Periodicity of spaces of walks

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These unpolished notes correspond to what I wrote on the board during the talk, plus some of the things I said out loud and a few things I'd hoped to do but didn't have time for.

This is very much work in progress—to put it politely.

Progression of talk:

1 A particular situation (will spend a long time on this)

2 A general theory

3 ? (puzzlement, questions)

1 One particular type of walk

This is not about *random* walks. There will be nothing random in this talk.

Consider walks on $\mathbb{N} = \{0, 1, \ldots\}$. Rules:

- start at some position $n \in \mathbb{N}$ and walk forever
- every second, take one step left or right (unless you're at 0)
- if you're at 0, stay there. (You could impose a different rule at 0, and we'll come back to that. Actually, for the moment it makes no difference what the rule at 0 is.)

Let W_n be the 'space' of walks starting at n. (We'll come back to the meaning of 'space' too.) Then $W_0 = \{\star\}$, and for $n \geq 1$,

$$W_n \cong W_{n-1} + W_{n+1}$$

(whatever '$\cong$' and '$+$' mean), by conditioning on the first step of the walk.

So we have a sequence $W_0, W_1, \ldots$ of spaces. W_0 is a one-point space and the rest are infinite; let's ignore W_0 for the moment and think about the others.

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Which are the same and which are different? (Up to suitable iso.) If we were on $\mathbb{Z}$ (not $\mathbb{N}$), they'd all be the same by homogeneity; but we're on $\mathbb{N}$. Maybe all the W_n's are isomorphic. Maybe none of them are. Maybe some are the same and some different, in some pattern.

Proposition 1.1 *Up to isomorphism,* $(W_n)_{n\geq 1}$ *has period* 6.

I like to think of this as follows: $\mathbb{Z}$ is a flat expanse of water, infinite in all directions, with no interruptions. It's totally homogeneous so nothing interesting can happen. $\mathbb{N}$ is an infinite expanse of water lapping up against a sea wall, and you get wave-like behaviour propagating from the wall.

"Proof" Change subscripts to superscripts:

$$W^n \cong W^{n-1} + W^{n+1} \quad \forall n \geq 1.$$

Regard as powers and cancel:

$$W \cong 1 + W^2.$$

Solve: $W \cong e^{\pm\pi i/3}$. So $W^6 \cong 1$, so $W^{n+6} \cong W^n$ for all $n \geq 0$, so $W_{n+6} \cong W_n$ for all $n \geq 1$. (Why not for $n = 0$? Because that would clearly be wrong! W_6 is infinite but W_0 has only one point.) □

There are at least two things this nonsense might remind you of. First, as discovered by Andreas Blass, there's an explicit bijection between binary trees and 7-tuples of binary trees. (Reference: Blass, 'Seven trees in one'.)

But at a much more basic level, it might remind you of linear recurrence relations / difference equations, which you solve by writing down the 'auxiliary equation' (a polynomial equation) and solving it. You can take the proof that this method for difference equations works and try to apply it in the current situation. What you get is better than the previous "proof", but still nonsense.

'Proof' Let

$$\Omega = \{\text{sequences } X = (X_0, X_1, \ldots) \text{ of spaces}\}.$$

Define $D : \Omega \longrightarrow \Omega$ by $D(X) = (X_1, X_2, \ldots)$. Then

$$\begin{aligned} D(W) &= W + D^2(W) \\ \text{i.e.} \quad (1 - D + D^2)(W) &= 0 \\ \text{so} \quad (1 + D - D^3 - D^4)(1 - D + D^2)(W) &= 0 \\ \text{i.e.} \quad (1 - D^6)(W) &= 0 \\ \text{so} \quad W_n &\cong W_{n+6} \quad \forall n \geq 1 \end{aligned}$$

(again, choosing to ignore the wrong conclusion that $W_0 \cong W_6$). □

This is nonsense because we're subtracting spaces. However, there's a general procedure for taking a proof that uses subtraction and converting it to one that doesn't (see later). If you apply it here, you get something like the following:

Proof (Genuine!) For $n \geq 1$,

$$\begin{aligned} W_n &\cong W_{n-1} + W_{n+1} \\ &\cong W_{n-1} + W_n + W_{n+2} \\ &\cong W_{n-1} + W_n + W_{n+1} + W_{n+3} \\ &\cong W_n + W_n + W_{n+3} \\ &\cong \ldots \\ &\cong W_{n+6} \end{aligned}$$

(in 18 steps). □

It's not clear *what* we've proved, since we haven't assigned meanings to 'space', '$\cong$' or '+'. Let's try to do that now.

We want to give them meanings in such a way that the following is true:

$$\text{for } m, n \geq 1, \quad W_m \cong W_n \iff m \equiv n \pmod 6. \qquad (*)$$

If you just want $\Leftarrow$, you can take 'space' to mean 'set' and '+' to mean disjoint union—but then in fact the period is 1. To make $\Rightarrow$ true you need some extra structure on the set. Let's look closely at what structure the W_n's carry, by looking at how they're defined.

Write

$$W_{n,l} = \{\text{walks of length } l \text{ starting at position } n\},$$

($n, l \in \mathbb{N}$), a finite set. Then

$$W_n = \varprojlim(\cdots \longrightarrow W_{n,2} \longrightarrow W_{n,1} \longrightarrow W_{n,0}).$$

So W_n acquires a profinite topology. But that's not enough! In other words, if 'space' means topological space and '$\cong$' means homeomorphic then the $\Rightarrow$ of $(*)$ fails.

(The general point is that you can have two very different sequences of finite sets whose limits just happen to be homeomorphic. E.g. if n denotes an n-element set then the sequences

$$\cdots \longrightarrow 2^2 \longrightarrow 2^1 \longrightarrow 2^0$$

and

$$\cdots \longrightarrow 3^2 \longrightarrow 3^1 \longrightarrow 3^0$$

have respective limits $2^{\mathbb{N}}$ and $3^{\mathbb{N}}$, with the product topologies, which are homemorphic for not-quite-trivial reasons.)

We're only interested in sequences of finite sets and surjections, so let's write

$$\mathcal{S} = \{\text{diagrams } \cdots \twoheadrightarrow S_1 \twoheadrightarrow S_0 \text{ of finite sets and surjections}\}.$$

Let $\sim$ be the equivalence relation on $\mathcal{S}$ generated by:

- if $S \cong S'$ then $S \sim S'$
- $(\cdots \twoheadrightarrow S_1 \twoheadrightarrow S_0) \sim (\cdots \twoheadrightarrow S_2 \twoheadrightarrow S_1)$
- if $S \sim S'$ then $S + T \sim S' + T$ for all $T \in \mathcal{S}$ (where $+$ denotes disjoint union).

Certainly $S \sim S' \Rightarrow \lim_{\leftarrow} S$ homeomorphic to $\lim_{\leftarrow} S'$. In fact, $\sim$ is the right notion of sameness of walk-spaces. At present it's a bit hard to see what $\sim$ looks like, so before we go any further, here's a concrete description.

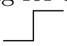
Given $S \in \mathcal{S}$, there is a metric d on $\lim_{\leftarrow} S$:

$$d(x, y) = 2^{-\sup\{n \in \mathbb{N} \mid \mathrm{pr}_n(x) = \mathrm{pr}_n(y)\}}$$

where pr_n is projection onto S_n. This makes each W_n into a metric space.

Clearly this metric is artificial, so we seek a notion of sameness of metric spaces coarse enough to get rid of this artificiality. A map $f : A \longrightarrow B$ of metric spaces is a **scaling** if

$$(\exists \lambda \geq 0)(\forall a, a' \in A) \quad d(f(a), f(a')) = \lambda d(a, a'),$$

and a **local scaling** if there is an open cover $(U_i)_{i \in I}$ of A such that $f|_{U_i}$ is a scaling for all i. Let $\mathbf{Met_{LS}}$ be the category of metric spaces and local scalings. (E.g.  $\cong$  in $\mathbf{Met_{LS}}$; they're locally isometric, in fact.) Then:

Lemma 1.2 *For $S, S' \in \mathcal{S}$,*

$$S \sim S' \iff \lim_{\leftarrow} S \cong \lim_{\leftarrow} S' \text{ in } \mathbf{Met_{LS}}.$$

□

$\mathbf{Met_{LS}}$ has coproducts, $+$, and $W_n \cong W_{n-1} + W_{n+1}$ in $\mathbf{Met_{LS}}$ for all $n \geq 1$.

Finally, we can state a precise version of our result:

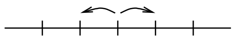
Proposition 1.3 *For $m, n \geq 1$,*

$$W_m \cong W_n \text{ in } \mathbf{Met_{LS}} \iff m \equiv n \pmod 6.$$

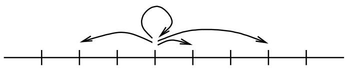
We've already proved $\Leftarrow$. The proof of $\Rightarrow$ uses the Lemma; it's some work (omitted here).

2 General theory of walks

So far, we've considered walks of this type:



Now we'll consider different walk-types, e.g.



A **walk-type** is a finite family $(q_1, \ldots, q_r)$ of integers together with an $N \in \mathbb{N}$ such that $N + q_i \geq 0$ for all i. Interpretation: if you're at position $n \geq N$, you can move q_1 or q_2 or ... or q_r steps right (where of course a negative number of steps right means some steps left); if you're at $n < N$, stay there. As before, this gives a sequence (W_n) of metric spaces, and

$$W_n \cong W_{n+q_1} + \cdots + W_{n+q_r} \quad \forall n \geq N.$$

(I make no claim not to be stretching the 'walks' metaphor too far.)

If all the q_i's were a multiple of 100, say, then for the whole walk we'd be congruent mod 100 to our starting position: we'd never see any of the rest of the natural numbers. So we might as well assume this kind of thing doesn't happen. Also, if the q_is were all non-negative or all non-positive then we'd only be able to move in one direction, and the situation would be very different. Putting the two together, we'll assume that the walk-type is **transitive**, that is, $q_1, \ldots, q_r$ have no non-trivial common factor and $\min_i q_i < 0 < \max_i q_i$. Together, these guarantee that to the right of N, you can get from anywhere to anywhere else. (To the left of N is covered in glue.)

We'd like to be able to argue as follows. Changing subscripts to superscripts, $W^n \cong \sum_{i=1}^r W^{n+q_i}$ for all $n \geq N$, i.e. $W^N \cong \sum W^{N+q_i}$. Write $g(x) = \sum x^{N+q_i} \in \mathbb{N}[x]$ and suppose that

$$\text{for } x \in \mathbb{C}, \quad x^N = g(x) \Rightarrow x^p = 1.$$

(for some p). Then $W^p = 1$, so $W_n \cong W_{n+p}$ for all $n \geq N$.

Our main theorem makes this possible:

Theorem 2.1 *Let $(q_1, \ldots, q_r; N)$ be a transitive walk-type. Write (W_n) for the resulting sequence of walk-spaces and $g(x) = \sum x^{n+q_i}$. For $m, n \geq N$, the following are equivalent:*

(a) $W_m \cong W_n$ *in* $\mathbf{Met_{LS}}$

(b) $(g(x) - x^N) | (x^m - x^n)$ *in* $\mathbb{Z}[x]$.

Under mild further hypotheses, these are also equivalent to:

(c) *for* $x \in \mathbb{C}$, $x^N = g(x) \Rightarrow x^{m-n} = 1$.

This really restricts what the pattern of sameness and difference in the sequence (W_n) can be. E.g. if $W_{10} \cong W_{14}$ then $(g(x) - x^N)$ divides $(x^{10} - x^{14})$, so also divides $(x^{11} - x^{15})$, so $W_{11} \cong W_{15}$, etc; hence the sequence $W_{10}, W_{11}, W_{12}, W_{13}$ just gets repeated over and over.

Corollary 2.2 *For a transitive walk-type,* either $(W_n)_{n \geq N}$ *is periodic* or *the spaces* $(W_n)_{n \geq N}$ *are all different.* □

This is bizarre! If W_n were a function of W_{n-1} (i.e. depended only on W_{n-1}) then this would obviously be true. But since that's not the case, it's a surprise. Maybe there's some way of looking at it that makes it obvious.

The main ingredient in the proof of the Theorem concerns rigs (= ri*n*gs without *n*egatives, or semirings):

Proposition 2.3 (joint with Marcelo Fiore) *Let* $g(x), p(x), q(x) \in \mathbb{N}[x]$ *and* $N \in \mathbb{N}$*, satisfying mild conditions. Suppose that*

$$\text{for all rings } A \text{ and } a \in A, \qquad a^N = g(a) \Rightarrow p(a) = q(a).$$

Then

$$\text{for all rigs } A \text{ and } a \in A, \qquad a^N = g(a) \Rightarrow p(a) = q(a).$$

Loosely, 'if you can prove it with the help of subtraction, you can also prove it without'. A version of the proposition with $N = 1$ appears in Fiore and Leinster, 'Objects of categories as complex numbers'.

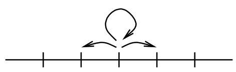
Examples 2.4

a. The result of §1 now becomes an example. Take the original walk-type $W_n = W_{n-1} + W_{n+1}$, or formally, $(q_1, q_2; N) = (-1, 1; 1)$. Then $N = 1$, $g(x) = 1 + x^2$, and

$$g(x) - x^N = (1 - x + x^2) | (1 - x^6)$$

(corresponding to the fact that the complex roots of $1 - x + x^2$ are 6th roots of unity). So $(W_n)_{n \geq 1}$ has period 6, as before.

b. You may have the option to stay still:

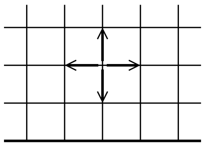


Then $W_n = W_{n-1} + W_n + W_{n+1}$ and $N = 1$. Thus $g(x) = 1 + x + x^2$ and

$$g(x) - x^N = (1 + x^2) | (1 - x^4),$$

so $(W_n)_{n \geq 1}$ has period 4.

c. Let W_n be the space of walks on the upper half-plane starting at $(0, n)$, of the type suggested by the following picture:



Then $W_n \cong W_{n-1} + 2W_n + W_{n+1}$ and we can take $N = 1$. Thus $g(x) = 1 + 2x + x^2$ and

$$g(x) - x^N = (1 + x + x^2) | (1 - x^3),$$

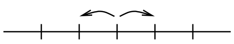
so $(W_n)_{n \geq 1}$ has period 3.

d. Let's do a d-dimensional version of (a) (where $d = 1$) and (c) (where $d = 2$). The 1-dimensional case has period 6, and the 2-dimensional case period 3, so we might guess that the 3-dimensional case has period 0. This would mean that $W_m \cong W_n$ if and only if $0|(m - n)$, i.e. $m = n$; in other words, that no two of the W_n's are isomorphic. It may be coincidence, but this is indeed the case!

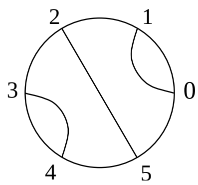
In fact, the same thing happens for all $d \geq 3$. It's easy to show that $g(x) - x$ has some root that is not a root of unity, so $g(x) - x$ does not divide $x^p - 1$ for any p, so (W_n) is not periodic, so (by the Corollary) $W_m \not\cong W_n$ whenever $m \neq n$.

(After the talk, Wolfgang Pitsch pointed out to me that there is also a division between *random* walks in dimension ≤ 2 and in dmiension > 3: in the former case, you almost surely return to your starting point, and in the latter case, you almost surely don't. It would be very nice if there were some connection.)

e. Finally, let's take the original walk-style



but change the boundary condition: if at 0, step right. Then $W_0 \cong W_1$. It turns out that (W_n) still has period 6, but there are some repetitions within each cycle:



(meaning that $W_0 \cong W_1$, $W_2 \cong W_5$, $W_3 \cong W_4$, and these three spaces are all different).

3 What can you do with a polynomial? (A substitute for a satisfactory ending)

More precisely: take $g(x) \in \mathbb{N}[x]$ and $N \in \mathbb{N}$. What can you do with the equation $x^N = g(x)$?

- Solve it (in $\mathbb{C}$) and study properties of the roots. E.g. for $x = 1 + x^2$: roots are $e^{\pm \pi i/3}$ and satisfy $x^6 = 1$. OR:
- Take the resulting recurrence relation: e.g. for $x = 1 + x^2$, it's

$$\begin{aligned} x_1 &= x_0 + x_2 \\ x_2 &= x_1 + x_3 \\ &\vdots \end{aligned}$$

  Then:

  - solve that (in $\mathbb{C}$, or another rig). E.g. for $x = 1 + x^2$, the solutions are 6-periodic. Or:
  - consider the resulting sequence (W_n) of walk-spaces. This is the terminal coalgebra for a certain endofunctor T of $\mathbf{Top}^{\mathbb{N}}$, e.g. for $x = 1 + x^2$,

$$T(X_0, X_1, X_2, \ldots) = (X_0, X_0 + X_2, X_1 + X_3, \ldots).$$

    Or:
  - consider the resulting 'Jónsson–Tarski topos', in which an object is a sequence $(X_0, X_1, \ldots)$ of sets together with (in our running example) bijections

$$\begin{aligned} X_0 &\xrightarrow{\sim} X_0 \\ X_1 &\xrightarrow{\sim} X_0 \times X_2 \\ X_2 &\xrightarrow{\sim} X_1 \times X_3 \\ &\vdots \end{aligned}$$

    (A topos is a topological object too!)

So there are various structures you can derive from a polynomial equation of this kind. How are they related? I don't know.