Solutions to exercises from Lecture 8

8.23 For each $Y \in \mathcal{V}$ we have an adjunction $- \otimes Y \dashv [Y, -]$, whose counit ε^Y has components

$$\varepsilon^Y_Z : [Y, Z] \otimes Y \longrightarrow Z$$

($Z \in \mathcal{V}$). (When $\mathcal{V}$ is **Set** with its usual product structure, ε is evaluation: $(f, y) \longmapsto f(z)$.) Since $\mathcal{V}$ is symmetric, we also have for each $X, Y \in \mathcal{V}$ a symmetry isomorphism $\gamma_{X,Y} : X \otimes Y \longrightarrow Y \otimes X$.

Given $X \in \mathcal{V}$, maps $X \longrightarrow X^{**} = [X^*, I]$ are in natural bijection with maps $X \otimes X^* \longrightarrow I$, and we may therefore define α_X to be the map $X \longrightarrow X^{**}$ corresponding to the composite

$$X \otimes X^* \xrightarrow{\gamma_{X,X^*}} X^* \otimes X \xrightarrow{\varepsilon^X_I} I.$$

In the spirit of this course, naturality of α is an inevitable consequence of the fact that it was built entirely of natural materials; alternatively, because no arbitrary choices were made. (To check it rigorously you first have to work out how the functor $(\ )^*$ is defined on maps. As usual, there is only one possibility: given $f : X \longrightarrow Y$, the map $f^* : Y^* \longrightarrow X^* = [X, I]$ is the transpose of the composite

$$[Y, I] \otimes X \xrightarrow{1 \otimes f} [Y, I] \otimes Y \xrightarrow{\varepsilon^Y_I} I$$

under the adjunction $- \otimes X \dashv [X, -]$.)

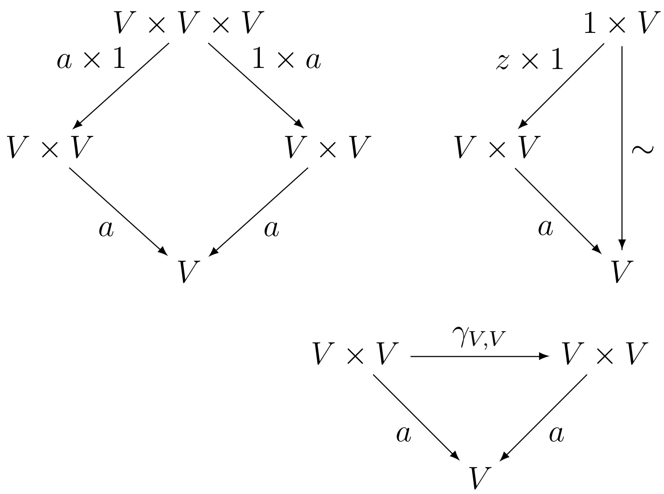
There are many examples where α is not an isomorphism. For instance, if $(\mathcal{V}, \otimes, I) = (\mathbf{Set}, \times, 1)$ then $X^* \cong 1$ for all X, so $X^{**} \cong 1$ for all X, so α_X is not an isomorphism unless $X \cong 1$, so α is not a natural isomorphism.

8.24 Part of the point of this question is that the theory of modules over a *fixed* ring (e.g. over $\mathbb{C}$) is an algebraic theory like any other. There is also a ('2-sorted') algebraic theory of ring-module pairs, a model of which in **Set** is a pair (R, M) consisting of a ring R and an R-module M, and it is possible to define 'ring-module pair' inside any category with finite products. But that is not the kind of thing required here.

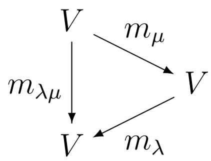
So, a **complex vector space** in $\mathcal{V}$ consists of an object $V \in \mathcal{V}$, maps

$$V \times V \xrightarrow{a} V, \qquad 1 \xrightarrow{z} V, \qquad V \xrightarrow{n} V,$$

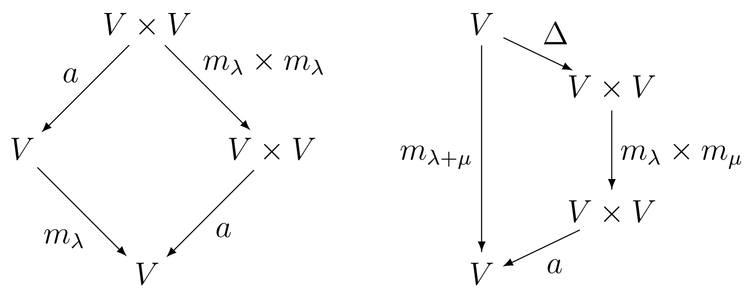
and one map $V \xrightarrow{m_\lambda} V$ for each complex number λ, satisfying the following axioms. (Think of a as addition, z as the zero element, n as the negative map $v \longmapsto -v$, and m_λ as scalar multiplication by λ.) First, the abelian group axioms: the diagrams



commute, where in the last diagram, $\gamma_{V,V}$ is the symmetry map (whose first component is second projection and *vice versa*). Second, the axioms '$1v = v$' and '$(\lambda\mu)v = \lambda(\mu v)$' are that $m_1 = 1_V$ and the diagram



commutes for all $\lambda, \mu \in \mathbb{C}$. Finally, the distributivity axioms, '$\lambda(v + w) = \lambda v + \lambda w$' and '$(\lambda + \mu)v = \lambda v + \mu v$': the diagrams



commute, the former for all $\lambda \in \mathbb{C}$ and the latter for all $\lambda, \mu \in \mathbb{C}$.