

Lecture 1

Categories and functors

Definition 1.1 A category \mathcal{A} consists of

- a collection $\text{ob}(\mathcal{A})$ (whose elements are called the **objects** of \mathcal{A})
- for each $A, B \in \text{ob}(\mathcal{A})$, a collection $\mathcal{A}(A, B)$ (whose elements are called the **maps** or **morphisms** or **arrows** from A to B)
- for each $A, B, C \in \text{ob}(\mathcal{A})$, a function

$$\begin{array}{ccc} \mathcal{A}(B, C) \times \mathcal{A}(A, B) & \longrightarrow & \mathcal{A}(A, C), \\ (g, f) & \longmapsto & g \circ f \end{array}$$

(called **composition**)

- for each $A \in \text{ob}(\mathcal{A})$, an element $1_A \in \mathcal{A}(A, A)$ (the **identity** on A)

satisfying

- **associativity:** $(h \circ g) \circ f = h \circ (g \circ f)$ for all $A, B, C, D \in \text{ob}(\mathcal{A})$, $f \in \mathcal{A}(A, B)$, $g \in \mathcal{A}(B, C)$, and $h \in \mathcal{A}(C, D)$
- **unit laws:** $f \circ 1_A = f = 1_B \circ f$ for all $A, B \in \text{ob}(\mathcal{A})$ and $f \in \mathcal{A}(A, B)$.

Notation 1.2 We often write:

$$\begin{array}{l} A \in \mathcal{A} \quad \text{to mean} \quad A \in \text{ob}(\mathcal{A}) \\ f : A \longrightarrow B \text{ or } A \xrightarrow{f} B \quad \text{to mean} \quad f \in \mathcal{A}(A, B) \\ gf \quad \text{to mean} \quad g \circ f. \end{array}$$

People often write $\mathcal{A}(A, B)$ as $\text{Hom}_{\mathcal{A}}(A, B)$ or $\text{Hom}(A, B)$.

Remarks 1.3 a. Loosely, a category is a system of objects and arrows in which any string of arrows

$$A_0 \xrightarrow{f_1} \cdots \xrightarrow{f_n} A_n$$

gives rise to precisely one arrow $A_0 \longrightarrow A_n$. When $n = 2$, this is composition; when $n = 0$, it is the formation of identities; when $n = 3$, the ‘precisely one’ part implies the associativity law; and when $n = 0$, it implies the unit laws.

- b. I will say as little as you let me about set theory. It suffices to make a naive distinction between **small** and **large** collections, which can be interpreted as meaning ‘sets’ and ‘proper classes’ respectively. A category \mathcal{A} is **locally small** if $\mathcal{A}(A, B)$ is a small collection for each A and B . (Many authors build locally small into their definition of category.) A category \mathcal{A} is **small** if it is locally small and the collection $\text{ob}(\mathcal{A})$ is small.
- c. A harmless convention is that $\mathcal{A}(A, B) \cap \mathcal{A}(A', B') = \emptyset$ unless $A = A'$ and $B = B'$. If $f \in \mathcal{A}(A, B)$, the **domain** of f is A and the **codomain** of f is B .

The most obvious examples of categories come under the banner ‘categories of mathematical structures’.

Example 1.4 There is a category **Set** in which the objects are sets and the maps are functions. Similarly:

- **Top** is topological spaces and continuous maps
- **Gp** is groups and homomorphisms
- **Ab** is abelian groups and homomorphisms
- **k -Mod** is (left) k -modules and homomorphisms, for any ring k .

Example 1.5 There is an obvious notion of **subcategory**. For instance, there is a subcategory of **Ab** consisting of all abelian groups with between 50 and 60 elements and all surjective homomorphisms between them.

A map $f : A \longrightarrow B$ in a category is an **isomorphism** if there exists $f' : B \longrightarrow A$ satisfying $f'f = 1_A$ and $ff' = 1_B$. There is at most one such f' (exercise), so we may write $f' = f^{-1}$ and call it the **inverse** of f . We also call A and B isomorphic and write $A \cong B$.

The next examples are ‘categories as mathematical structures’.

Example 1.6 A **(partial) order** on a set A is a binary relation \leq on A that is reflexive, transitive, and antisymmetric ($a \leq b \leq a$ implies $a = b$). Examples: $A = \mathbb{R}$ and \leq has the usual meaning; A is the set of subsets of some fixed set and \leq is \subseteq ; $A = \mathbb{N}$ and $a \leq b$ means $a|b$.

An ordered set (A, \leq) can be regarded as a category \mathcal{A} in which each ‘hom-set’ $\mathcal{A}(a, b)$ has at most one element. The objects of \mathcal{A} are the elements of A , and there is an arrow $a \longrightarrow b$ if and only if $a \leq b$. It doesn’t matter what you call this arrow; you can think of it as ‘the assertion that $a \leq b$ ’. This example shows that the ‘maps’ in a category need not be remotely like ‘maps’ in the sense of functions.

Example 1.7 A **monoid** is a set equipped with an associative binary operation and a two-sided unit (e.g. $(\mathbb{N}, +, 0)$).

A small category with precisely one object is the same thing as a monoid. For if the object is called \star , say, then such a category consists of a single hom-set $\mathcal{A}(\star, \star)$ together with an associative binary operation (composition) and a two-sided unit (the identity on \star).

Example 1.8 In particular, a group is the same thing as a one-object small category in which every arrow is an isomorphism.

Example 1.9 A **groupoid** is a category in which every arrow is an isomorphism. Every topological space X has a **fundamental groupoid** $\Pi_1(X)$, whose objects are the points of X and whose arrows $x \longrightarrow y$ are the homotopy classes of paths from x to y .

Digression 1.10 You might have noticed that in many categories \mathcal{A} , the sets $\mathcal{A}(A, B)$ carry extra structure. For instance, if $\mathcal{A} = k\text{-Mod}$ then they are abelian groups, and if \mathcal{A} is a suitable category of spaces then they carry a topology. Such things are called ‘enriched categories’.

Homological algebra works with ‘abelian categories’. An **Ab-category** is a category in which each $\mathcal{A}(A, B)$ has the structure of an abelian group and composition is bilinear. An **additive category** is an **Ab-category** satisfying further conditions. An **abelian category** is an additive category satisfying further conditions still, enabling one to define and manipulate exact sequences inside the category. The basic example is $k\text{-Mod}$ where k is a commutative ring.

Definition 1.11 Let \mathcal{A} and \mathcal{B} be categories. A **functor** $F : \mathcal{A} \longrightarrow \mathcal{B}$ consists of

- a function

$$\begin{array}{ccc} \text{ob}(\mathcal{A}) & \longrightarrow & \text{ob}(\mathcal{B}), \\ A & \longmapsto & FA \end{array}$$

- for each $A, A' \in \text{ob}(\mathcal{A})$, a function

$$\begin{array}{ccc} \mathcal{A}(A, A') & \longrightarrow & \mathcal{B}(FA, FA'), \\ f & \longmapsto & Ff \end{array}$$

such that

- $F(f' \circ f) = Ff' \circ Ff$ for all $A \xrightarrow{f} A' \xrightarrow{f'} A''$ in \mathcal{A}

- $F1_A = 1_{FA}$ for all $A \in \mathcal{A}$.

Loosely, a functor $\mathcal{A} \longrightarrow \mathcal{B}$ is something that associates to every object A of \mathcal{A} an object FA of \mathcal{B} and to every string of arrows

$$A_0 \xrightarrow{f_1} \cdots \xrightarrow{f_n} A_n$$

in \mathcal{A} precisely one arrow $FA_0 \longrightarrow FA_n$. When $n = 1$ this says what the (Ff) s are; when $n = 2$ and $n = 0$ it implies that F preserves composition and identities.

Example 1.12 ‘Forgetful functors’ (an informal term) are functors that forget structure or properties. For instance, there is a functor $\mathbf{Gp} \longrightarrow \mathbf{Set}$ sending every group to its underlying set; it ‘forgets’ the group structure and that homomorphisms are homomorphisms. There is a forgetful functor $\mathbf{Ab} \longrightarrow \mathbf{Gp}$, which might also be called an inclusion; it forgets the property of being abelian.

Example 1.13 In the other direction, ‘free functors’ add in structure or properties freely. For instance, there is a functor $\mathbf{Set} \longrightarrow \mathbf{Gp}$ that forms the free group on each set, and a functor $F : \mathbf{Gp} \longrightarrow \mathbf{Ab}$ that sends each group to its largest abelian quotient: $F(X)$ is $X^{\text{ab}} = X/[X, X]$, the **abelianization** of X .

Example 1.14 Any monoid M (e.g. a group) can be regarded as a one-object category (1.7). A functor $M \longrightarrow \mathbf{Set}$ is just a set with a left A -action. Similarly, a functor from $A \longrightarrow k\text{-Mod}$ is a k -linear representation of A .

Some functors reverse the direction of arrows: an arrow $A \xrightarrow{f} A'$ in \mathcal{A} gives rise to an arrow $FA \xleftarrow{Ff} FA'$ in \mathcal{B} . This can be made precise as follows. Given a category \mathcal{A} , the **opposite** or **dual** category \mathcal{A}^{op} is defined by $\text{ob}(\mathcal{A}^{\text{op}}) = \text{ob}(\mathcal{A})$ and $\mathcal{A}^{\text{op}}(A', A) = \mathcal{A}(A, A')$; composition and identities are as in \mathcal{A} , but reversed. A functor $\mathcal{A}^{\text{op}} \longrightarrow \mathcal{B}$ (or equivalently, $\mathcal{A} \longrightarrow \mathcal{B}^{\text{op}}$) is called a **contravariant functor**; ordinary functors are sometimes called **covariant**, for emphasis.

Example 1.15 Taking duals defines a functor

$$\begin{array}{ccc} \mathbf{Vect}_k^{\text{op}} & \longrightarrow & \mathbf{Vect}_k, \\ V & \longmapsto & V^*, \end{array}$$

where \mathbf{Vect}_k is the category of vector spaces over a field k .

Example 1.16 Homology defines a functor $H_* : \mathbf{Top} \longrightarrow \mathbf{GrAb} =$ (graded abelian groups); cohomology defines a functor $H^* : \mathbf{Top}^{\text{op}} \longrightarrow \mathbf{GrAb}$.

Example 1.17 Fix a topological space X . The set $\mathbf{Open}(X)$ of open subsets of X can be ordered by inclusion, and so forms a category (1.6). A functor $\mathbf{Open}(X)^{\text{op}} \longrightarrow \mathbf{Set}$ is called a **presheaf** (of sets) on X . Concretely, a presheaf on X consists of a set $F(U)$ for each open set U , and, for each pair $U' \subseteq U$ of open sets, a function $F(U) \longrightarrow F(U')$, satisfying some axioms. Example: $F(U)$ is the set of continuous maps $U \longrightarrow \mathbb{R}$, and the functions $F(U) \longrightarrow F(U')$ are given by restriction.

More generally, a **presheaf** on a category \mathcal{A} is a functor $\mathcal{A}^{\text{op}} \longrightarrow \mathbf{Set}$.

Functors are the structure-preserving maps of categories; they can be composed, so there is a (large) category \mathbf{Cat} consisting of small categories and functors. Informally, there is also a (huge) category \mathbf{CAT} consisting of all categories and functors.

In the next lecture we'll see that there is a further notion of map between functors.

Exercises

1.18 There is a category \mathbf{Toph} whose objects are topological spaces and whose arrows $X \longrightarrow Y$ are homotopy classes of continuous maps from X to Y . What would you need to know about homotopy in order to prove that this is a category? What does it mean for two objects of \mathbf{Toph} to be isomorphic?

1.19 Two categories \mathcal{A} and \mathcal{B} are **isomorphic**, written $\mathcal{A} \cong \mathcal{B}$, if they are isomorphic as objects of \mathbf{CAT} . Prove that any group, regarded as a one-object category, is isomorphic to its opposite. Find a monoid not isomorphic to its opposite.

1.20 Prove that functors preserve isomorphism.