Noncommutative Geometry An Elementary Introduction

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Algebras as generalised spaces

- Traditional objects of geometry: Point sets X with additional structures - measure spaces, topological spaces, smooth manifolds, algebraic varieties...
- Algebraic geometry, global analysis, physics: Fix some ground ring k and study X through suitable functions $X \to k$ that respect the considered structure - measurable, continous, smooth, regular functions... Typically, these functions form an algebra that contains the full information about X. As long as k is commutative, these algebras are all commutative.
- Example: If X is compact Hausdorff, then $A := C(X, \mathbb{C})$ is a commutative C^*-algebra. Any such one arises in this way (up to isomorphism), and X is determined by A up to homeomorphism (Gelfand-Naimark theorem).
- Upshot: General C^*-algebras can be viewed as mathematical objects generalising compact Hausdorff spaces.

Projective modules as generalised vector bundles

- X compact Hausdorff, $\pi : E \to X$ finite-dimensional complex vector bundle. Then the set $\Gamma(E)$ of continous sections

$$s : X \to E, \quad \pi \circ s = \mathrm{id}$$

  is a module over $A = C(X, \mathbb{C})$, with action given by

$$(fs)(x) = f(x)s(x), \quad f \in A, s \in \Gamma(E), x \in X.$$

  This module characterises E up to isomorphism.
- There exists a second bundle E' such that

$$E \oplus E' \simeq X \times \mathbb{C}^n.$$

  Therefore:

$$\Gamma(E) \oplus \Gamma(E') \simeq \Gamma(X \times \mathbb{C}^n) \simeq A^n.$$

  Thus $\Gamma(E)$ is finitely generated projective. Finally: Any finitely generated projective A-module is (up to iso) of this form.
- Upshot: F.g.p. modules generalise vector bundles.

K-theory

- The direct sum turns the isomorphism classes of f.g.p. A-modules into a semigroup $P(A)$. Its Grothendieck gorup (formal differences of elements in $P(A)$) is denoted $K_0(A)$.
- Example: $A = k$ - a field. Then any module is free and characterised up to isomorphism by its dimension, so $P(A) \simeq \mathbb{N}$. The K-group is $K_0(k) \simeq \mathbb{Z}$, via the usual construction of the integers as formal differences of natural numbers.
- Warning: In general, $P(A)$ does not have cancellation, $[M] \oplus [N] = [M'] \oplus [N] \not\Rightarrow [M] = [M']$.
- There exist higher K-groups $K_i(A)$ that give altogether some sort of homology theory, but in fact even several variants of them for various types of rings. Simplest case for C^*-algebras, only one more group $K_1(A)$, and $K_i(A) := K_{i-2}(A)$ for higher i yields a nice theory with all desired properties. $K^i(X) := K_i(C(X, \mathbb{C}))$ is Atiyah's topological K-theory of compact Hausdorff spaces.

The Connes-Chern character - I

- Classically: The Chern character is a morphism

$$\mathrm{ch} : K^i(X) \to \bigoplus_{n \geq 0} H^{2n+i}(X, \mathbb{Q}).$$

- If $[E] \in K^0(X)$ is the class of a vector bundle $\pi : E \to X$ as above, then $\mathrm{ch}([E]) \in \bigoplus_{n \geq 0} H^{2n}(X, \mathbb{Q})$ measures roughly speaking something like an intrinsic curvature of the bundle. In particular, $\mathrm{ch}(E) \neq 0$ implies the nontriviality of E.
- Connes: The Chern character can be generalised to noncommutative algebras. The main problem is to deifne the correct target for it to land in - cyclic homology.

Hochschild homology

- k - a field, A - a unital associative k-algebra.
- Define $b : C_n \to C_{n-1}$, $C_n := A \otimes \bar{A}^{\otimes n}$, $\bar{A} := A/k$, by

$$b(a_0 \otimes \ldots \otimes a_n) = \sum_{i=0}^{n-1} (-1)^i a_0 \otimes \ldots \otimes a_i a_{i+1} \otimes \ldots \otimes a_n, \\ +(-1)^n a_n a_0 \otimes a_1 \otimes \ldots \otimes a_{n-1}.$$

  Then $b \circ b = 0$. The resulting homology $HH_\bullet(A) = \ker b / \mathrm{im}\, b$ of $(C_\bullet, b)$ is the Hochschild homology of A.
- Hochschild-Kostant-Rosenberg: For the coordinate ring $A = k[X]$ of a smooth affine variety one has $HH_n(A) \simeq \Omega^n(X)$, the Kähler differentials on X.

Cyclic homology - I

- Define $B : C_n \to C_{n+1}$ by

$$B(a_0 \otimes \ldots \otimes a_n) = \sum_{i=0}^{n} (-1)^{ni} \otimes a_i \otimes \ldots \otimes a_n \otimes \bar{a}_0 \otimes \ldots \otimes a_{i-1},$$

  where $\bar{a}_0 \in \bar{A}$ is the class of $a_0 \in A$.

- Then one has

$$B \circ B = b \circ B + B \circ b = 0 \quad \Rightarrow \quad (b + B) \circ (b + B) = 0.$$

Cyclic homology - II

- Therefore, $b + B$ turns

$$C_{\text{even}} := \bigoplus_{n \geq 0} C_{2n}, \quad C_{\text{odd}} := \bigoplus_{n \geq 0} C_{2n+1}$$

  into a $\mathbb{Z}_2$-graded chain complex whose homology is the (periodic) cyclic homology $HP_\bullet(A)$ of A.
- For $A = k[X]$ as in the Hochschild-Kostant-Rosenberg theorem, one has

$$HP_{\text{even/odd}}(A) := \bigoplus_{n \geq 0} H^{2n/2n+1}(\Omega^\bullet(X), d),$$

  where $d : \Omega^n(X) \to \Omega^{n+1}(X)$ is Cartan's exterior differential.

The Connes-Chern charcter - II

- For $M \oplus N = A^n$, let $p \in M_n(A)$ represent the projection $\pi : A \to A$, $\pi \circ \pi = \pi$, $\mathrm{im}\, \pi = M$, $\ker \pi = N$. Put

$$c_{2n} := (-1)^n \frac{(2n)!}{n!} \sum_{i_0,\ldots,i_n} (p_{i_0,i_1} - \frac{1}{2}\delta_{i_0,i_1} \otimes p_{i_1,i_2} \otimes \ldots \otimes p_{i_n,i_0}).$$

  Then

$$c := (c_0, c_2, \ldots) \in C_{\mathrm{even}}$$

  is a cycle, $(b + B)(c) = 0$, and its homology class $\mathrm{ch}([M]) := [c] \in HP_{\mathrm{even}}(A)$ depends only on $[M] \in K_0(A)$.
- The Chern character in odd degree is defined similarly using invertible matrices in $M_n(A)$.

Towards a motivation of all that - I

- B - bounded operators on a separable Hilbert space, $K \subset B$ - the ideal of compact operators. $F \in B$ is Fredholm if its image in B/K is invertible. This implies that it has a well-defined index

$$\mathrm{ind}(F) := \dim \ker F - \dim \mathrm{coker}\, F \in \mathbb{Z}.$$

- Example:

$$H = \ell^2(\mathbb{N}) = \{(x_0, x_1, x_2, \dots) \mid x_i \in \mathbb{C}, \sum_i \|x_i\|^2 < \infty\},$$

$$F : (x_0, x_1, \dots) \mapsto (0, x_0, x_1, \dots)$$

is Fredholm with index -1.

Towards a motivation for all that - II

- X - compact oriented smooth manifold, D - elliptic pseudodifferential operator acting on smooth sections of a vector bundle E over X. Then D is Fredholm and the Atiyah-Singer index theorem computes its index in terms of purely topological data attached to D, E, X.
- Example: Cartan's exterior differential is elliptic. Its index is the Euler characteristic

$$\chi(X) := \sum_i (-1)^i \dim H^i(X, \mathbb{R})$$

of X. For 2-dimensional X the Atiyah-Singer index theorem reduces to the Gauss-Bonnet formula.

A motivation for all that

- The family index: The index of a family D of elliptic operators acting along the fibres of a fibration $\pi : X \to Y$ is

$$[\ker D] - [\operatorname{coker} D] \in K^0(Y).$$

- Connes: What about operators D that act on the leaves of a foliation for which there is no quotient space?
- Ansatz: Describe the quotient space by a noncommutative algebra A and define an index as an element in $K_0(A)$. Develop index theorems that compute numerical invariants of these elements.

Example: The noncommutative torus

- The foliation is the Kronecker foliation of the 2-torus T^2 with irrational angle $\theta \in \mathbb{R} \setminus \mathbb{Q}$: The noncommutative space we are considering is the quotient of T^2 by $\mathbb{R}$ which acts as

$$t \triangleright (x, y) = (x + t, y + t\theta), \quad t \in \mathbb{R}, x, y \in S^1 = \mathbb{R} \text{ mod } \mathbb{Z},$$

  or equivalently the quotient $S^1/\mathbb{Z}$ with respect to the action

$$t \triangleright x = x + t\theta, \quad t \in \mathbb{Z}, x \in S^1 = \mathbb{R} \text{ mod } \mathbb{Z}.$$

- The algebra A used to describe this quotient is the crossed product $C(S^1) \rtimes \mathbb{Z}$, that is, the universal C^*-algebra generated by two unitaries u, v satisfying $uv = qvu$, $q = e^{2\pi i\theta}$.

Spectral triples

- Aim: Do Riemannian geometry on noncommutative spaces.
- Ansatz: Consider triples (A, H, D), where A is an algebra represented on a Hilbert space H, and D is a self-adjoint operator with compact resolvent such that

$$df = [D, f], \quad f \in A$$

  is bounded.
- Example: X - compact spin manifold, $A = C^\infty(X, \mathbb{C})$, $H = L^2(S)$ - the square integrable spinor fields, A acting by multiplication operators. D - the Dirac operator. Then df is the differential of f that acts via Clifford multiplication on spinors.

An application: The standard model

- The Einstein-Hilbert action can be generalised to the formalism of spectral triples. Classically, this is the action functional on the space of metrics on a given 4-manifold whose associated Euler-Lagrange equations are the vacuum Einstein field equations

  $$R_{\mu\nu} - \frac{1}{2} R \, g_{\mu\nu} = 0.$$

  Here $R_{\mu\nu}$ is the Ricci tensor (the trace of the curvature tensor), R is the scalar curvature, and $g_{\mu\nu}$ the metric tensor.
- Applying this machinery to the tensor product of $C^\infty(X, \mathbb{C})$ with the small noncommuttive space $\mathbb{C} \oplus \mathbb{H} \oplus M_3(\mathbb{C})$ allows to treat traditional gravity plus the three other fundamental forces as described by the standard model in a unified way as pure gravity on a noncommutative space.

The Atiyah-Singer index theorem

- Atiyah-Singer: X compact oreiented smooth manifold, $D : E \to E$ elliptic pseudodifferential operator. Then D or rather

$$F = \frac{D}{|D|}$$

is Fredholm and its index can be computed as

$$\int_{[X]} \tau_!(\mathrm{ch}(\sigma_D)) \cup \mathrm{td}(X).$$

Here $\tau_! : H_c^\bullet(T^*X) \to H_c^\bullet(X)$ is the inverse of the Thom isomorphism, $\sigma_D \in K_0(T^*X)$ is the symbol of D, $\mathrm{td}(X)$ is the Todd class of X, and $[X] \in H_\bullet(X)$ is the fundamental class of X that gives the orientation.