

NERVES OF ALGEBRAS

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INTERESTED IN...

Monads $T = (T, \mu, \eta)$ on cats \mathcal{B} , where:

- $\mathcal{B} \simeq [\mathcal{B}^{\text{op}}, \text{Set}]$ (some small \mathcal{B})
- T preserves connected limits
- μ and η are cartesian.

E.g.

- $\mathcal{B} = \text{Set}$, $T = \text{theory of monoids}$
- $\mathcal{B} = \text{Set}$, $T = \text{theory of } M\text{-sets (some monoid } M)$
- $\mathcal{B} = \text{DirGph}$, $T = \text{theory of cats}$
- $\mathcal{B} = n\text{-Gph}$, $T = \text{theory of strict } n\text{-cats}$
- $\mathcal{B} = n\text{-Gph}$, $T = \text{theory of weak } n\text{-cats}$

REPRESENTABLE FUNCTORS

Let \mathcal{A} be a cat such that every limit-preserving functor out of \mathcal{A} has a left adjoint. E.g.: $\mathcal{A} \simeq [A^{\text{op}}, \text{Set}]$.

Prop'n: For $T: \mathcal{A} \rightarrow \text{Set}$,

T is representable

$\Leftrightarrow T$ preserves limits.

FAMILIALLY REPRESENTABLE FUNCTORS (I)

(Diers, Carboni - Johnstone)

$T: \mathcal{A} \rightarrow \text{Set}$ is familially representable

if $T = \sum_{i \in I} \mathcal{A}(T_i, -)$ for some family $(T_i)_{i \in I}$

of objects of \mathcal{A} .

Prop'n: For $T: \mathcal{A} \rightarrow \text{Set}$,

T is familially representable

$\Leftrightarrow T$ preserves connected limits.

FAMILIALLY REPRESENTABLE FUNCTORS (II)

$T: \mathcal{A} \longrightarrow \hat{\mathcal{B}} := [\mathcal{B}^{\text{op}}, \text{Set}]$ is familially representable if for all $b \in \mathcal{B}$,

$$T(-)(b): \mathcal{A} \longrightarrow \text{Set}$$

is familially representable. So

$$(TX)(b) = \sum_{i \in I_b} \hat{\mathcal{B}}(T_{b,i}, X)$$

($X \in \mathcal{A}$, $b \in \mathcal{B}$).

Prop'n: For $T: \mathcal{A} \longrightarrow \hat{\mathcal{B}}$,

T is familially representable
 $\Leftrightarrow T$ preserves connected limits.

FAMILIALLY REPRESENTABLE MONADS

A monad (T, μ, η) on $\hat{\mathcal{B}}$ is familially representable if T is familially representable and μ and η are cartesian.

Can say explicitly (in terms of representing families):

- what μ and η are
- what "cartesian" means
- what the monad axioms are
- what T-algebras are.

GENERALIZATION OF Δ

Let T be a fam. rep. monad on $\hat{\mathbb{B}}$.

Define a cat Δ_T by

- $\text{ob } \Delta_T = \{ (b, i) \mid b \in \mathbb{B}, i \in I_b \}$
- $\Delta_T((b, i), (c, j)) = \hat{\mathbb{B}}^T(FT_{b,i}, FT_{c,j})$
 $= \hat{\mathbb{B}}(T_{b,i}, T(T_{c,j}))$.

E.g. $\mathbb{B} = (\cdot \rightrightarrows \cdot)$, $T = \text{free cat}$: $\Delta_T \simeq \Delta$.

Have

$$\begin{array}{ccc} \Delta_T & \hookrightarrow & \hat{\mathbb{B}}^T \\ (b, i) & \mapsto & FT_{b,i} \end{array}$$

hence nerve functor

$$N: \hat{\mathbb{B}}^T \longrightarrow \hat{\Delta}_T$$

where

$$(NX)(b, i) = \hat{\mathbb{B}}(T_{b,i}, X).$$

E.g. $T = \text{free cat}$: N is standard nerve functor.

NERVES OF ALGEBRAS

Let T be a fam. rep. monad on $\hat{\mathbb{B}}$.

Thm: $N: \hat{\mathbb{B}}^T \rightarrow \hat{\Delta}_T$ defines an equivalence between $\hat{\mathbb{B}}^T$ and the full subcat

$\{Z: \Delta_T^{\text{op}} \rightarrow \text{Set} \mid Z \text{ preserves certain limits}\}$
of $\hat{\Delta}_T$.

(Those limits: one for each $b \in \mathbb{B}$, $i \in I_b$, $\gamma \in \hat{\mathbb{B}}(T_{b,i}, I)$,
of shape $\text{El}(T_{b,i})$.)

E.g.

• $\mathbb{B} = (\cdot \rightrightarrows \cdot)$, $T = \text{free cat}$: condition is

$$Z_{n_1 + \dots + n_k} \xrightarrow{\sim} Z_{n_1} \times_{Z_0} \dots \times_{Z_0} Z_{n_k}$$

• $\mathbb{B} = \mathbf{1}$, $T = \text{free monoid}$: $\Delta_T = (\text{Lawvere theory})^{\text{op}}$

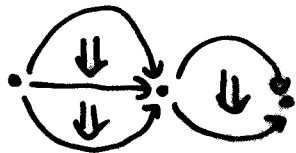
• $\mathbb{B} = \mathbf{1}$, $T = \text{free } M\text{-set}$: $\Delta_T = M^{\text{op}}$.

EXAMPLE : STRICT n-CATS

$$\mathbb{B} = (\cdot \rightrightarrows \dots \rightrightarrows \cdot), \quad \hat{\mathbb{B}} = n\text{-Gph},$$

$T = \text{free strict } n\text{-cat}$

- Object of Δ_T : globular pasting diagram, e.g.



- Map in Δ_T : strict n -functor.

Have

$$N : \text{Str-}n\text{-Cat} \longleftrightarrow \hat{\Delta}_T,$$

$$(NX) \left(\text{diagram} \right) = \{ \text{labelled diagrams of shape } \text{diagram} \text{ in } X \}$$

giving

$$\text{Str-}n\text{-Cat} \cong \{ Z \in \hat{\Delta}_T \mid Z \text{ preserves certain limits} \}.$$

In fact, $\Delta_T \cong \ominus$ (Joyal, Berger, Makkai-Zawadowski).

DEFINITIONS OF WEAK n-CAT

Two styles:

- ① an n-cat is an algebra for a fam. rep. monad
- ② an n-cat is a presheaf with properties.

When $n=1$:

- ① a cat is a graph with structure
- ② a cat is a simplicial set with properties.

Idea in ②: presheaves are taken on some $\mathbb{D} \subseteq \text{Str-n-Cat}$, and "the functor is

$$\mathbb{D}^{\text{op}} \longrightarrow \text{Set}$$

$$S \mapsto \{\text{weak functors } S \rightarrow X\}$$

where X is our weak n-cat."

$$\textcircled{1} \longleftrightarrow \textcircled{2}$$

$$\mathbb{B} = (\cdot \rightrightarrows \dots \rightrightarrows \cdot), \quad \hat{\mathbb{B}} = n\text{-Gph},$$

$T = \text{free weak } n\text{-cat}$

• Object of Δ_T : globular pasting diagram

• A map



in Δ_T is a strict n -functor

$$F_{wk} \left(\textcircled{\downarrow \rightarrow} \textcircled{\downarrow \rightarrow} \right) \longrightarrow F_{wk} \left(\textcircled{\downarrow \rightarrow} \textcircled{\downarrow \rightarrow} \textcircled{\downarrow \rightarrow} \right).$$

Have

$$N: Wk\text{-}n\text{-Cat} \longleftrightarrow \hat{\Delta}_T,$$

$$(NX) \left(\textcircled{\downarrow \rightarrow} \textcircled{\downarrow \rightarrow} \right) = \{ \text{strict } n\text{-functors } F_{wk} \left(\textcircled{\downarrow \rightarrow} \textcircled{\downarrow \rightarrow} \right) \rightarrow X \}$$

$$= \{ \text{weak } n\text{-functors } F_{str} \left(\textcircled{\downarrow \rightarrow} \textcircled{\downarrow \rightarrow} \right) \rightarrow X \}."$$