Lecture 2
Natural transformations and equivalence

Example 2.1 π_1 and H_1 both give ways of turning a space into a group. Precisely, there is a functor

$$\pi_1 : \mathbf{Top}_* \longrightarrow \mathbf{Gp}$$

where $\mathbf{Top}_*$ is the category of spaces with distinguished basepoint and continuous basepoint-preserving maps, and there is a functor

$$\begin{array}{rccc} H_1' : & \mathbf{Top}_* & \longrightarrow & \mathbf{Gp}, \\ & (X, x) & \longmapsto & H_1(X). \end{array}$$

For each $(X, x) \in \mathbf{Top}_*$ there is a canonical map $\alpha_{(X,x)} : \pi_1(X, x) \longrightarrow H_1(X) = H_1'(X, x)$ (the Hurewicz map), and α defines a natural transformation $\pi_1 \longrightarrow H_1'$.

Definition 2.2 Let $\mathcal{A}$ and $\mathcal{B}$ be categories and $F, G : \mathcal{A} \longrightarrow \mathcal{B}$ functors. A **natural transformation** $\alpha : F \longrightarrow G$ (sometimes drawn $\begin{tikzcd} \mathcal{A} \arrow[r, bend left, "F"] \arrow[r, bend right, "G"'] \arrow[r, phantom, "\Downarrow\alpha"] & \mathcal{B} \end{tikzcd}$) is a family $\left(FA \xrightarrow{\alpha_A} GA\right)_{A \in \mathcal{A}}$ of maps in $\mathcal{B}$ such that for every map $A \xrightarrow{f} A'$ in $\mathcal{A}$, the square

$$\begin{tikzcd} FA \arrow[r, "Ff"] \arrow[d, "\alpha_A"'] & FA' \arrow[d, "\alpha_{A'}"] \\ GA \arrow[r, "Gf"'] & GA' \end{tikzcd} \tag{2}$$

commutes.

Loosely, a natural transformation $\alpha : F \longrightarrow G$ is something that associates to each arrow $A \xrightarrow{f} A'$ in $\mathcal{A}$ precisely one arrow $FA \longrightarrow GA'$ in $\mathcal{B}$. When $f = 1_A$, this latter arrow is α_A. For general f, it is the diagonal of the naturality square (2), and 'precisely one' implies that the square commutes.

Example 2.3 For any vector space V, there is an evaluation map

$$\begin{array}{rccc} \varepsilon_V : & V & \longrightarrow & V^{**}, \\ & v & \longmapsto & (\text{evaluate at } v). \end{array}$$

(When V is finite-dimensional, ε_V is an isomorphism.) Since $(\ )^*$ is contravariant, $(\ )^{**}$ is covariant, and for any linear map $f : V \longrightarrow W$, the square

$$\begin{tikzcd} V \arrow[r, "f"] \arrow[d, "\varepsilon_V"'] & W \arrow[d, "\varepsilon_W"] \\ V^{**} \arrow[r, "f^{**}"'] & W^{**} \end{tikzcd}$$

commutes. So we have a natural transformation

$$\begin{tikzcd} \mathbf{Vect} \arrow[r, bend left, "\mathrm{id}"] \arrow[r, bend right, "(\ )^{**}"'] \arrow[r, phantom, "\Downarrow\varepsilon"] & \mathbf{Vect}. \end{tikzcd}$$

(There is no question of there being a natural transformation $\mathrm{id} \longrightarrow (\ )^*$, as $(\ )^*$ is not even a functor $\mathbf{Vect} \longrightarrow \mathbf{Vect}$: it's contravariant.)

Natural transformations are a kind of map, so you would expect to be able to compose them. Fix categories $\mathcal{A}$ and $\mathcal{B}$.

- Given functors and natural transformations
$$\begin{tikzcd} \mathcal{A} \arrow[r, bend left=60, "F"] \arrow[r, "G"] \arrow[r, bend right=60, "H"'] \arrow[r, phantom, shift left=4, "\Downarrow\alpha"] \arrow[r, phantom, shift right=4, "\Downarrow\beta"] & \mathcal{B} \end{tikzcd}$$
, there is a composite natural transformation
$$\begin{tikzcd} \mathcal{A} \arrow[r, bend left=50, "F"] \arrow[r, bend right=50, "H"'] \arrow[r, phantom, "\Downarrow\beta\circ\alpha"] & \mathcal{B} \end{tikzcd}$$
defined by $(\beta \circ \alpha)_A = \beta_A \circ \alpha_A$ $(A \in \mathcal{A})$.
- Given a functor $\mathcal{A} \xrightarrow{F} \mathcal{B}$, there is an identity natural transformation
$$\begin{tikzcd} \mathcal{A} \arrow[r, bend left=50, "F"] \arrow[r, bend right=50, "F"'] \arrow[r, phantom, "\Downarrow 1_F"] & \mathcal{B} \end{tikzcd}$$
defined by $(1_F)_A = 1_{FA}$ $(A \in \mathcal{A})$.

So there is a category $[\mathcal{A}, \mathcal{B}]$ whose objects are functors

$\mathcal{A} \longrightarrow \mathcal{B}$ and whose maps are natural transformations between them. It is called the **functor category** from $\mathcal{A}$ to $\mathcal{B}$.

Example 2.4 If G is a group and k is a ring then $[G, k\text{-}\mathbf{Mod}]$ is the category of (left) k-linear representations of G.

Example 2.5 If X is a topological space then $[\mathbf{Open}(X)^{\mathrm{op}}, \mathbf{Set}]$ is the category of presheaves (or more precisely, presheaves of sets) on X.

Everyday phrases such as '*the* cyclic group of order 6' and '*the* product of two spaces' reflect the fact that given two isomorphic objects of a category, we usually neither know nor care whether they are actually equal. In particular, given two functors $F, G : \mathcal{A} \longrightarrow \mathcal{B}$, we usually do not care whether they are equal (which would imply that the objects FA and GA of $\mathcal{B}$ were equal for all A). What really matters is whether they are naturally isomorphic.

Definition 2.6 A **natural isomorphism** of functors from $\mathcal{A}$ to $\mathcal{B}$ is an isomorphism in $[\mathcal{A}, \mathcal{B}]$.

There is an equivalent, more explicit, definition:

Lemma 2.7 *Let* $\begin{tikzcd} \mathcal{A} \arrow[r, bend left, "F"] \arrow[r, bend right, "G"'] \arrow[r, phantom, "\Downarrow\alpha"] & \mathcal{B} \end{tikzcd}$ *be a natural transformation. Then α is a natural transformation if and only if $\alpha_A : FA \longrightarrow GA$ is an isomorphism for all $A \in \mathcal{A}$.*

Proof Elementary. $\square$

If there is a natural isomorphism $\begin{tikzcd} \mathcal{A} \arrow[r, bend left, "F"] \arrow[r, bend right, "G"'] \arrow[r, phantom, "\Downarrow"] & \mathcal{B} \end{tikzcd}$ then we write $F \cong G$. Another way of saying '$F \cong G$' is '$FA \cong GA$ **naturally in** A'.

Example 2.8 For finite-dimensional vector spaces V, we have $V \cong V^{**}$ naturally in V. That is, the two functors $\mathbf{FDVect} \underset{(\ )^{**}}{\overset{\mathrm{id}}{\rightrightarrows}} \mathbf{FDVect}$ are naturally isomorphic. (Proof: 2.3 + 2.7.)

Example 2.9 Let G be a group and k a ring: then a functor $G \longrightarrow k$-**Mod** is a representation of G, and two such are naturally isomorphic if and only if there is an isomorphism between the two vector spaces compatible with the actions by G.

Example 2.10 There exist functors $\mathcal{A} \overset{F}{\underset{G}{\rightrightarrows}} \mathcal{B}$ such that $FA \cong GA$ for all A, but not *naturally* in A. Exercise 2.19 is an example from elementary combinatorics.

Two elements of a set are either equal or not. Two objects of a category might be equal, or isomorphic, or not. But for categories themselves, even isomorphism is an unreasonably strict relation: for if $\mathcal{A} \cong \mathcal{B}$ then there are functors

$$\mathcal{A} \underset{G}{\overset{F}{\rightleftarrows}} \mathcal{B}$$

such that $G(F(A))$ is *equal* to A for all $A \in \mathcal{A}$. The most useful notion of sameness of categories is 'equivalence'.

Definition 2.11 An **equivalence** between categories $\mathcal{A}$ and $\mathcal{B}$ is a pair $\mathcal{A} \underset{G}{\overset{F}{\rightleftarrows}} \mathcal{B}$ of functors together with natural isomorphisms $\eta : 1_{\mathcal{A}} \xrightarrow{\sim} G \circ F$ and $\varepsilon : F \circ G \xrightarrow{\sim} 1_{\mathcal{B}}$. We then say that $\mathcal{A}$ and $\mathcal{B}$ are **equivalent** and write $\mathcal{A} \simeq \mathcal{B}$. We also say that the functor F (or G) is an **equivalence**.

There is a more practical criterion for a functor to be an equivalence.

Definition 2.12 A functor $F : \mathcal{A} \longrightarrow \mathcal{B}$ is

- **full** if for each $A, A' \in \mathcal{A}$, the function

$$\begin{array}{ccc} \mathcal{A}(A, A') & \longrightarrow & \mathcal{B}(FA, FA') \\ f & \longmapsto & Ff \end{array}$$

  is surjective

- **faithful** if for each $A, A' \in \mathcal{A}$, this function is injective

- **essentially surjective on objects** if for all $B \in \mathcal{B}$, there exists $A \in \mathcal{A}$ such that $FA \cong B$.

Lemma 2.13 *A functor is an equivalence if and only if it is full, faithful, and essentially surjective on objects.*

Proof Omitted. 'If' uses a strong form of the axiom of choice. $\square$

Example 2.14 A subcategory $\mathcal{B}$ of a category $\mathcal{A}$ is **full** if the inclusion functor $\mathcal{B} \hookrightarrow \mathcal{A}$ is full, that is, $\mathcal{B}(B, B') = \mathcal{A}(B, B')$ for all $B, B' \in \mathcal{B}$. (So a full subcategory is determined by its objects.) Suppose that $\mathcal{B}$ is a full subcategory of $\mathcal{A}$ containing at least one object from every isomorphism class of objects of $\mathcal{A}$. Then the inclusion $\mathcal{B} \hookrightarrow \mathcal{A}$ is full, faithful (like any inclusion of subcategories), and essentially surjective on objects, so $\mathcal{B} \simeq \mathcal{A}$.

Example 2.15 A **total order** on a set A is an order $\leq$ such that for all $a, a' \in A$, either $a \leq a'$ or $a' \leq a$. A function $f : (A, \leq) \longrightarrow (B, \leq)$ is **order-preserving** if $a \leq a' \Rightarrow f(a) \leq f(a')$.

Let $\mathcal{D}$ be the category of nonempty finite totally ordered sets and order-preserving maps. Write $[n]$ for the set $\{0, \ldots, n\}$ with its usual ordering, and let Δ be the full subcategory of $\mathcal{D}$ with objects $[0], [1], \ldots$. (A functor $\Delta^{\mathrm{op}} \longrightarrow \mathbf{Set}$ is called a **simplicial set**.) Then $\Delta \simeq \mathcal{D}$ by the previous example.

Example 2.16 More substantially, if k is an algebraically closed field then

$$\begin{aligned} & \text{(affine varieties over } k)^{\mathrm{op}} \\ \simeq \ & \text{(finitely generated reduced } k\text{-algebras).} \end{aligned}$$

Example 2.17 Let $\mathcal{C}$ be the full subcategory of **Cat** consisting of the small categories with precisely one object. There is a functor $U : \mathcal{C} \longrightarrow$ **Monoid** sending a one-object category to its unique hom-set, and U is an equivalence. This makes precise the statement (1.7) that one-object categories are the same as monoids.

Example 2.18 This example shows that there's no point having a category of sets-with-structure if the maps don't preserve the structure. Let $\mathcal{A}$ be the category whose objects are groups and whose maps are all functions between them. Every nonempty set admits at least one group structure, so the forgetful functor from $\mathcal{A}$ to the category of nonempty sets is an equivalence.

Exercises

2.19 Let $\mathcal{B}$ be the category of finite sets and bijections. For $X \in \mathcal{B}$, let $\mathbf{Sym}(X)$ be the set of bijections $X \longrightarrow X$ and let $\mathbf{Ord}(X)$ be the set of total orders on X.

a. Give a definition of $\mathbf{Sym}$ on morphisms of $\mathcal{B}$ so that $\mathbf{Sym}$ becomes a functor $\mathcal{B} \longrightarrow \mathbf{Set}$. Do the same for $\mathbf{Ord}$. (Both your definitions should be canonical. This means, roughly, that you should not use the word 'choose'.)

b. Show that there is no natural transformation $\mathbf{Sym} \longrightarrow \mathbf{Ord}$. (Hint: consider the identity permutation.)

Conclude that $\mathbf{Sym}(X) \cong \mathbf{Ord}(X)$ for all $X \in \mathcal{B}$, but not naturally in $X \in \mathcal{B}$.

2.20 Let G be a group. Homomorphisms $\mathbb{Z} \longrightarrow G$ can be identified with elements of G (by considering where $1 \in \mathbb{Z}$ is sent). So regarding groups as one-object categories, an element of G is the same as a functor $\mathbb{Z} \longrightarrow G$; hence natural isomorphism defines an equivalence relation on the elements of G. What is that equivalence relation? (One-word answer.)

2.21 (Linear algebra can be done equivalently with matrices or linear transformations...)

Let k be a field. Let **Mat** be the category whose objects are natural numbers and with

$$\mathbf{Mat}(m, n) = \{n \times m \text{ matrices over } k\}.$$

Prove that **Mat** is equivalent to **FDVect**, the category of finite-dimensional vector spaces over k. (Hint: use 2.13.) Does your equivalence involve a *canonical* functor $\mathbf{Mat} \longrightarrow \mathbf{FDVect}$, or $\mathbf{FDVect} \longrightarrow \mathbf{Mat}$?