

## Solutions to exercises from Lecture 6

**6.19** By Theorem 6.5, it is enough to show that none of these forgetful functors preserves colimits. In fact, none of them even preserves initial objects. For each of the categories **Gp**,  $k\text{-Mod}$ , **LieAlg**, and **Ring** does have an initial object (respectively:  $\{1\}$ ,  $\{0\}$ ,  $\{0\}$ , and  $\mathbb{Z}$ ), and in each case the (underlying set of the) initial object is nonempty. But the initial object of **Set** is  $\emptyset$ .

**6.20** Take a category  $\mathcal{A}$ , a diagram  $D : \mathbb{I} \longrightarrow \mathcal{A}$  that has a colimit, and an object  $A \in \mathcal{A}$ . The functor  $D$  gives rise to a functor  $D^{\text{op}} : \mathbb{I}^{\text{op}} \longrightarrow \mathcal{A}^{\text{op}}$ , and a colimit of  $D$  in  $\mathcal{A}$  is exactly a limit of  $D^{\text{op}}$  in  $\mathcal{A}^{\text{op}}$ , so  $D^{\text{op}}$  has a limit. Equation (7) applied to  $\mathcal{A}^{\text{op}}$ ,  $A$  and  $D^{\text{op}}$  is

$$\mathcal{A}^{\text{op}}(A, \lim_{\leftarrow I} D^{\text{op}} I) \cong \lim_{\leftarrow I} \mathcal{A}^{\text{op}}(A, D^{\text{op}} I),$$

that is,

$$\mathcal{A}(\lim_{\rightarrow I} DI, A) \cong \lim_{\leftarrow I} \mathcal{A}(DI, A),$$

as required.

**6.21** Briefly put: epics are characterized by colimits, so epics in  $[\mathbb{A}^{\text{op}}, \mathbf{Set}]$  are computed pointwise; hence  $\alpha$  is epic exactly when each  $\alpha_A$  is surjective.

More slowly: let  $\alpha : X \longrightarrow Y$  be a map in  $[\mathbb{A}^{\text{op}}, \mathbf{Set}]$  (thus, a natural transformation). Then  $\alpha$  is epic if and only if the square

$$\begin{array}{ccc}
 & Y & \\
 \alpha \nearrow & & \searrow 1 \\
 X & & Y \\
 \alpha \searrow & & \nearrow 1 \\
 & Y &
 \end{array}$$

is a pushout, by Proposition 5.7. Corollary 6.14 tells us that pushouts in

$[\mathbb{A}^{\text{op}}, \mathbf{Set}]$  are computed pointwise, so this square is a pushout if and only if

$$\begin{array}{ccc} & YA & \\ \alpha_A \nearrow & & \searrow 1 \\ XA & & YA \\ \alpha_A \searrow & & \nearrow 1 \\ & YA & \end{array}$$

is a pushout for each  $A \in \mathbb{A}$ . By Proposition 5.7 again, this is true if and only if each  $\alpha_A$  is epic. But epics in  $\mathbf{Set}$  are just surjections, so  $\alpha$  is epic if and only if each  $\alpha_A$  is surjective.

I do not know of any more direct proof, although there is one for the analogous fact about monics.