]^{-1}[\beta 56789]) \cdot [1345]^{-1} * ([4567]^{-1}[\beta 567]) \\
= \quad & [13456789]^{-1} * ([4567]^{-1}[\beta 567]) \cdot [1345]^{-1} * ([456789]^{-1}[\beta 56789]) \qquad (12)
\end{aligned}$$

$$\begin{aligned}
& [124567]^{-1} * ([456789]^{-1}[\beta 56789]) \cdot [1245]^{-1} * ([4567]^{-1}[\beta 567]) \\
= \quad & [12456789]^{-1} * ([4567]^{-1}[\beta 567]) \cdot [1245]^{-1} * ([456789]^{-1}[\beta 56789]) \qquad (13)
\end{aligned}$$

We can also conjugate Relation 3 by C_1 and then by C_2, and then do the same for Relation 3, to obtain four more relations:

$$\begin{aligned}
& [134567]^{-1} * ([456789]^{-1}[\beta 56789]) \cdot [1346]^{-1} * ([4567]^{-1}[\beta 567]) \\
= \quad & [13456789]^{-1} * ([4567]^{-1}[\beta 567]) \cdot [1346]^{-1} * ([456789]^{-1}[\beta 56789]) \qquad (14)
\end{aligned}$$

$$\begin{aligned}
& [124567]^{-1} * ([456789]^{-1}[\beta 56789]) \cdot [1246]^{-1} * ([4567]^{-1}[\beta 567]) \\
= \quad & [12456789]^{-1} * ([4567]^{-1}[\beta 567]) \cdot [1246]^{-1} * ([456789]^{-1}[\beta 56789]) \qquad (15)
\end{aligned}$$

$$\begin{aligned}
& [134567]^{-1} * ([456789]^{-1}[\beta 56789]) \cdot [1347]^{-1} * ([4567]^{-1}[\beta 567]) \\
= \quad & [13456789]^{-1} * ([4567]^{-1}[\beta 567]) \cdot [1347]^{-1} * ([456789]^{-1}[\beta 56789]) \qquad (16)
\end{aligned}$$

$$\begin{aligned}
& [124567]^{-1} * ([456789]^{-1}[\beta 56789]) \cdot [1247]^{-1} * ([4567]^{-1}[\beta 567]) \\
= \quad & [12456789]^{-1} * ([4567]^{-1}[\beta 567]) \cdot [1247]^{-1} * ([456789]^{-1}[\beta 56789]) \qquad (17)
\end{aligned}$$

We finish off the generalized B-relations by repeating this for Relations 4 - 11.

$$\begin{aligned} & [134568]^{-1} * ([456789]^{-1}[\beta 56789]) \cdot [1348]^{-1} * ([4568]^{-1}[\beta 568]) \\ = \; & [13456789]^{-1} * ([4568]^{-1}[\beta 568]) \cdot [1348]^{-1} * ([456789]^{-1}[\beta 56789]) \end{aligned} \tag{18}$$

$$\begin{aligned} & [124568]^{-1} * ([456789]^{-1}[\beta 56789]) \cdot [1248]^{-1} * ([4568]^{-1}[\beta 568]) \\ = \; & [12456789]^{-1} * ([4568]^{-1}[\beta 568]) \cdot [1248]^{-1} * ([456789]^{-1}[\beta 56789]) \end{aligned} \tag{19}$$

$$\begin{aligned} & [134569]^{-1} * ([456789]^{-1}[\beta 56789]) \cdot [1349]^{-1} * [4569]^{-1}[\beta 569]) \\ = \; & [13456789]^{-1} * ([4569]^{-1}[\beta 569]) \cdot [1349]^{-1} * ([456789]^{-1}[\beta 56789]) \end{aligned} \tag{20}$$

$$\begin{aligned} & [124569]^{-1} * ([456789]^{-1}[\beta 56789]) \cdot [1249]^{-1} * [4569]^{-1}[\beta 569]) \\ = \; & [12456789]^{-1} * ([4569]^{-1}[\beta 569]) \cdot [1249]^{-1} * ([456789]^{-1}[\beta 56789]) \end{aligned} \tag{21}$$

$$\begin{aligned} & [134578]^{-1} * ([456789]^{-1}[\beta 56789]) \cdot [1348]^{-1} * ([4578]^{-1}[\beta 578]) \\ = \; & [13456789]^{-1} * ([4578]^{-1}[\beta 578]) \cdot [1348]^{-1} * ([456789]^{-1}[\beta 56789]) \end{aligned} \tag{22}$$

$$\begin{aligned} & [124578]^{-1} * ([456789]^{-1}[\beta 56789]) \cdot [1248]^{-1} * ([4578]^{-1}[\beta 578]) \\ = \; & [12456789]^{-1} * ([4578]^{-1}[\beta 578]) \cdot [1248]^{-1} * ([456789]^{-1}[\beta 56789]) \end{aligned} \tag{23}$$

$$\begin{aligned} & [134678]^{-1} * ([456789]^{-1}[\beta 56789]) \cdot [1348]^{-1} * ([4678]^{-1}[\beta 678]) \\ = \; & [13456789]^{-1} * ([4678]^{-1}[\beta 678]) \cdot [1348]^{-1} * ([456789]^{-1}[\beta 56789]) \end{aligned} \tag{24}$$

$$
\begin{aligned}
& [124678]^{-1} * ([456789]^{-1}[\beta 56789]) \cdot [1248]^{-1} * ([4678]^{-1}[\beta 678]) \\
= \; & [12456789]^{-1} * ([4678]^{-1}[\beta 678]) \cdot [1248]^{-1} * ([456789]^{-1}[\beta 56789]) \qquad (25)
\end{aligned}
$$

$$
\begin{aligned}
& [134579]^{-1} * ([456789]^{-1}[\beta 56789]) \cdot [1349]^{-1} * ([4579]^{-1}[\beta 579]) \\
= \; & ([13456789]^{-1} * [4579]^{-1}[\beta 579])([1349]^{-1} * [456789]^{-1}[\beta 56789]) \qquad (26)
\end{aligned}
$$

$$
\begin{aligned}
& [124579]^{-1} * ([456789]^{-1}[\beta 56789]) \cdot [1249]^{-1} * ([4579]^{-1}[\beta 579]) \\
= \; & ([12456789]^{-1} * [4579]^{-1}[\beta 579])([1249]^{-1} * [456789]^{-1}[\beta 56789]) \qquad (27)
\end{aligned}
$$

$$
\begin{aligned}
& [134589]^{-1} * ([456789]^{-1}[\beta 56789]) \cdot [1349]^{-1} * ([4589]^{-1}[\beta 589]) \\
= \; & [13456789]^{-1} * ([4589]^{-1}[\beta 589]) \cdot [1349]^{-1} * ([456789]^{-1}[\beta 56789]) \qquad (28)
\end{aligned}
$$

$$
\begin{aligned}
& [124589]^{-1} * ([456789]^{-1}[\beta 56789]) \cdot [1249]^{-1} * ([4589]^{-1}[\beta 589]) \\
= \; & [12456789]^{-1} * ([4589]^{-1}[\beta 589]) \cdot [1249]^{-1} * ([456789]^{-1}[\beta 56789]) \qquad (29)
\end{aligned}
$$

$$
\begin{aligned}
& [134689]^{-1} * ([456789]^{-1}[\beta 56789]) \cdot [1349]^{-1} * ([4689]^{-1}[\beta 689]) \\
= \; & [13456789]^{-1} * ([4689]^{-1}[\beta 689]) \cdot [1349]^{-1} * ([456789]^{-1}[\beta 56789]) \qquad (30)
\end{aligned}
$$

$$
\begin{aligned}
& [124689]^{-1} * ([456789]^{-1}[\beta 56789]) \cdot [1249]^{-1} * ([4689]^{-1}[\beta 689]) \\
= \; & [12456789]^{-1} * ([4689]^{-1}[\beta 689]) \cdot [1249]^{-1} * ([456789]^{-1}[\beta 56789]) \qquad (31)
\end{aligned}
$$

$$
\begin{aligned}
& [134789]^{-1} * ([456789]^{-1}[\beta 56789]) \cdot [1349]^{-1} * ([4789]^{-1}[\beta 789]) \\
= \; & [13456789]^{-1} * ([4789]^{-1}[\beta 789]) \cdot [1349]^{-1} * ([456789]^{-1}[\beta 56789]) \qquad (32)
\end{aligned}
$$

$$
\begin{aligned}
& [124789]^{-1} * ([456789]^{-1}[\beta 56789]) \cdot [1249]^{-1} * ([4789]^{-1}[\beta 789]) \\
= \; & [12456789]^{-1} * ([4789]^{-1}[\beta 789]) \cdot [1249]^{-1} * ([456789]^{-1}[\beta 56789]) \qquad (33)
\end{aligned}
$$

The reader who attempts to perform such a calculation in higher genus or for the commutativity relations will quickly appreciate the desirability of using a computer for such a task.

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