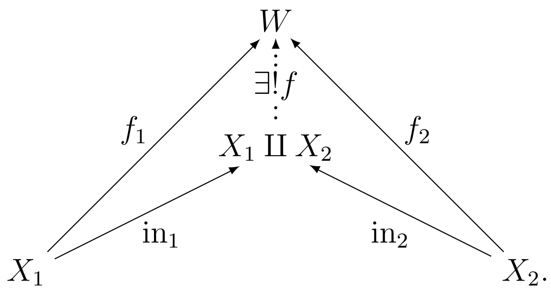
Solutions to exercises from Lecture 0

0.12 Here are three possibilities:

a. Take sets X_1 and X_2, their disjoint union $X_1 \amalg X_2$, and the inclusions

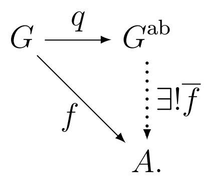
$$X_1 \xrightarrow{\mathrm{in}_1} X_1 \amalg X_2 \xleftarrow{\mathrm{in}_2} X_2.$$

Then the following universal property holds:



(Compare Example 0.5.)

b. Take a group G, its abelianization $G^{\mathrm{ab}} = G/[G,G]$, and the natural map $q : G \longrightarrow G^{\mathrm{ab}}$. Then $G \xrightarrow{q} G^{\mathrm{ab}}$ is the universal homomorphism from G to an abelian group. That is, if $G \xrightarrow{f} A$ is a homomorphism from G to an abelian group A, there is a unique homomorphism $\overline{f} : G^{\mathrm{ab}} \longrightarrow A$ such that $\overline{f} \circ q = f$:

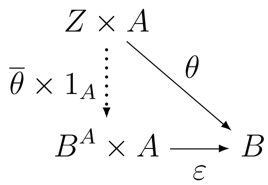


(Compare Example 0.1.)

c. Take sets A and B, the set B^A of functions from A to B, and the evaluation map

$$\begin{array}{rccc} \varepsilon : & B^A \times A & \longrightarrow & B \\ & (f, a) & \longmapsto & f(a). \end{array}$$

For any set Z and function $\theta : Z \times A \longrightarrow B$, there is a unique function $\overline{\theta} : Z \longrightarrow B^A$ such that

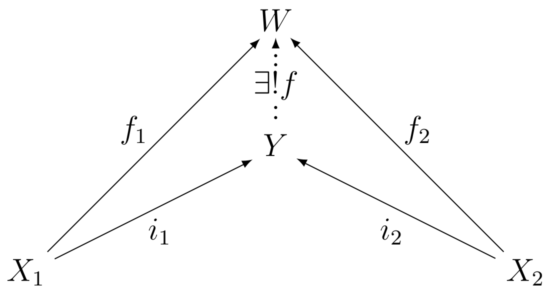


commutes. (This $\overline{\theta}$ is given by $(\overline{\theta}(z))(a) = \theta(z, a)$.)

0.13 I'll do (a). Fix sets X_1 and X_2. Suppose that

$$X_1 \xrightarrow{i_1} Y \xleftarrow{i_2} X_2$$

has the universal property

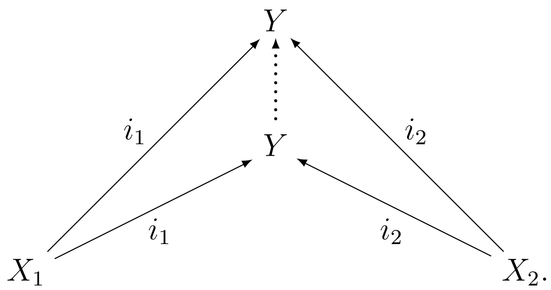


(1)

and that

$$X_1 \xrightarrow{i'_1} Y' \xleftarrow{i'_2} X_2$$

does too. Take $(W, f_1, f_2) = (Y', i'_1, i'_2)$ in (1): then we obtain $f : Y \longrightarrow Y'$ such that $fi_1 = i'_1$ and $fi_2 = i'_2$. Similarly, we obtain $f' : Y' \longrightarrow Y$ such that $f'i'_1 = i_1$ and $f'i'_2 = i_2$. But then $(f'f)i_1 = 1_Y i_1$ and $(f'f)i_2 = 1_Y i_2$; in other words, the following diagram commutes when the dotted arrow is either $f'f$ or 1_Y:



So by the uniqueness part of the universal property, $f'f = 1_Y$. Similarly, $ff' = 1_{Y'}$. So $Y \cong Y'$.