

Dirac Operators

Lecture 4: From $S^2 = \mathbb{C}P^1$ to Borel-Weil

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- Given a compact $2n$ -dimensional Hermitian manifold and

$$A = C^\infty(X, \mathbb{C}), \quad V = \text{Der}_{\mathbb{C}}(A, A),$$

we have seen that there is a canonical spinor module

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where $V = V_+ \oplus V_-$ is the decomposition of the vector fields V into holomorphic and antiholomorphic ones.

- Today we will compute all these structures explicitly for the simple example of $S^2 \simeq \mathbb{C}P^1$.

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- From this one sees that $\mathbb{C}P^1$ is the 1-point compactification of \mathbb{C} which is also $X = S^2$. Identifying $X \simeq \mathbb{C}P^1$ gives it the structure of a complex manifold.

$X \simeq SL(2, \mathbb{C})/B$

- The group

$$G := SL(2, \mathbb{C}) := \{g \in M_2(\mathbb{C}) \mid \det g = 1\}$$

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so we get a realisation of X as a homogeneous space

$$X \simeq G/B.$$

$X \simeq SU(2)/U(1)$

- By the Iwasawa decomposition, every $g \in G = SL(2, \mathbb{C})$ can be uniquely written as $g_0 g_1$ with

$$g_0 \in SU(2) := \{g \in G \mid g^\dagger = g^{-1}\}, \quad g^\dagger = \bar{g}^T$$

and

$$g_1 \in AN := \left\{ \begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix} \mid a \in (0, \infty), b \in \mathbb{C} \right\}.$$

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- Since $AN \subset B$ acts trivially on $[1 : 0]$, $SU(2)$ acts already transitively on X . Since

$$B \cap SU(2) = \left\{ \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} \mid a \in U(1) \right\}$$

we get a final identification

$$X \simeq SU(2)/U(1).$$

The Lie algebras and $\mathfrak{g}/\mathfrak{b}$

- The Lie algebra $\mathfrak{g} = \mathfrak{sl}(2, \mathbb{C}) = \{x \in M_2(\mathbb{C}) \mid \operatorname{tr} x = 0\}$ has a vector space basis

$$H := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, E := \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, F := \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

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- The tangent space of G/B at 1 can be identified with $\mathfrak{g}/\mathfrak{b}$ as a **representations of B** : this acts on \mathfrak{g} via the adjoint action

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which leaves \mathfrak{b} invariant so that the quotient $\mathfrak{g}/\mathfrak{b}$ is a B -module.

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- V_+ is the module of sections of $(G \times \mathfrak{g}/\mathfrak{b})/B$. To determine this explicitly we have to understand $\mathfrak{g}/\mathfrak{b}$ a bit better.

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- It is easily seen that $L_n^* = L_{-n}$, so $V_- = L_2$.

A general remark about bundles on X

Lemma

F -bundles over X are classified by $\pi_1(\text{Aut}(F))$.

Proof. Let $U_{\pm} := \text{im } \iota_{\pm} \simeq \mathbb{C}$ be the two charts. U_{\pm} is contractible, so every bundle is trivial over them.

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Two functions give the same bundle iff they are homotopy equivalent, and $U_+ \cap U_-$ is contractible to S^1 (the equator of S^2). \square

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where g' is the matrix g with the first column rescaled by $\frac{1}{\det g}$.

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The spinor modules on $\mathbb{C}P^1$

Theorem

The spinor modules over X are given by

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From the proof of Application 2 we see this has trivial (as matrix algebra bundle) endomorphism bundle iff j is even. Since $A \oplus L_{-2}$ is a spinor module (the canonical one) we know that $\text{Cl}(V, q)$ is this trivial bundle, hence all spinor modules must be of the form $A \oplus L_{2j}$. \square

Representations of $G = SL(2, \mathbb{C})$

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- In a suitable basis $\{v_n^d\}$ the corresponding rep of \mathfrak{g} is given by

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- In particular, each of these reps is self-dual, that is, there exists a symmetric nondegenerate G -invariant bilinear form $\langle \cdot, \cdot \rangle$ on ρ_d .

The Peter-Weyl theorem

- Define the functions

$$t_{mn}^d : G \rightarrow \mathbb{C}, \quad \langle \bar{v}_m^d, \rho_d(g)v_n^d \rangle,$$

where $\{\bar{v}_m^d\}$ is the basis dual to $\{v_n^d\}$ with respect to the invariant form,

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- For background and proof read Klimyk, Schmüdgen, “Quantum groups and their irreducible representations”, Section 11.2.

Peter-Weyl continued

- The Peter-Weyl basis is designed to describe the representations $(\rho(g)f)(h) := f(hg)$, $(\lambda(g)f)(h) := f(g^{-1}h)$, $g, h \in G, f \in \mathbb{C}[G]$ of G on $\mathbb{C}[G]$:

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$$\begin{aligned}(\rho(g)t_{mn})(h) &= t_{mn}(hg) = \langle \bar{v}_m, \rho_d(hg)v_n \rangle = \langle \bar{v}_m, \rho_d(h)\rho_d(g)v_n \rangle \\ &= \langle \bar{v}_m, \rho_d(h) \sum_{i=0}^d \langle \bar{v}_i, \rho_d(g)v_n \rangle v_i \rangle \\ &= \sum_{i=0}^d t_{mi}(h)t_{in}(g).\end{aligned}$$

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- In Hopf-algebraic language this can be rephrased by saying that for the coproduct on $\mathbb{C}[G]$ dual to the group law one has

$$\Delta(t_{mn}^d) = \sum_{i=0}^d t_{mi}^d \otimes t_{in}^d.$$

- In particular, we can read off the action of \mathfrak{g} on the t_{mn}^d :

$$Ht_{mn}^d = \sum_i t_{mi}^d \langle \bar{v}_i^d, H v_n^d \rangle = (d - 2n)t_{mn}^d$$

and similarly

$$Et_{mn}^d = (d - n + 1)t_{mn-1}^d, \quad Ft_{mn}^d = (n + 1)t_{mn+1}^d.$$

The holomorphic elements in L_n

- Sections of the $(G \times \chi_n)/B$ can be identified with functions

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Back to $SU(2)/U(1)$

- Restricting the t_{mn}^d to $SU(2)$ also gives a vector space basis of the ring $\hat{A} \subset A$ of complex-valued polynomials on $SU(2)$. Taking now the $U(1)$ -equivariant elements gives the algebraic sections $\hat{L}_N \subset L_n$ of our line bundles. Explicitly, we get:

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Corollary

$$E(\hat{L}_n) \subset \hat{L}_{n-2}, F(\hat{L}_n) \subset \hat{L}_{n+2}.$$

A (the?) Dirac operator

- There is no time to construct D properly from a connection, so I give an ad hoc definition. If we pick a G -invariant metric and connection, then D will be a first-order operator commuting with the G -action on spinors that is given by the λ -action of G on $\mathbb{C}[G]$.

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- Using our last corollary allows us to define on $S_n = L_n \oplus L_{n-2}$

$$D := \begin{pmatrix} 0 & D_+ \\ D_- & 0 \end{pmatrix}, \quad D_+ := E, \quad D_- := F.$$

by which we mean that D maps the subbundles L_n, L_{n-2} into each other using E, F .

The Index of D

- When talking about the index of D we mean the one defined as for even Fredholm modules,

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- Note that for each n one has either $\ker D_+ = 0$ or $\ker D_- = 0$.
- S_{-1} comes from a spin structure (not only $\operatorname{spin}^{\mathbb{C}}$). For general n , the index is the pairing of the K-homology class of this D_{spin} with the K-theory class of L_{n+1} by which we have to twist the spin structure to get to S_n ,

Theorem

$$\operatorname{ind} D = \langle [D_{\operatorname{spin}}], [L_{n+1}] \rangle = n + 1 \text{ on } S_n.$$

The Borel-Weil theorem

D commutes with the λ -action of G , so $\ker D, \operatorname{coker} D$ are representations of G . The index can be defined more abstractly not as a number but as an element in the representation ring of G which is the equivariant K-theory of a point. $\ker D$ is just the irreducible representation of G of dimension $|n - 1|$.

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Outlook

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- Obviously, this is only one of many many applications of Diracs, see the literature below.

- Bass: “Clifford algebras and spinor norms over a commutative ring”
- Friedrich: “Dirac operators in Riemannian geometry” .
- Berline, Getzler, Vergne, “Heat kernels and Dirac operators” .