

## Lecture 3

# Adjoint

‘Adjoint functors arise everywhere’ (Mac Lane).

**Example 3.1** Given any set  $S$ , there is a vector space  $F(S)$  with basis  $S$ ; its elements can be thought of as formal linear combinations  $\sum_i \lambda_i s_i$  of elements of  $S$ . It has a universal property (Example 0.2): any function from  $S$  to a vector space  $V$  extends uniquely to a linear map  $F(S) \longrightarrow V$ .

This can be rephrased as follows. Let  $U : \mathbf{Vect} \longrightarrow \mathbf{Set}$  be the forgetful functor. Then for any set  $S$  and vector space  $V$ , there is a function

$$\text{restriction} : \mathbf{Vect}(F(S), V) \longrightarrow \mathbf{Set}(S, U(V)),$$

and the unique extension property says exactly that this function is a bijection. So we have functors  $\mathbf{Set} \xrightleftharpoons[U]{F} \mathbf{Vect}$  such that for each  $S \in \mathbf{Set}$  and  $V \in \mathbf{Vect}$ ,

$$\mathbf{Vect}(F(S), V) \cong \mathbf{Set}(S, U(V)).$$

This isomorphism is (in a sense that can be made precise) natural in  $S$  and  $V$ .

**Definition 3.2** Take categories  $\mathcal{A}$  and  $\mathcal{B}$  and functors  $\mathcal{A} \begin{matrix} \xrightarrow{F} \\ \xleftarrow{G} \end{matrix} \mathcal{B}$ . If

$$\mathcal{B}(FA, B) \cong \mathcal{A}(A, GB) \quad (3)$$

naturally in  $A \in \mathcal{A}$  and  $B \in \mathcal{B}$ , we say that  $F$  is **left adjoint** to  $G$ , and  $G$  is **right adjoint** to  $F$ , and write  $F \dashv G$ . An **adjunction** between  $F$  and  $G$  is a choice of natural isomorphism (3).

What this says is that for any  $A \in \mathcal{A}$  and  $B \in \mathcal{B}$ , a map  $FA \longrightarrow B$  is the same thing as a map  $A \longrightarrow GB$ . The correspondence is written as  $\overline{(\quad)}$  (**transpose**) in both directions:

$$\begin{aligned} (FA \xrightarrow{g} B) &\longmapsto (A \xrightarrow{\overline{g}} GB), \\ (FA \xrightarrow{\overline{f}} B) &\longleftarrow (A \xrightarrow{f} GB), \end{aligned}$$

and  $\overline{\overline{f}} = f$ ,  $\overline{\overline{g}} = g$ . In the examples that follow, I'll be content to interpret the 'naturally' part of the definition informally, but for the record, naturality amounts

to the equation

$$\overline{\left(FA' \xrightarrow{Fp} FA \xrightarrow{g} B \xrightarrow{q} B'\right)} = \left(A' \xrightarrow{p} A \xrightarrow{\bar{g}} GB \xrightarrow{Gq} GB'\right).$$

A given functor  $G$  may not have a left adjoint, but if it does, it is unique up to isomorphism, so we may speak of ‘*the* left adjoint of  $G$ ’ (and dually for right adjoints). We prove this uniqueness next lecture.

**Example 3.3 (Free  $\dashv$  forgetful)** Forgetful functors between categories of algebras always have left adjoints:

$$\begin{array}{cccccc} \mathbf{Vect} & \mathbf{Gp} & \mathbf{Ring} & \mathbf{AssAlg} & \mathbf{Ab} & \mathbf{Gp} \\ F_1 \uparrow \dashv \downarrow U_1 & F_2 \uparrow \dashv \downarrow U_2 & F_3 \uparrow \dashv \downarrow U_3 & F_4 \uparrow \dashv \downarrow U_4 & F_5 \uparrow \dashv \downarrow U_5 & F_6 \uparrow \dashv \downarrow U_6 \\ \mathbf{Set} & \mathbf{Set} & \mathbf{Monoid} & \mathbf{LieAlg} & \mathbf{Gp} & \mathbf{Monoid}. \end{array}$$

- $F_1 \dashv U_1$  is the adjunction of Example 3.1.
- $F_2 \dashv U_2$  is very similar;  $F_2$  forms the free group on a set.
- $U_3$  forgets the additive structure of a ring, and

$$F_3(M) = \mathbb{Z}M = \{\text{formal sums } \sum \lambda_i m_i \text{ with } \lambda_i \in \mathbb{Z}, m_i \in M\}$$

(which when  $M$  is a group is called the **group ring** of  $M$ ).

- $U_4$  forgets everything about an associative algebra  $B$  except its underlying vector space and its commutator; thus  $U_4(B)$  is a Lie algebra with bracket  $[b_1, b_2] = b_1b_2 - b_2b_1$ . The left adjoint  $F_4$  assigns to a Lie algebra its **universal enveloping algebra**.
- $U_5$  forgets that an abelian group is abelian; its left adjoint  $F_5$  is abelianization (Example 1.13), and adjointness says that a map  $G \longrightarrow A$  is the same thing as a map  $G^{\text{ab}} \longrightarrow A$  (for  $G \in \mathbf{Gp}$  and  $A \in \mathbf{Ab}$ ).
- $U_6$  forgets that a group has inverses; its left adjoint  $F_6$  turns a monoid  $M$  into a group by throwing in a formal inverse  $m^{-1}$  for each element  $m$  of  $M$ . This left adjoint is crucial in  $K$ -theory.

Warning: categories such as (fields) and (torsion abelian groups) are not categories of algebras in the sense used here, and their forgetful functors to **Set** do not have left adjoints.

**Example 3.4 ( $\otimes$  and  $\mathbf{Hom}$ )** For modules  $A$ ,  $B$ , and  $C$  over a commutative ring  $k$ , there is an isomorphism

$$k\text{-Mod}(A \otimes B, C) \cong k\text{-Mod}(A, \mathbf{Hom}_k(B, C))$$

where  $\mathbf{Hom}_k(B, C)$  is the set  $k\text{-Mod}(B, C)$  with its evident  $k$ -module structure. So for every  $k$ -module  $B$  gives rise to an adjunction

$$\begin{array}{ccc} & k\text{-Mod} & \\ & \uparrow \dashv \downarrow & \\ - \otimes B & & \mathbf{Hom}_k(B, -) \\ & \downarrow & \\ & k\text{-Mod} & \end{array}$$

A slightly different situation is ‘change of base’: any map  $\phi : k \longrightarrow k'$  of commutative rings induces functors and adjunctions

$$\begin{array}{ccc} & k\text{-Mod} & \\ & \uparrow \dashv \downarrow & \\ k' \otimes_k - & & \mathbf{Hom}_k(k', -) \\ & \downarrow \phi^* \downarrow & \\ & k'\text{-Mod} & \end{array}$$

**Example 3.5 (Topology)** Miscellaneous topological adjunctions:

$$\begin{array}{ccc}
 \mathbf{Top} & \mathbf{CptHff} & \mathbf{CompleteMet} \\
 D \uparrow \dashv U \dashv I & F_7 \uparrow \dashv U_7 & F_8 \uparrow \dashv U_8 \\
 \mathbf{Set} & \mathbf{Top} & \mathbf{Met.}
 \end{array}$$

- $U$  sends a topological space to its underlying set, and  $D$  and  $I$  endow a set with the discrete and indiscrete topology, respectively.
- $U_7$  is the inclusion of compact Hausdorff spaces into all spaces, and  $F_7$  is Stone-Ćech compactification (more on which below).
- $\mathbf{Met}$  is the category of metric spaces and uniformly continuous maps,  $\mathbf{CompleteMet}$  is the full subcategory consisting of the complete metric spaces,  $U_8$  is inclusion, and  $F_8$  is metric space completion.

**Example 3.6 (Galois connections)** Take ordered

sets  $A$  and  $B$  and an adjoint pair

$$\begin{array}{ccc} & B^{\text{op}} & \\ f \uparrow & \dashv & \downarrow g \\ & A & \end{array}$$

This means that  $f$  and  $g$  are order-reversing functions and

$$b \leq f(a) \iff a \leq g(b).$$

Such an adjunction  $f \dashv g$  is called a **Galois connection** because of the following example: for fields  $K \subseteq M$ , we have

$$\begin{array}{ccc} & (\text{subgroups of } \text{Gal}(M : K))^{\text{op}} & \\ \text{invariant subgroup} \uparrow & \dashv & \downarrow \text{fixed field} \\ & (\text{intermediate fields of } M : K) & \end{array}$$

For any Galois connection  $f \dashv g$  there is an order-reversing isomorphism

$$(\text{image}(f))^{\text{op}} \cong \text{image}(g).$$

In the example given (Galois theory), the existence of the Galois connection is trivial; identifying the images

is the hard part, and to do it we usually make simplifying assumptions such as normality and separability.

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There are two other ways of saying what an adjunction is. First:

**Proposition 3.7** *Take categories and functors  $\mathcal{A} \xrightleftharpoons[G]{F} \mathcal{B}$ . Then there is a one-to-one correspondence between*

- adjunctions  $F \dashv G$
- pairs  $(1_{\mathcal{A}} \xrightarrow{\eta} G \circ F, F \circ G \xrightarrow{\varepsilon} 1_{\mathcal{B}})$  of natural transformations satisfying the **triangle identities**: the triangles

$$\begin{array}{ccc}
 F(A) & \xrightarrow{F\eta_A} & FGF(A) \\
 \searrow 1_{F(A)} & & \downarrow \varepsilon_{F(A)} \\
 & & F(A)
 \end{array}
 \qquad
 \begin{array}{ccc}
 G(B) & \xrightarrow{\eta_{G(B)}} & GFG(B) \\
 \searrow 1_{G(B)} & & \downarrow G\varepsilon_B \\
 & & G(B)
 \end{array}$$

commute for all  $A \in \mathcal{A}$  and  $B \in \mathcal{B}$ .

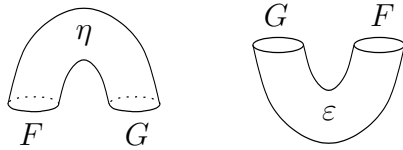
**Sketch proof** An adjunction  $F \dashv G$  consists of a bijection

$$\mathcal{B}(FA, B) \xrightarrow{\sim} \mathcal{A}(A, GB)$$

for each  $A$  and  $B$ , satisfying naturality axioms. For each  $A \in \mathcal{A}$  we can take  $B = FA$  and the identity  $1_{FA} \in \mathcal{B}(FA, FA)$  to obtain a map  $\eta_A = \overline{1_{FA}} : A \longrightarrow GFA$ . This defines a natural transformation  $\eta : 1 \longrightarrow GF$ , the **unit** of the adjunction. Dually, there is a **counit**  $\varepsilon : FG \longrightarrow 1$ . The triangle identities follow from naturality.

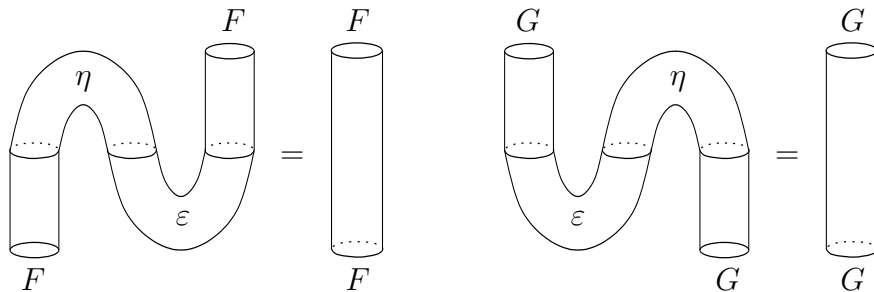
Conversely, the whole adjunction can be recovered from the unit and counit: with the usual notation,  $\overline{g} = (Gg) \circ \eta_A$  and  $\overline{f} = \varepsilon_B \circ (Ff)$ .  $\square$

**Digression 3.8** The unit and counit can sensibly be drawn as



and the triangle identities then become the topologi-

cally plausible



Moreover, suppose you're in some category such as  $\mathbf{Vect}_k$  that comes with a tensor product  $\otimes$  and a unit object  $k$  for the tensor: then you can sensibly define a **(left) dual** of an object  $V$  as an object  $V^*$  together with maps  $\eta : k \longrightarrow V \otimes V^*$  and  $\varepsilon : V^* \otimes V \longrightarrow k$  satisfying equations like the triangle identities. A nice explanation of these connections is in Joachim Kock, *Frobenius Algebras and 2D Topological Quantum Field Theories*, Cambridge, 2003.

So far, both formulations of adjointness look symmetric. But the third, which you'll probably see most often, does not.

**Example 3.9** Consider once again the universal property of a vector space  $F(S)$  with basis  $S$ . There is

a map  $\eta_S : S \longrightarrow U(F(S))$  picking out the basis elements, and the ‘unique extension property’ mentioned previously can be drawn as

$$\begin{array}{ccc}
 S & \xrightarrow{\eta_S} & U(F(S)) \\
 & \searrow f & \downarrow U(\exists! \bar{f}) \\
 & & U(V).
 \end{array}$$

That is, for any vector space  $V$  and function  $f$  from  $S$  to (the underlying set of)  $V$ , there is a unique linear map  $\bar{f} : F(S) \longrightarrow V$  extending  $f$  to  $F(S)$ .

This statement turns out to be precisely equivalent to adjointness. In general:

**Proposition 3.10** *The following are equivalent for a functor  $G : \mathcal{B} \longrightarrow \mathcal{A}$ :*

- $G$  has a left adjoint
- for each  $A \in \mathcal{A}$ , there exist an object  $FA \in \mathcal{B}$  and a map  $\eta_A : A \longrightarrow G(FA)$  with the universal

*property*

$$\begin{array}{ccc}
 A & \xrightarrow{\eta_A} & G(FA) \\
 & \searrow f & \downarrow G(\exists! \bar{f}) \\
 & & G(B).
 \end{array}$$

**Sketch proof** If  $G$  has a left adjoint  $F$  then the universal property follows (eventually) from naturality of the adjunction. Conversely, suppose we have an object  $FA$  and a map  $\eta_A$  for each  $A \in \mathcal{A}$ , with the given universal property. Then  $F$  can be made into a functor in a unique way such that  $\eta_A$  is a natural transformation, and the universal property tells us that for any  $A$  and  $B$ , the function

$$\begin{array}{ccc}
 \mathcal{B}(FA, B) & \longrightarrow & \mathcal{A}(A, GB) \\
 g & \longmapsto & (Gg) \circ \eta_A
 \end{array}$$

is bijective. So  $F$  is left adjoint to  $G$ . □

Often right adjoints are easily described but the left adjoint is a lot of trouble. There are existence theorems of the form ‘any functor satisfying certain conditions has a left adjoint’—and of course, that left adjoint is unique when it exists.

Let  $G : \mathcal{B} \longrightarrow \mathcal{A}$  be a functor between locally small categories, and suppose that  $\mathcal{B}$  has all limits and  $G$  preserves them. (This terminology will be defined later.)

**Theorem 3.11 (General Adjoint Functor Theorem)**

*If  $G$  satisfies certain further conditions then  $G$  has a left adjoint.*

This implies that forgetful functors between categories of algebras, such as those in Example 3.3, always have left adjoints.

**Theorem 3.12 (Special Adjoint Functor Theorem)**

*If  $\mathcal{B}$  satisfies certain further conditions then  $G$  has a left adjoint.*

This gives, for instance, Stone-Čech compactification (Example 3.5). The Theorem does not merely imply that the left adjoint exists, but actually constructs it: it tells us that the Stone-Čech compactification of a space  $A$  is the closure of the image of the canonical map  $A \longrightarrow [0, 1]^{\mathbf{Top}(A, [0, 1])}$ .

## Exercises

**3.13** Find three more examples of adjunctions.

**3.14** Let  $G$  be a group.

- a. What interesting functors are there (in either direction) between **Set** and the category  $[G, \mathbf{Set}]$  of sets with a left  $G$ -action? What adjunctions are there between these functors?

(Hints: before you dive into it formally, ask yourself ‘given a set with a  $G$ -action, what sets arise from it?’ and *vice versa*.)

I have six functors and four adjunctions in mind. It might help to observe that  $\mathbf{Set} \cong [1, \mathbf{Set}]$ , where  $1$  is the trivial group, and to know that for any functor  $T : \mathbb{A} \longrightarrow \mathbb{B}$  between small categories, the functor

$$\begin{array}{ccc} T^* : [\mathbb{B}, \mathbf{Set}] & \longrightarrow & [\mathbb{A}, \mathbf{Set}] \\ Y & \longmapsto & Y \circ T \end{array}$$

always has both a left and a right adjoint.)

- b. What happens if we replace **Set** by **Vect**<sub>*k*</sub>, so that we're looking for adjunctions between **Vect**<sub>*k*</sub> and the category  $[G, \mathbf{Vect}_k]$  of *k*-linear representations of *G*?