

Every subspace of  $(\mathbb{R}^n, \|\cdot\|_p)$  is positive definite,  
for  $1 \leq p \leq 2$

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These notes consist of a statement and proof of the assertion in the title. They are not intended for publication. The proof is given in full detail—probably more detail than most people would want. It is wholly self-contained except at one point (4.3), where we quote a theorem of Bernstein and refer to the literature for a proof.

The proof presented here emerged from discussions at the  $n$ -Category Café [Wi+] and Math Overflow [CS, LM+]. Most parts of the argument are well known in certain circles of analysis; the major difficulty was in locating and assembling them. I am grateful to all those who contributed, including Neal Bez, Yemon Choi, David Corfield, Mark Lewko, Mark Meckes, Josh Shadlen, David Speyer, Simon Wassermann, Stuart White and Simon Willerton.

What I believe to be the most original part of the proof, Section 5, is due to Mark Meckes [LM+]. I have not found his argument anywhere else in the literature. Some improvements in other parts of the proof are also due to Meckes.

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# 1 Statement of the main theorem

**Definition 1.1** Let  $m \in \mathbb{N} = \{0, 1, \dots\}$ . A real square  $m \times m$  matrix  $Z$  is **positive definite** if  $\sum_{i,j} c_i Z_{ij} c_j \geq 0$  for all  $\mathbf{c} \in \mathbb{R}^m$ , with equality only if  $\mathbf{c} = \mathbf{0}$ .

**Definition 1.2** Let  $A = \{a^1, \dots, a^m\}$  be a finite metric space. (These are superscripts, not powers.) The **similarity matrix**  $Z_A$  of  $A$  is the  $m \times m$  real symmetric matrix  $Z_A$  defined by

$$(Z_A)_{ij} = e^{-d(a^i, a^j)}.$$

The metric space  $A$  is **positive definite** if  $Z_A$  is positive definite. (This property does not depend on the order in which the points of  $A$  were listed.)

**Definition 1.3** Let  $n \in \mathbb{N}$  and  $p \in [1, \infty]$ . We write  $\ell_p^n$  for  $\mathbb{R}^n$  equipped with the metric induced by the  $\ell_p$ -norm  $\|\cdot\|_p$ .

We will prove:

**Theorem 1.4 (Main Theorem)** *Let  $n \in \mathbb{N}$  and  $p \in [1, 2]$ . Then every finite metric subspace of  $\ell_p^n$  is positive definite.*

**Remarks 1.5** Let  $A$  be a finite metric space for which  $Z_A$  is invertible. The **magnitude** of  $A$  is  $\sum_{i,j} (Z_A^{-1})_{ij}$ , the sum of all  $m^2$  entries of  $Z_A^{-1}$ . (See [L+], where magnitude is called cardinality, and [LW].) Since every positive definite matrix is invertible, every positive definite metric space has well-defined magnitude. So, every finite subspace of  $\ell_p^n$  ( $1 \leq p \leq 2$ ) has well-defined magnitude. In particular, this is true for the Euclidean metric,  $p = 2$ .

In fact, positive definite metric spaces have especially convenient properties as far as magnitude is concerned: see Section 2 of [L1] and the comments at [Wi+].

Here is a very brief sketch of the proof of Theorem 1.4. In analytical language, the theorem states that for  $1 \leq p \leq 2$ , the function  $\mathbf{x} \mapsto e^{-\|\mathbf{x}\|_p}$  on  $\mathbb{R}^n$  is strictly positive definite. A result of Wendland tells us that this is true if the Fourier transform of this function is everywhere positive. Using a result of Bernstein, we can show that this is indeed the case.

# 2 Fourier transforms

Fix  $n \in \mathbb{N}$  throughout this section.

**Definition 2.1** Let  $f \in L_1(\mathbb{R}^n)$  ( $= L^1(\mathbb{R}^n)$ ). The **Fourier transform** of  $f$  is the function  $\widehat{f}$  on  $\mathbb{R}^n$  defined by

$$\widehat{f}(\boldsymbol{\xi}) = \int_{\mathbb{R}^n} f(\mathbf{x}) e^{-2\pi i \boldsymbol{\xi} \cdot \mathbf{x}} \, d\mathbf{x}$$

( $\boldsymbol{\xi} \in \mathbb{R}^n$ ).

**Examples 2.2** i. Let  $f(\mathbf{x}) = e^{-\|\mathbf{x}\|_1}$ . Then

$$\widehat{f}(\boldsymbol{\xi}) = \prod_{r=1}^n \frac{2}{1 + 4\pi^2 \xi_r^2}.$$

ii. Let  $a > 0$  and  $f(\mathbf{x}) = e^{-\pi a^2 \|\mathbf{x}\|_2^2}$ . Then

$$\widehat{f}(\boldsymbol{\xi}) = a^{-n} e^{-\pi \|\boldsymbol{\xi}\|_2^2 / a^2}.$$

We use the following standard facts about Fourier transforms.

**Proposition 2.3** *Let  $f \in L_1(\mathbb{R}^n)$ . Then:*

i. *The function  $\widehat{f}$  on  $\mathbb{R}^n$  is continuous.*

ii. *For all  $g \in L_1(\mathbb{R}^n)$  we have  $\widehat{f \cdot g} \in L_1(\mathbb{R}^n)$  and  $f \cdot \widehat{g} \in L_1(\mathbb{R}^n)$ , where  $\widehat{f \cdot g}$  denotes the function  $\boldsymbol{\xi} \mapsto \widehat{f}(\boldsymbol{\xi})g(\boldsymbol{\xi})$ , etc. Moreover,*

$$\int_{\mathbb{R}^n} \widehat{f \cdot g} = \int_{\mathbb{R}^n} f \cdot \widehat{g}.$$

iii. *Suppose that  $f$  is continuous and  $\widehat{f} \in L_1(\mathbb{R}^n)$ . Then*

$$f(\mathbf{x}) = \int_{\mathbb{R}^n} \widehat{f}(\boldsymbol{\xi}) e^{2\pi i \boldsymbol{\xi} \cdot \mathbf{x}} d\boldsymbol{\xi}$$

for all  $\mathbf{x} \in \mathbb{R}^n$ . □

**Lemma 2.4** *Let  $f \in L_1(\mathbb{R}^n)$ . Suppose that  $f$  is bounded and  $\widehat{f}$  is everywhere nonnegative. Then  $\widehat{f} \in L_1(\mathbb{R}^n)$ .*

**Proof** The following proof is extracted from the proof of Lemma 4 of Wendland [We]. For each  $\varepsilon > 0$ , let  $g_\varepsilon$  be the function on  $\mathbb{R}^n$  defined by

$$g_\varepsilon(\boldsymbol{\xi}) = e^{-\pi \varepsilon^2 \|\boldsymbol{\xi}\|_2^2}.$$

We will apply the Monotone Convergence Theorem to the family  $(\widehat{f \cdot g})_{\varepsilon > 0}$  of functions, which is decreasing in  $\varepsilon$  since  $\widehat{f}$  is everywhere nonnegative. Using Proposition 2.3(ii) and Example 2.2(ii), for all  $\varepsilon > 0$  we have  $\widehat{f \cdot g_\varepsilon} \in L_1(\mathbb{R}^n)$  and

$$\begin{aligned} \int_{\mathbb{R}^n} \widehat{f \cdot g_\varepsilon} &= \int_{\mathbb{R}^n} f \cdot \widehat{g_\varepsilon} \\ &= \int_{\mathbb{R}^n} f(\mathbf{x}) \varepsilon^{-n} e^{-\pi \|\mathbf{x}\|_2^2 / \varepsilon^2} d\mathbf{x} \\ &= \int_{\mathbb{R}^n} f(\varepsilon \mathbf{y}) e^{-\pi \|\mathbf{y}\|_2^2} d\mathbf{y}. \end{aligned}$$

But  $f$  is bounded, so for all  $\varepsilon > 0$ ,

$$\int_{\mathbb{R}^n} \widehat{f} \cdot g_\varepsilon \leq (\sup f) \int_{\mathbb{R}^n} e^{-\pi\|\mathbf{y}\|_2^2} d\mathbf{y} < \infty.$$

Also

$$\lim_{\varepsilon \rightarrow 0} (\widehat{f}(\boldsymbol{\xi})g_\varepsilon(\boldsymbol{\xi})) = \widehat{f}(\boldsymbol{\xi})$$

for all  $\boldsymbol{\xi} \in \mathbb{R}^n$ . So by the Monotone Convergence Theorem,  $\widehat{f} \in L_1(\mathbb{R}^n)$ .  $\square$

**Proposition 2.5** *Let  $f \in L_1(\mathbb{R}^n)$ . Suppose that  $f$  is continuous and bounded and that  $\widehat{f}$  is everywhere nonnegative. Then*

$$f(\mathbf{x}) = \int_{\mathbb{R}^n} \widehat{f}(\boldsymbol{\xi})e^{2\pi i\boldsymbol{\xi} \cdot \mathbf{x}} d\boldsymbol{\xi}$$

for all  $\mathbf{x} \in \mathbb{R}^n$ .

**Proof** Proposition 2.3(iii) plus Lemma 2.4.  $\square$

### 3 Strictly positive definite functions

Fix  $n \in \mathbb{N}$  throughout this section.

**Definition 3.1** A function  $f : \mathbb{R}^n \rightarrow \mathbb{R}$  is **strictly positive definite** if for all distinct  $\mathbf{x}^1, \dots, \mathbf{x}^m \in \mathbb{R}^n$ , the  $m \times m$  matrix  $(f(\mathbf{x}^i - \mathbf{x}^j))_{i,j}$  is positive definite.

(Here and later, **distinct** means pairwise distinct.)

**Remarks 3.2** Most of the literature concentrates on (non-strictly) positive definite functions, that is, those for which this matrix is positive semidefinite. Clearly the terminology is not ideal, but it seems quite firmly entrenched.

To be clear: I call a real number  $x$  **positive** if  $x > 0$ , and **nonnegative** if  $x \geq 0$ .

In this language, Theorem 1.4 states that for all  $p \in [1, 2]$ , the function  $\mathbf{x} \mapsto e^{-\|\mathbf{x}\|_p}$  on  $\mathbb{R}^n$  is strictly positive definite. We will prove it, eventually, using the following sufficient condition.

**Proposition 3.3 (Wendland)** *Let  $f \in L_1(\mathbb{R}^n)$ . Suppose that  $f$  is continuous and bounded, and that  $\widehat{f}$  is everywhere positive. Then  $f$  is strictly positive definite.*

**Remarks 3.4** i. This result appears to be due to Wendland ([We], Lemma 4). The proof given here is his, slightly restructured. Note that in [We], Wendland calls a function ‘positive definite’ where we (and most other authors) would call it ‘strictly positive definite’. Also, the statement of his Lemma 4 requires that the function is radial, but this requirement is not needed.

- ii. The proposition can be sharpened considerably and a converse stated; Wendland's Lemma 4 does both. However, we need no more than what the proposition says.
- iii. Related results on *non*-strictly positive definite functions are much better known: see the literature on Bochner's Theorem.

**Proof of Proposition 3.3** By Proposition 2.5,

$$f(\mathbf{x}) = \int_{\mathbb{R}^n} \widehat{f}(\boldsymbol{\xi}) e^{2\pi i \boldsymbol{\xi} \cdot \mathbf{x}} d\boldsymbol{\xi}$$

for all  $\mathbf{x} \in \mathbb{R}^n$ . Let  $\mathbf{x}^1, \dots, \mathbf{x}^m$  be distinct points of  $\mathbb{R}^n$ , and let  $\mathbf{c} \in \mathbb{R}^m$ . Then

$$\begin{aligned} \sum_{j,k=1}^m c_j f(\mathbf{x}^j - \mathbf{x}^k) c_k &= \int_{\mathbb{R}^n} \widehat{f}(\boldsymbol{\xi}) \sum_{j,k=1}^m c_j e^{2\pi i \boldsymbol{\xi} \cdot (\mathbf{x}^j - \mathbf{x}^k)} c_k d\boldsymbol{\xi} \\ &= \int_{\mathbb{R}^n} \widehat{f}(\boldsymbol{\xi}) \left| \sum_{j=1}^m c_j e^{2\pi i \boldsymbol{\xi} \cdot \mathbf{x}^j} \right|^2 d\boldsymbol{\xi} \\ &\geq 0 \end{aligned}$$

since  $\widehat{f}$  is everywhere nonnegative.

Now suppose that equality holds. Since  $\widehat{f}$  is continuous (Proposition 2.3(i)) and everywhere positive,  $\sum_j c_j e^{2\pi i \boldsymbol{\xi} \cdot \mathbf{x}^j} = 0$  for all  $\boldsymbol{\xi} \in \mathbb{R}^n$ . But  $\mathbf{x}^1, \dots, \mathbf{x}^m$  are distinct, so the functions  $\boldsymbol{\xi} \mapsto e^{2\pi i \boldsymbol{\xi} \cdot \mathbf{x}^j}$  on  $\mathbb{R}^n$  ( $1 \leq j \leq m$ ) are linearly independent (Proposition A.1). Hence  $\mathbf{c} = \mathbf{0}$ .  $\square$

We will apply Proposition 3.3 to the function  $f : \mathbf{x} \mapsto e^{-\|\mathbf{x}\|_p}$ , as follows.

**Proposition 3.5** *Let  $p \in [1, \infty]$  and define  $f_p : \mathbb{R}^n \rightarrow \mathbb{R}$  by  $f_p(\mathbf{x}) = e^{-\|\mathbf{x}\|_p}$ . Then  $f_p \in L_1(\mathbb{R}^n)$ . Moreover, if  $\widehat{f}_p$  is everywhere positive then every finite metric subspace of  $\ell_p^n$  is positive definite.*

**Proof** It is easy to show that  $f_1 \in L_1(\mathbb{R}^n)$ ; indeed,

$$\int_{\mathbb{R}} \dots \int_{\mathbb{R}} e^{-\|\mathbf{x}\|_1} dx_1 \dots dx_n = \left( \int_{\mathbb{R}} e^{-|x|} dx \right)^n = 2^n.$$

Now, all norms on the finite-dimensional vector space  $\mathbb{R}^n$  are equivalent, so there is a constant  $\lambda > 0$  such that  $\|\mathbf{x}\|_p \geq \lambda \|\mathbf{x}\|_1$  for all  $\mathbf{x} \in \mathbb{R}^n$ . Then

$$0 \leq f_p(\mathbf{x}) = e^{-\|\mathbf{x}\|_p} \leq e^{-\lambda \|\mathbf{x}\|_1} = f_1(\lambda \mathbf{x})$$

for all  $\mathbf{x}$ , and  $f_1 \in L_1(\mathbb{R}^n)$ , so  $f_p \in L_1(\mathbb{R}^n)$ .

For 'moreover', certainly  $f_p$  is continuous and bounded; the result then follows from Proposition 3.3.  $\square$

We can now prove the main theorem (1.4) in the special cases  $p = 1$  and  $p = 2$ . The case  $p = 1$  can also be proved by much more elementary means: see [L2]. However, I know of no easier proof of the case  $p = 2$ .

This result is not logically necessary for the proof of the main theorem, since that proof will handle all  $p \in [1, 2]$  uniformly.

**Proposition 3.6** *Let  $p \in \{1, 2\}$ . Then every finite subspace of  $\ell_p^n$  is positive definite.*

**Proof** Write  $f_p$  for the function  $\mathbf{x} \mapsto e^{-\|\mathbf{x}\|_p}$  on  $\mathbb{R}^n$ . By Proposition 3.5, it is enough to prove that  $\widehat{f}_p$  is everywhere positive. This is true for  $p = 1$  by Example 2.2(i). For  $p = 2$ , we have

$$\widehat{f}_2(\boldsymbol{\xi}) = \Gamma\left(\frac{n+1}{2}\right) \frac{2^n \pi^{(n-1)/2}}{(1 + 4\pi^2 \|\boldsymbol{\xi}\|_2^2)^{(n+1)/2}} > 0$$

for all  $\boldsymbol{\xi} \in \mathbb{R}^n$ , by Theorem 1.14 of Stein and Weiss [SW]. □

The general case cannot be proved in this way, because there is no known explicit formula for the Fourier transform of  $\mathbf{x} \mapsto e^{-\|\mathbf{x}\|_p}$  when  $1 < p < 2$ .

## 4 Completely monotone functions

We need to be able to handle the function

$$\mathbf{x} \mapsto e^{-\|\mathbf{x}\|_p} = e^{-(\sum_r |x_r|^p)^{1/p}}$$

on  $\mathbb{R}^n$ , where  $1 \leq p \leq 2$ . For this reason, and for another that will emerge later, we study the function  $t \mapsto e^{-t^\alpha}$  on  $[0, \infty)$ , where  $0 < \alpha \leq 1$ . We show that this function is ‘completely monotone’, and deduce that it can be represented in a way that makes it easier to work with.

**Definition 4.1** A function  $\phi : [0, \infty) \rightarrow \mathbb{R}$  is **completely monotone** if  $\phi$  is continuous on  $[0, \infty)$  and infinitely differentiable on  $(0, \infty)$ , and

$$(-1)^n \phi^{(n)}(t) \geq 0$$

for all  $n \geq 0$  and  $t \in (0, \infty)$ .

**Examples 4.2** The function  $t \mapsto e^{-at}$  is completely monotone for every  $a \geq 0$ . The class of completely monotone functions is closed under linear combinations with nonnegative coefficients: hence

$$t \mapsto \sum \lambda_i e^{-a_i t}$$

is completely monotone for any  $a_1, \lambda_1, a_2, \lambda_2, \dots \geq 0$ . Similarly, the function

$$t \mapsto \int_{[0, \infty)} e^{-\tau t} d\mu(\tau)$$

is completely monotone for any finite measure  $\mu$  on  $[0, \infty)$ . (For us, a **measure** is by definition nonnegative.) The following result says that *every* completely monotone function has this form.

**Theorem 4.3 (Bernstein)** *Let  $\phi$  be a completely monotone function on  $[0, \infty)$ . Then there exists a finite Borel measure  $\mu$  on  $[0, \infty)$  such that*

$$\phi(t) = \int_{[0, \infty)} e^{-\tau t} d\mu(\tau)$$

for all  $t \in [0, \infty)$ .

**Proof** See, for instance, Section XIII.4 of Feller [F]. □

**Lemma 4.4** *Let  $\alpha \in [0, 1]$ . Then the function  $t \mapsto e^{-t^\alpha}$  on  $[0, \infty)$  is completely monotone.*

This is stated, but not proved, in the proof of Lemma 2.27 of Koldobsky [K].

In the following proof (only), we will speak of completely monotone functions  $\theta$  on the open interval  $(0, \infty)$ . This will mean that  $\theta$  is infinitely differentiable and  $(-1)^n \theta^{(n)}(t) \geq 0$  for all  $n \geq 0$  and  $t \in (0, \infty)$ . There is no requirement that  $\theta$  extends continuously to  $[0, \infty)$ .

**Proof** Write  $\phi(t) = e^{-t^\alpha}$ . Certainly  $\phi$  is continuous on  $[0, \infty)$  and infinitely differentiable on  $(0, \infty)$ . Write

$$\phi^{(n)}(t) = \psi_n(t) e^{-t^\alpha}$$

( $n \in \mathbb{N}, t \in (0, \infty)$ ). I claim that the function  $(-1)^n \psi_n$  on  $(0, \infty)$  is completely monotone for all  $n \in \mathbb{N}$ . Since completely monotone functions are nonnegative, the result will follow.

The claim is proved by induction on  $n$ . For the base case, just observe that  $\psi_0(t) = 1$  for all  $t$ . Now let  $n \geq 1$ . We have

$$\psi_n(t) = \psi'_{n-1}(t) - \alpha t^{\alpha-1} \psi_{n-1}(t)$$

and so

$$(-1)^n \psi_n(t) = (-1)^n \psi'_{n-1}(t) + \alpha t^{\alpha-1} \cdot (-1)^{n-1} \psi_{n-1}(t).$$

Since  $\alpha \leq 1$ , the function  $t \mapsto t^{\alpha-1}$  is completely monotone. By inductive hypothesis,  $(-1)^{n-1} \psi_{n-1}$  is completely monotone; hence  $(-1)^n \psi'_{n-1}$  is too. The class of completely monotone functions is closed under sums, products, and multiplication by nonnegative scalars, so  $(-1)^n \psi_n$  is completely monotone, as required. □

**Proposition 4.5** *Let  $\alpha \in (0, 1]$ . Then there is a finite Borel measure  $\mu$  on  $(0, \infty)$  such that*

$$e^{-t^\alpha} = \int_{(0, \infty)} e^{-\tau t} d\mu(\tau) \quad (1)$$

for all  $t \in [0, \infty)$ .

Note that two of these intervals are open.

**Proof** By Theorem 4.3 and Lemma 4.4, we may choose a finite Borel measure  $\mu$  on  $[0, \infty)$  such that (1) holds with the *closed* interval  $[0, \infty)$  as the range of integration. Then  $e^{-t^\alpha} \geq \mu(\{0\})$  for all  $t \in [0, \infty)$ ; but since  $\alpha > 0$ , this implies that  $\mu(\{0\}) = 0$ . The result follows.  $\square$

## 5 Proof of the main theorem

The argument of this section is due to Mark Meckes [LM+].

By Proposition 3.5, the main theorem will be proved if we can show that for each  $p \in [1, 2]$ , the Fourier transform of the function  $\mathbf{x} \mapsto e^{-\|\mathbf{x}\|_p}$  is everywhere positive. We will do this using Proposition 4.5 twice over: once with  $\alpha = p/2$ , and once with  $\alpha = 1/p$ . Since  $\alpha \in (0, 1]$ , this restricts us to  $p \in [1, 2]$ .

Here is the first use of Proposition 4.5. Both the statement and the proof are taken directly from Koldobsky ([K], Lemma 2.27).

**Lemma 5.1 (Koldobsky)** *Let  $p \in (0, 2]$ . Then the Fourier transform of the function  $z \mapsto e^{-|z|^p}$  on  $\mathbb{R}$  is everywhere positive.*

This function does indeed have a Fourier transform, since it is in  $L_1(\mathbb{R})$ . For we have

$$e^{|z|^p} = \sum_{j=0}^{\infty} |z|^{jp} / j! \geq z^2 / 2$$

(since  $p > 0$ ), so  $e^{-|z|^p} \leq \min\{1, 2/z^2\}$ , and  $\min\{1, 2/z^2\}$  is in  $L_1(\mathbb{R})$ .

**Proof** Let  $\alpha = p/2 \in (0, 1]$  and choose a measure  $\mu$  as in Proposition 4.5: then for all  $z \in \mathbb{R}$ ,

$$e^{-|z|^p} = \int_{(0, \infty)} e^{-\tau z^2} d\mu(\tau).$$

So, writing  $g(z) = e^{-|z|^p}$ ,

$$\widehat{g}(\zeta) = \int_{(0, \infty)} \int_{\mathbb{R}} e^{-\tau z^2} e^{-2\pi i \zeta z} dz d\mu(\tau) \quad (2)$$

for all  $\zeta \in \mathbb{R}$ , by Fubini's Theorem. By Example 2.2(ii), for all  $\tau \in (0, \infty)$  and  $\zeta \in \mathbb{R}$  we have

$$\int_{\mathbb{R}} e^{-\tau z^2} e^{-2\pi i \zeta z} dz = \sqrt{\frac{\pi}{\tau}} e^{-\pi^2 \zeta^2 / \tau},$$

which is continuous in  $\tau$  and positive. And clearly  $\mu$  is not the zero measure, so (2) implies that  $\widehat{g}(\zeta) > 0$  for all  $\zeta \in \mathbb{R}$ , as required.  $\square$

**Proposition 5.2 (Meckes)** *Let  $n \in \mathbb{N}$  and  $p \in [1, 2]$ . Then the Fourier transform of the function  $\mathbf{x} \mapsto e^{-\|\mathbf{x}\|_p}$  on  $\mathbb{R}^n$  is everywhere positive.*

**Proof** Write  $f(\mathbf{x}) = e^{-\|\mathbf{x}\|_p}$  ( $\mathbf{x} \in \mathbb{R}^n$ ). Let  $\alpha = 1/p \in (0, 1]$ , and choose a measure  $\mu$  as in Proposition 4.5. Then for all  $\boldsymbol{\xi} \in \mathbb{R}^n$ ,

$$\begin{aligned}
\widehat{f}(\boldsymbol{\xi}) &= \int_{\mathbb{R}^n} e^{-\|\mathbf{x}\|_p} e^{-2\pi i \boldsymbol{\xi} \cdot \mathbf{x}} \, d\mathbf{x} \\
&= \int_{\mathbb{R}^n} \int_{(0, \infty)} e^{-\tau \sum_{r=1}^n |x_r|^p} \, d\mu(\tau) e^{-2\pi i \boldsymbol{\xi} \cdot \mathbf{x}} \, d\mathbf{x} \\
&= \int_{(0, \infty)} \left( \prod_{r=1}^n \int_{\mathbb{R}} e^{-\tau |x_r|^p} e^{-2\pi i \xi_r x_r} \, dx_r \right) d\mu(\tau) \\
&= \int_{(0, \infty)} \left( \prod_{r=1}^n \int_{\mathbb{R}} e^{-|z_r|^p} e^{-2\pi i (\tau^{-1/n} \xi_r) z_r} \tau^{-1/n} \, dz_r \right) d\mu(\tau) \\
&= \int_{(0, \infty)} \tau^{-1} \left( \prod_{r=1}^n \widehat{g}(\tau^{-1/n} \xi_r) \right) d\mu(\tau) \tag{3}
\end{aligned}$$

where  $g$  is the function  $z \mapsto e^{-|z|^p}$  on  $\mathbb{R}$ . But  $\widehat{g}$  is everywhere positive by Lemma 5.1, and continuous by Proposition 2.3(i). Hence the integrand in (3) is everywhere positive and continuous in  $\tau$ . Since  $\mu$  is not the zero measure,  $\widehat{f}(\boldsymbol{\xi}) > 0$ .  $\square$

**Proof of the main theorem (1.4)** Follows from Propositions 3.5 and 5.2.  $\square$

**Remarks 5.3** i. Many results about positive definite functions hold for  $p \leq 2$  but fail for  $p > 2$ : see, for example, Schoenberg [S] and Koldobsky [K]. So, I conjecture that for every  $p \in (2, \infty]$ , there exist  $n \in \mathbb{N}$  and a finite subspace of  $\ell_p^n$  that is not positive definite. This is certainly true for  $p = \infty$ : for as shown in Section 7 of [S], *every* finite metric space occurs as a subspace of  $\ell_\infty^n$  for some  $n$ , and it is known [L+] that not every finite metric space is positive definite.

ii. Meckes observes that the main theorem can also be extended to  $p \in (0, 1)$ . For such  $p$ , the formula  $d(\mathbf{a}, \mathbf{b}) = \|\mathbf{a} - \mathbf{b}\|_p$  ( $\mathbf{a}, \mathbf{b} \in \mathbb{R}^n$ ) does not define a metric. However, the formula

$$d(\mathbf{a}, \mathbf{b}) = \|\mathbf{a} - \mathbf{b}\|_p^p = \sum_{r=1}^n |a_r - b_r|^p$$

does, and  $\ell_p^n$  is defined to be  $\mathbb{R}^n$  equipped with this metric. Then every finite subspace of  $\ell_p^n$  is positive definite. This is proved by a simplified version of the argument above: apply Lemma 5.1 and Fubini's Theorem.

## A Linear independence of the exponential functions

Fix  $n \in \mathbb{N}$ . For each  $\mathbf{x} \in \mathbb{R}^n$ , let  $\chi_{\mathbf{x}}$  be the function  $\xi \mapsto e^{2\pi i \xi \cdot \mathbf{x}}$  on  $\mathbb{R}^n$ . In this appendix we prove:

**Proposition A.1** *The family  $(\chi_{\mathbf{x}})_{\mathbf{x} \in \mathbb{R}^n}$  of functions is linearly independent over  $\mathbb{C}$ .*

This is a consequence of the linear independence of characters, as stated in the following standard result. (See Theorem 12 of [A], for instance.)

**Proposition A.2** *Let  $A$  be a commutative monoid, written additively, and let  $k$  be a field. Then the family of all monoid homomorphisms  $(A, +, 0) \rightarrow (k, \cdot, 1)$  is linearly independent over  $k$ .*

**Proof** The proposition states that for any  $m \geq 1$ , any distinct homomorphisms  $\chi_1, \dots, \chi_m$ , and any scalars  $\lambda_1, \dots, \lambda_m \in k$  such that  $\sum \lambda_i \chi_i = 0$ , we have  $\lambda_1 = \dots = \lambda_m = 0$ . We prove this by induction on  $m$ .

For the base case, suppose that  $\lambda_1 \chi_1 = 0$ . Since  $\chi_1$  is a homomorphism,  $\chi_1(0) = 1$ . But  $\lambda_1 \chi_1(0) = 0$ , so  $\lambda_1 = 0$ .

Now let  $m \geq 2$  and suppose that  $\sum_{i=1}^m \lambda_i \chi_i = 0$ . For all  $a \in A$  we have

$$\sum_{i=1}^m \lambda_i \chi_i(a) \chi_i = \sum_{i=1}^m \lambda_i \chi_i(a + -) = 0,$$

but also

$$\sum_{i=1}^m \lambda_i \chi_m(a) \chi_i = 0,$$

and so, subtracting,

$$\sum_{i=1}^{m-1} \lambda_i (\chi_i(a) - \chi_m(a)) \chi_i = 0.$$

By inductive hypothesis,  $\lambda_i (\chi_i(a) - \chi_m(a)) = 0$  for all  $a \in A$ . But  $\chi_1 \neq \chi_m$  (since  $m \geq 2$ ), so there exists  $a \in A$  such that  $\chi_1(a) \neq \chi_m(a)$ . Hence  $\lambda_1 = 0$ . Similarly,  $\lambda_2 = \dots = \lambda_m = 0$ , completing the induction.  $\square$

**Lemma A.3** *Let  $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$ . Then  $\chi_{\mathbf{x}} = \chi_{\mathbf{y}}$  if and only if  $\mathbf{x} = \mathbf{y}$ .*

**Proof** Suppose that  $\chi_{\mathbf{x}} = \chi_{\mathbf{y}}$ . Then for all  $\xi \in \mathbb{R}^n$  we have  $\chi_{\mathbf{x}-\mathbf{y}}(\xi) = 1$ , that is,  $\xi \cdot (\mathbf{x} - \mathbf{y}) \in \mathbb{Z}$ . By continuity,  $\xi \cdot (\mathbf{x} - \mathbf{y}) = 0$  for all  $\xi$ . Hence  $\mathbf{x} = \mathbf{y}$ .  $\square$

**Proof of Proposition A.1** For each  $\mathbf{x} \in \mathbb{R}^n$ , the function  $\chi_{\mathbf{x}}$  is a monoid homomorphism  $(\mathbb{R}^n, +, 0) \rightarrow (\mathbb{C}, \cdot, 1)$ . The result follows from Proposition A.2 and Lemma A.3.  $\square$

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