HARMONIC MEASURES FOR DISTRIBUTIONS WITH FINITE SUPPORT ON THE MAPPING CLASS GROUP ARE SINGULAR

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ABSTRACT. Kaimanovich and Masur [14] showed that a random walk on the mapping class group for an initial distribution whose support generates a non-elementary subgroup, when projected into Teichmüller space converges almost surely to a point in the space $\mathscr{PMF}$ of projective measured foliations on the surface. This defines a harmonic measure on $\mathscr{PMF}$. Here, we show that when the initial distribution has finite support, the corresponding harmonic measure is singular with respect to the natural Lebesgue measure class on $\mathscr{PMF}$.

CONTENTS

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1. Introduction.

Let S be an orientable surface of finite type. The mapping class group of S is the group G of orientation preserving diffeomorphisms of S modulo those isotopic to identity. The Teichmüller space $T(S)$ of S is the space of marked conformal structures on S modulo biholomorphisms isotopic to identity. The mapping class group G acts on $T(S)$ by changing the marking. The Teichmüller space is homeomorphic to an open ball in $\mathbb{R}^{6g-6+2m}$, where g is the genus and m is the number of punctures of S. Thurston showed that $T(S)$ can be compactified by the space $\mathscr{PMF}$ of projective measured foliations on S. The action of G extends continuously to $\mathscr{PMF}$.

In [14], Kaimanovich and Masur considered random walks on G with some initial distribution μ. It is possible to project the random walk into $T(S)$ by choosing a base-point and then using the action of G. They showed that if the subgroup of G generated by the support of μ is non-elementary, then almost every sample path converges to some uniquely ergodic foliation in $\mathscr{PMF}$. This means that there is a well defined hitting measure h on $\mathscr{PMF}$ coming from the random walk. Moreover, the measure h is a harmonic measure in the sense that if A is a measurable set of $\mathscr{PMF}$ then

$$h(A) = \sum_{g\in G} \mu(g)h(g^{-1}A).$$

Complete train tracks on S define an atlas of charts on the space $\mathscr{MF}$ of measured foliations on S. The set of integer weights on a train track correspond to mutli-curves on S carried by it. So the transition functions between charts have to preserve the integer weights. As a result, even though projectivization does not necessarily define a global *Lebesgue* measure on $\mathscr{PMF}$, there is still a well-defined Lebesgue measure class.

The main theorem here is:

Theorem 1.1. *If μ is a finitely supported initial probability distribution on G such that the subgroup of G generated by the support of μ is non-elementary, then the induced harmonic measure h on $\mathscr{PMF}$ is singular with respect to the Lebesgue measure class.*

Conversely, one can ask if there is a Lebesgue measure ℓ on $\mathscr{PMF}$ such that ℓ is a harmonic measure for some initial distribution on G? If such ℓ exists, the associated distribution has infinite support as a corollary to our theorem. A recent announcement by Eskin, Mirzakhani and Rafi asserts that such a distribution on G exists. However, it seems that this distribution does not have finite first moment.

1.2. **Strategy of the proof.** The proof of the theorem works by an explicit construction of a singular set for h i.e., a measurable set in $\mathscr{PMF}$ that has full Lebesgue measure and zero harmonic measure.

As a preliminary example, the group $PSL(2,\mathbb{Z})$ is quasi-isometric to the trivalent tree dual to the Farey tessellation of $\mathbb{H}$. Fixing a forward direction in the tree from a base-point, at each vertex one turns either left or right. Infinite paths in the forward cone converge to irrational points in an interval of $S^1 = \partial\mathbb{H}$ giving a symbolic coding in terms of L and R whose tail is exactly the continued fraction expansion of the limit irrational point. The coefficients in a continued fraction expansion have a $1/n^2$-distribution in the Lebesgue measure. For a non-backtracking random walk in the forward cone, the coefficients are distributed as $1/2^n$ since at each vertex there are two directions forward to choose from. The construction of a singular set then follows from exploiting this discrepancy in a Borel-Cantelli setup. For general random walks on the trivalent tree, the coefficients have a uniform exponential distribution in the harmonic measure. So the construction goes through as before.

In the general situation, the symbolic coding of points in $\mathscr{PMF}$ is achieved by splitting sequences of complete train tracks with a single switch which we call complete non-classical interval exchanges. For this symbolic coding, unlike $PSL(2,\mathbb{Z})$, one cannot write down the distribution (in either measure) for every type of "cylinder" sets. However, for certain combinatorial types of complete non-classical interval exchanges there are cylinder sets corresponding to Dehn twists in vertex cycles for which the distributions can be written down. In the Lebesgue measure, these cylinder sets have a $1/\text{poly}(n)$ distribution and for the harmonic measure they have an exponential distribution. Given this discrepancy of the distributions, the construction of a singular set follows similar lines. However, it is a harder technical task to get the measure theory to work out in this full generality.

1.3. **General groups.** The study of boundary phenomena for random processes on groups was initiated by Furstenberg in the 60's. He showed that Brownian motion on semi-simple non-compact Lie groups has a natural boundary at infinity to which almost every Brownian path converges. This defines a harmonic measure on this geometric boundary giving the distribution of the limit points of Brownian paths. Moreover, he showed that for any lattice in the Lie group, the Brownian motion can be discretized to a random walk on the lattice which converges to the same geometric boundary as the Lie group. This led to the first rigidity results.

In general, when a group is the fundamental group of a manifold with a geometric boundary (for instance, non-positively curved manifolds), in addition to the harmonic measure for Brownian motion, there are two other naturally defined measures on the group boundary: the visual or Lebesgue measure, and the Patterson-Sullivan measure. For compact negatively curved manifolds of dimension 2 there is measure rigidity: either the manifold is locally symmetric in which case the three measure classes coincide, or the measures are all mutually singular [18]. The measure rigidity conjecture is still open in higher dimensions.

The random walk on the group coming from Brownian motion is infinitely supported. When restricted to finitely supported distributions, little is known about absolute continuity/singularity dichotomy of the associated harmonic measures. For a non-uniform lattice G in $SL(2,\mathbb{R})$ ($\mathbb{H}/G$ is a finite area surface/orbifold with cusps), Guicarc'h and LeJan [11] showed that a harmonic measure on S^1 coming from an initial distribution satisfying a certain moment condition (finitely supported distributions being a special case) is singular with respect to the Lebesgue measure class on S^1. This result has been generalized to certain types of finitely generated groups of circle diffeomorphisms by Deroin, Kleptsyn and Navas [6]. In a loose sense, the main theorem here is an analog of the Guivarc'h-LeJan result. In higher rank, Kaimanovich and LePrince [13] showed that any Zariski dense countable subgroup of $SL(d,\mathbb{R})$ carries a non-degenerate finitely supported distribution whose induced harmonic measure on the flag space is singular. They conjecture that harmonic measures for finitely supported distributions on discrete subgroups are singular at infinity.

On the other side of the dichotomy, Lyons [19] produced examples of finite graphs with non-constant vertex degrees for which the simple random walk on its universal covering tree gives a harmonic measure that is absolutely continuous with respect to the visual measure. Solomyak [32] showed that for almost every $\lambda > 1/2$, the associated Bernoulli convolution is absolutely continuous. See also [31]. Barany-Pollicott-Simon [1] and McMullen [29], have provided examples of non-discrete subgroups of $SL(2,\mathbb{R})$ with explicit finitely supported distributions such that the corresponding harmonic measures on S^1 are absolutely continuous. However, the general picture remains unclear.

1.4. **Outline of the paper.** Section 2 begins with preliminaries from Teichmüller theory, and defines the Lebesgue measure class on $\mathscr{PMF}$. Section 3 provides background on random walks on groups followed by the statements of Theorem 3.1 by Kaimanovich and Masur for random walks on mapping class groups, and the main theorem of this paper. Section 4 states and proves the key measure theory result (a consequence of the quasi-independent Borel-Cantelli lemma) that we exploit in the construction of the singular set. Section 5 explains the construction of the singular set in the special case of a torus, and indicates how this should generalize. In the process, it introduces techniques of interval exchanges. Section 6 discusses complete non-classical interval exchanges, the parameter spaces of which provides charts on $\mathscr{PMF}$. The key technical theorem, Theorem 6.9 is stated here. Section 7 constructs the particular combinatorial types of complete non-classical interval exchanges needed for the construction. It also establishes the necessary estimates for the Lebesgue measure. Section 8 provides some background on the curve complex, and states Klarreich's Theorem 8.1 which implies that it does not matter which space the mapping class group random walk is projected into so long as the boundaries are measurably isomorphic. Section 9 introduces the marking complex and relative space, and provide some background on sub-surface projections defined on these spaces. Section 10 outlines the key facts about half-spaces in the relative space. Section 11 states and proves Maher's theorem, Theorem 11.6, the main decay estimate for harmonic measure for nesting under a sub-surface projection. Section 12 outlines a technical trick that provides the setup necessary to apply Theorem 11.6. Finally, Section 13 puts together all the ingredients to construct a singular set.

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2. Preliminaries from Teichmüller theory.

For a detailed introduction to Teichmüller theory and mapping class groups see [8]. Let $S_{g,m}$ be an orientable surface with genus g and m punctures. For brevity, we will drop the subscripts and call it just S. The mapping class group of S is the group G of orientation preserving diffeomorphisms of S that preserve the set of punctures modulo those isotopic to identity.

2.1. Teichmüller space.

The Teichmüller space of S is the space of marked conformal structures on S modulo biholomorphisms isotopic to identity. We denote the Teichmüller space by $T(S)$. By the uniformization theorem, there is a unique marked hyperbolic metric in each marked conformal class. This means that $T(S)$ can also be thought of as the space of marked hyperbolic metrics on S modulo isometries isotopic to identity. The mapping class group G acts on $T(S)$ by changing the marking.

It is a classical theorem that $T(S)$ is homeomorphic to an open ball in $\mathbb{R}^{6g-6+2m}$, where g is the genus and m is the number of punctures of S. Thurston showed that $T(S)$ can be compactified by the space $\mathscr{PMF}$ of projective measured foliations on S, such that the action of G on $T(S)$ extends to a continuous action on the boundary $\mathscr{PMF} = \partial T(S)$. The space $\mathscr{PMF}$ is homeomorphic to a sphere of dimension $6g-7+2m$. It is also the same as the space of projective measured laminations $\mathscr{PML}$ on S by a homeomorphism that is mapping class group equivariant.

2.2. Lebesgue measure on $\mathscr{PMF}$.

For a detailed account of the theory of train tracks see [30]. A train track on S is an embedded 1-dimensional CW complex with additional structure. The edges are called *branches* and the vertices are called *switches*. The branches are smoothly embedded on the interiors, and there is a common line of tangency to all branches meeting at a switch. Thus, in a cyclic order at a switch coming from the orientation on S, the branches incident on it split into two disjoint intervals of subsets, arbitrarily assigned as incoming and outgoing. It is natural to assume that the valence of each switch is at least three and that no complementary region in S is a bigon or a once-punctured nullgon. A train track on S is *large* if all complementary regions are polygons or once-punctured polygons.

A *train route* is a regular smooth path in the train track. In particular, a train route traverses a switch only by passing from an incoming edge to an outgoing edge or vice versa. An essential simple closed curve or multi-curve is said to be *carried* by a train track if it is isotopic to a closed train route. The number of times a carried multi-curve passes over a branch gives us an assignment of non-negative integral weights to the branches satisfying the switch conditions that at every switch the sum of weights on incoming branches is equal to the sum of weights on outgoing branches. In general, the set of non-negative weight assignments satisfying the switch conditions are called *transverse measures* carried by the train track and can be identified with a set of measured foliations on S. These measured foliations are said to be *carried* by the train track.

A train track is *recurrent* if it is possible to assign weights satisfying the switch conditions, such that all weights are positive. Similarly, a train track on S is *transversely recurrent* if for every branch there is an essential simple closed curve on S hitting the train track efficiently (i.e., the complement of the union of the train track and the curve has no bigons) such that it intersects the branch at least once. A track that is both recurrent and transversely recurrent is called *birecurrent*. A train track on S is *maximal* if it is not a proper subtrack of any other track. A maximal birecurrent train track is called *complete*. It is easy to see that if a train track is complete then all of its complementary regions in S are ideal triangles or once punctured monogons. A generic measured foliation carried by a complete train track can be realized as the vertical foliation of a quadratic differential in the principle stratum: it has a simple zero at each three pronged singularity of the foliation and a simple pole at every puncture.

The set of transverse measures carried by a complete train track has the structure of a cone (i.e., a homogeneous set) in $\mathbb{R}^{6g-6+2m}_{\geqslant 0}$, over a convex polytope of dimension $6g-7+2m$ with finitely many extremal vertices. Moreover, the directions determined by the extremal vertices are rational; hence we can find a minimal integer point on each of them. These integer points represent simple closed curves on S, and are called the *vertex cycles* of the train track. In general, integer points in the cone represent multi-curves carried by the train track.

The set of transverse measures carried by complete train tracks on S defines an atlas of charts on the space $\mathscr{MF}$ of measured foliations on S. In each chart, a choice of projectivization such as normalizing the sum of weights to be 1, gives a chart on $\mathscr{PMF}$. However, the transition functions between charts need not preserve this normalization. So we do not get a global measure on $\mathscr{PMF}$ by this process. However, the transition functions have to preserve the multi-curves i.e., the integer points. Hence, there is a well defined measure class on $\mathscr{PMF}$. We call the measure class on $\mathscr{PMF}$ coming from these charts the *Lebesgue* measure class.

3. Random walks.

Let G be a group and μ a probability distribution on G. A random walk on G is a Markov chain with transition probabilities $p(g,h) = \mu(g^{-1}h)$. It is assumed that one starts at the identity element in G at time zero. Denote the group element generated in n steps of the random walk by ω_n i.e., the group element ω_n is the product $g_1 g_2 \cdots g_n$ where each group element g_i is sampled by μ. The distribution of ω_n is given by the n-fold convolution $\mu^{(n)}$ of the initial distribution μ. The *path space* for the random walk is the space $G^{\mathbb{Z}_+}$ of one-sided infinite sequences of elements of G with the probability measure $\mathbb{P}$ that is the push-forward of the product measure on $G^{\mathbb{Z}_+}$ under the map $(g_1, g_2, g_3, \cdots) \to (\omega_1, \omega_2, \omega_3, \cdots)$. The measure $\mathbb{P}$ has marginals $\mu^{(n)}$. The group G acts on the path space on the left, as opposed to the increments g_n from ω_{n-1} to ω_n, which get multiplied on the right. For general background about random walks on infinite groups, see [34].

From now on, let G be the mapping class group. A subgroup of G is *non-elementary* if it contains a pair of psuedo-Anosov elements with distinct stable and unstable measured foliations. Kaimanovich and Masur proved the following theorem in [14].

Theorem 3.1 (Kaimanovich-Masur)**.** *If μ is a probability measure on the mapping class group G such that the group generated by its support is non-elementary, then there exists a unique μ-stationary probability measure h on $\mathscr{PMF}$, which is purely non-atomic and concentrated on the subset $\mathscr{UE} \subset \mathscr{PMF}$ of uniquely ergodic foliations. For any $X \in T(S)$, and almost every sample path $\omega = \{\omega_n\}$ determined by (G,μ), the sequence $\omega_n X$ converges to a limit $F(\omega)$ in $\mathscr{UE}$, and the distribution of the limit points $F(\omega)$ is given by h.*

The measure h on $\mathscr{PMF}$ is μ-stationary in the sense that if A is a measurable set in $\mathscr{PMF}$ then the measure h satisfies

$$h(A) = \sum_{g \in G} \mu(g) h(g^{-1}A). \tag{3.2}$$

The measure h is called a *harmonic* measure because of the above property.

Remark 3.3. Kaimanovich and Masur state the result only for closed surfaces, but the proof works for surfaces with punctures also, as pointed out in [9]. Kaimanovich and Masur also show that if, in addition to the hypothesis of Theorem 3.1, the initial distribution μ has finite entropy and finite first logarithmic moment with respect to the Teichmüller metric, then the measure space $(\mathscr{PMF}, h)$ is the Poisson boundary of (G,μ).

The main theorem we prove is

Theorem 3.4. *If μ is a finitely supported probability distribution on the mapping class group G such that the subgroup of G generated by the support is non-elementary, then the induced harmonic measure h on $\mathscr{PMF}$ is singular with respect to the Lebesgue measure class.*

To construct a singular set for h i.e., a measurable set of $\mathscr{PMF}$ that has full Lebesgue measure but zero harmonic measure, the crucial point is to understand the action of the reducible elements, in particular Dehn twists, from the measure theoretic point of view. We construct one-switch complete train tracks on S with the property that a positive Dehn twist in one of its vertex cycles can be realized as a splitting sequence of the train track. After normalizing so that the measures of the original chart are 1, we show that the Lebesgue measure of the charts obtained after applying the Dehn twist splitting sequence n successive times, are $\approx 1/n^k$, for some positive integer k. On the other hand, by a theorem of Maher, the harmonic measures of the same charts, are $\leqslant \exp(-n)$. It is this discrepancy that we exploit to give a construction of the singular set. The key measure theoretic tool is Proposition 4.4, a consequence of the quasi-independent Borel-Cantelli lemma that we state and prove in the next section.

4. The Borel-Cantelli Setup.

In this section, we state and prove the key proposition in measure theory that we use in our construction in Section 13. We state the quasi-independent Borel-Cantelli Lemma [33], a generalization of the classical Borel-Cantelli lemma to the case when events are *pairwise almost independent.*

Lemma 4.1 (Borel-Cantelli)**.** *Let $(\Omega,\mathscr{B},\mu)$ be a probability space and $\{X_n\}_1^\infty$ be a sequence of $\mathscr{B}$-measurable sets such that there exists a positive integer d and a constant $c>1$ for which*

$$\mu(X_m\cap X_n)<c\mu(X_m)\mu(X_n),\qquad m\in\mathbb{N}\ \text{ and }\ n\geqslant m+d. \tag{4.2}$$

Then

$$\sum_{n=1}^{\infty}\mu(X_n)=\infty \implies \mu(\limsup_{n\to\infty}X_n)\geqslant\frac{1}{4c}.$$

On the other hand, with no constraint on $\mu(X_m\cap X_n)$ we have

$$\sum_{n=1}^{\infty}\mu(X_n)<\infty \implies \mu(\limsup_{n\to\infty}X_n)=0.$$

Remark 4.3. The condition in (4.2) is called *pairwise almost independence.* To simplify the discussion henceforth, we shall require $d=1$ in our definition of pairwise almost independent.

Now consider a measure space $(\Omega,\mathscr{B})$ with two $\mathscr{B}$-measures ℓ and h. Suppose that there is a sequence of measurable sets X_n such that the sets X_n are pairwise almost independent for the measure ℓ, and that for n large enough, the measures satisfy $\ell(X_n)\approx 1/n$ and $h(X_n)\leqslant\exp(-kn)$ for some uniform positive constant $k>0$. Then a direct application of Lemma 4.1 shows that $\ell(\limsup X_n)>0$ and $h(\limsup X_n)=0$. So the goal is to set up such a sequence of sets in our context.

In the construction in Section 13, instead of a direct construction of a sequence X_n with properties above, it turns out to be natural to construct a doubly indexed sequence of sets $Y_n^{(m)}$ that for different m, are pairwise almost independent for the Lebesgue measure ℓ i.e., when $m_1\neq m_2$ they satisfy the inequality (4.2). In addition, the sets $Y_n^{(m)}$ have the property that there are positive integers N, j independent of m, such that $\ell(Y_n^{(m)})\approx 1/n^j$ for $n>N$. For the measure h, there is a constant $k>0$ independent of m, such that $h(Y_n^{(m)})\leqslant\exp(-kn)$ for $n>N$.

Given such a doubly indexed sequence, the proposition below shows how to construct the sequence X_n, with the properties described above. The set $\limsup X_n$ is then a set with positive ℓ measure and zero h measure.

Proposition 4.4. *Let $(\Omega,\mathscr{B})$ be a probability space with measures ℓ and h. Let $Y_n^{(m)}$ be a doubly indexed sequence of measurable sets such that there exists a positive integer N and constants $j\in\mathbb{N}$, $a\in\mathbb{R}$ with $a>1$ and $r<1$ such that*

$$\frac{1}{an^j}<\ell(Y_n^{(m)})<\frac{a}{n^j},\qquad h(Y_n^{(m)})<r^n \tag{4.5}$$

for all m and for all $n > N$. Further, suppose that the sets $Y_n^{(m)}$ are pairwise almost independent i.e., for all pairs $(Y_{n_1}^{(m_1)}, Y_{n_2}^{(m_2)})$ of sets with $m_1 \neq m_2$, there exists a constant $c > 1$ such that

$$\ell(Y_{n_1}^{(m_1)} \cap Y_{n_2}^{(m_2)}) < c \cdot \ell(Y_{n_1}^{(m_1)})\ell(Y_{n_2}^{(m_2)}).$$

Then there is a set X such that $\ell(X) > 0$ and $h(X) = 0$.

Proof. Define a sequence of sets X_n as follows: for $n > N$, let $s(n) = \sum_{i=1}^{n} i^{j-1}$ and $t(n) = \sum_{i=1}^{n+1} i^{j-1}$ and define

$$X_n = \bigcup_{m=s(n)}^{m=t(n)} Y_n^{(m)}.$$

Then for the measure ℓ we get

$$\sum_{m=s(n)}^{m=t(n)} \ell(Y_n^{(m)}) - \sum_{s(n)\leqslant m_1<m_2\leqslant t(n)} \ell(Y_n^{(m_1)} \cap Y_n^{(m_2)}) \leqslant \ell(X_n) \leqslant \sum_{m=s(n)}^{m=t(n)} \ell(Y_n^{(m)}).$$

By the property of almost independence and the upper bound in Equation (4.5)

$$\ell(Y_n^{(m_1)} \cap Y_n^{(m_2)}) < c \cdot \ell(Y_n^{(m_1)})\ell(Y_n^{(m_2)}) < \frac{ca^2}{n^{2j}}.$$

Hence, by the using the bounds in Equation (4.5)

$$\sum_{m=s(n)}^{m=t(n)} \frac{1}{an^j} - \sum_{m_1<m_2} \frac{ca^2}{n^{2j}} < \ell(X_n) < \sum_{m=s(n)}^{m=t(n)} \frac{a}{n^j},$$

which reduces to

$$\frac{1}{an} - \frac{ca^2}{n^2} < \ell(X_n) \leqslant \frac{a}{n}.$$

This shows that there is positive integer N and a constant $A > a$ such that for all $n > N$

$$\frac{1}{An} < \ell(X_n) < \frac{A}{n}.$$

Now we show that the sets X_n are almost independent. For $n_1 < n_2$ notice that

$$X_{n_1} \cap X_{n_2} \subseteq \bigcup_{\substack{s(n_1)\leqslant p\leqslant t(n_1)\\ s(n_2)\leqslant q\leqslant t(n_2)}} Y_{n_1}^{(p)} \cap Y_{n_2}^{(q)}.$$

As a consequence, with $s(n_1) \leqslant p \leqslant t(n_1)$ and $s(n_2) \leqslant q \leqslant t(n_2)$ there is no overlap of superscripts and we get

$$\ell(X_{n_1} \cap X_{n_2}) \leqslant \ell\left(\bigcup_{p,q} Y_{n_1}^{(p)} \cap Y_{n_2}^{(q)}\right) \leqslant \sum_{p,q} \ell(Y_{n_1}^{(p)} \cap Y_{n_2}^{(q)}).$$

Using pairwise almost independence of the Y's, we get

$$\ell(X_{n_1} \cap X_{n_2}) < \sum_{p,q} c \cdot \ell(Y_{n_1}^{(p)})\ell(Y_{n_2}^{(q)}) < \sum_{p,q} \frac{ca^2}{n_1^j n_2^j} = \frac{ca^2}{n_1 n_2} < \frac{ca^2}{A^2}\ell(X_{n_1})\ell(X_{n_2}),$$

showing pairwise almost independence of the sequence X_n with respect to the measure ℓ.

For the measure h note that

$$h(X_n) \leqslant \sum_{m=s(n)}^{t(n)} h(Y_n^{(m)}) < \sum_{m=s(n)}^{t(n)} r^n = n^{j-1} r^n.$$

The above inequality implies that for N large enough, there is a constant $\rho < 1$ such that for all $n > N$, we have $h(X_n) < \rho^n$. Finally, by Lemma 4.1, we have $\ell(\limsup X_n) > 0$ and $h(\limsup X_n) = 0$. □

5. THE $PSL(2,\mathbb{Z})$ EXAMPLE.

We shall first explain the construction of the singular set in the special case when the group is $PSL(2,\mathbb{Z})$. The group $PSL(2,\mathbb{Z})$ is the mapping class group of the torus. The Teichmüller space of the torus is $\mathbb{H}$ and the group $PSL(2,\mathbb{Z})$ acts on it by fractional linear transformations. The projective class of a measure foliation on the torus is determined by its slope, and can be marked as a point on the boundary $\partial\mathbb{H} = \mathbb{R}\cup\{\infty\}$. The simple curves on the torus are the rational points. The Farey graph is constructed with the rational points as the vertex set with an edge between two rational points if the simple curves representing those points can be isotoped to intersect minimally i.e., in a single point. The group $PSL(2,\mathbb{Z})$ acts on the Farey graph and the trivalent tree dual to it and by the Švarc-Milnor lemma it is quasi-isometric to the trivalent tree. A part of the Farey graph and its dual tree are shown in Figure 1.

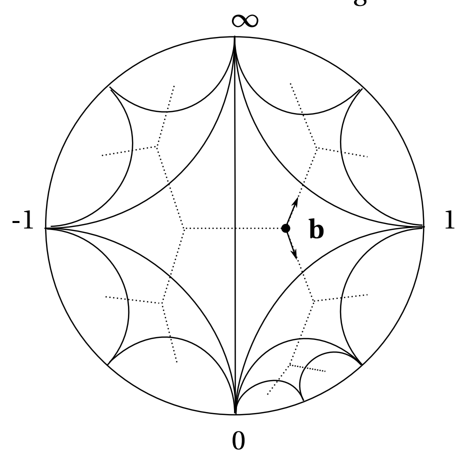

FIGURE 1. Farey graph and the dual tree

To illustrate the main point of the construction, first restrict to the nearest neighbor non-backtracking random walk on the trivalent tree starting from the base vertex **b** as shown in Figure 1. Moving forward in the tree from the base vertex, we can choose to move either "right" or "left" as shown by the arrows in the figure. The $PSL(2,\mathbb{Z})$ generators that move the base vertex forward by right and left can be taken to be

$$R = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \quad , \quad L = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \tag{5.1}$$

respectively. Now consider the interval $[0,1]$ in $\partial\mathbb{H} = \mathbb{R}\cup\{\infty\}$. Every irrational number $\mathbf{x}$ in $[0,1]$ corresponds to a unique non-backtracking sample path in the tree. This sample path can be written down as an infinite word $R^{a_1}L^{a_2}\cdots$, where all a_n are positive integers. In fact, it is easy to see that the numbers a_n are the coefficients in the continued fraction expansion of $\mathbf{x}$. The set of points in $[0,1]$ whose expansion begins with R^m with $m \geqslant n$ or equivalently $a_1 \geqslant n$ is $[0,1/n)$, whose Lebesgue measure is $1/n$. In fact, the classical Gauss map acts as a shift on the coefficients of the continued fraction expansion and has an invariant measure absolutely continuos with respect to Lebesgue measure on $[0,1]$. Hence, for the Gauss measure all coefficients are identically distributed and have the $1/n$ tail that a_1 has.

On the other hand, from the point of view of the nearest neighbor non-backtracking random walk, we are choosing R from the possible choices $\{R,L\}$, n times. So the set of infinite sample paths that begin with R^n have probability $1/2^n$. This indicates that near parabolic fixed points the Lebesgue and the harmonic measure scale differently under a repeated application of that parabolic element. Later, we exploit this discrepancy to show that the measures are mutually singular.

To give the explicit construction of the singular set restricted to $[0,1]$, we shall first interpret the continued fraction expansion of $\mathbf{x}$ as a splitting sequence of a classical interval exchange on 2 bands.

5.2. **Classical interval exchanges.** For a detailed exposition of classical interval exchanges and Rauzy-Veech renormalization, see [35].

In a classical interval exchange, an interval I is partitioned into d subintervals, which are permuted and then glued back preserving orientation, to get I. The result is a Lebesgue measure preserving map from I to itself. The data that completely determines a classical interval exchange map is: first, the widths of the subintervals and second, the permutation used for gluing. To work with projective classes of foliations, we normalize the length of I to be 1.

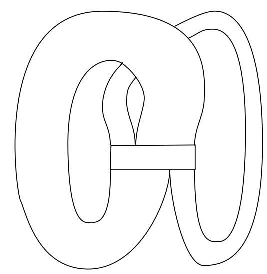

FIGURE 2. A classical interval exchange.

A classical interval exchange can be represented pictorially as described below. Here, we work with labeled interval exchanges which we now define. Let $\mathscr{A}$ denote an alphabet over d letters. In the definition that follows, the set $\mathscr{A}$ labels the bands. A classical interval exchange is determined by the lengths $(\lambda_\alpha), \alpha \in \mathscr{A}$ of the subintervals and bijections p_0 and p_1 from $\mathscr{A}$ to the set $\{1, \dots, d\}$ as follows: In the plane, draw the interval $I = [0, \sum_\alpha \lambda_\alpha)$ along the horizontal axis and then thicken it slightly in the vertical direction to get two copies, I_+ and I_-. Call them top interval and bottom interval respectively. Let $\epsilon : I_+ \sqcup I_- \to I_+ \sqcup I_-$ be the map that switches the intervals i.e., $\epsilon(I_+) = I_-$ and $\epsilon(I_-) = I_+$. Subdivide I_+ into d subintervals with widths $\lambda_{p_0^{-1}(1)}, \dots, \lambda_{p_0^{-1}(d)}$ from left to right. Subdivide I_- into d subintervals with widths $\lambda_{p_1^{-1}(1)}, \dots, \lambda_{p_1^{-1}(d)}$ from left to right. For each $\alpha \in \mathscr{A}$, join the $p_0(\alpha)$ subinterval of I_+ to the $p_1(\alpha)$ subinterval of I_- by a band of uniform width λ_α. The vertical flow along the bands from I_+ to I_-, followed by ϵ exhibits the classical interval exchange as a map from I_+ to I_+. Similarly, the inverse of the interval exchange is realized as a map from I_- to itself by flowing reverse along the bands, followed by ϵ. The ambiguity at the endpoints of the subintervals is removed by requiring the endpoint flow along the band that lies to the left.

5.2.1. *Rauzy induction.* We shall now describe an induction process on the space of interval exchanges called Rauzy induction. Call the positions on I_+ and I_- that are the rightmost *critical* positions. Let α_0 and α_1 be the labels of the bands in the critical positions with α_1 on I_+. First, suppose that $\lambda_{\alpha_1} > \lambda_{\alpha_0}$. Then we slice as shown in Figure 3 till we hit $I_+ \sqcup I_-$ for the first time.

The α_1 band remains in its critical position, but typically a band with a different label α_0' moves into the other critical position. Furthermore, the new width of α_1 is $\lambda_{\alpha_1} - \lambda_{\alpha_0}$. All other widths remain unchanged. If $\lambda_{\alpha_1} < \lambda_{\alpha_0}$ instead, we slice in the opposite direction in Figure 3. In either case, we get a new interval exchange with combinatorics and widths as described above. The operation we just described is called *Rauzy* induction.

Let $I' = [0, \sum_{\alpha \neq \alpha_0} \lambda_\alpha)$ in the first instance and $I' = [0, \sum_{\alpha \neq \alpha_1} \lambda_\alpha)$ in the second. Let I'_+ and I'_- denote the copies of I' in I_+ and I_- respectively. Then, Rauzy induction is the same as the first return map to the interval $I'_+ \sqcup I'_-$. Since Rauzy induction is represented pictorially by one band being split by another, it is also called a *split*. Iterations of Rauzy induction are called *splitting sequences*.

5.2.2. *Rauzy diagram.* A permutation $\phi : \{1, \cdots, d\}$ is *irreducible* if $\phi^{-1}\{1, \cdots, k\} \neq \{1, \cdots, k\}$ for all $k < d$. A labeled classical interval exchange is irreducible if $\phi = p_0 p_1^{-1}$ is irreducible. From the dynamical point of view, if the permutation is reducible, then for any widths of the subintervals, the interval exchange decomposes as a concatenation of two interval exchanges each with fewer bands. So, it makes sense to restrict to irreducible permutations. The Rauzy diagram is an oriented graph $\mathscr{G}$ constructed as follows: the nodes of the graph are combinatorial types ϕ of labeled irreducible classical interval exchanges with d bands with an arrow from ϕ to ϕ', if ϕ' results from splitting ϕ. For each node ϕ, there are exactly two arrows coming out of it. A splitting sequence gives us a directed path in $\mathscr{G}$. A finite splitting sequence starting from ϕ shall be called a *stage* in the expansion starting from ϕ. It turns out that every connected

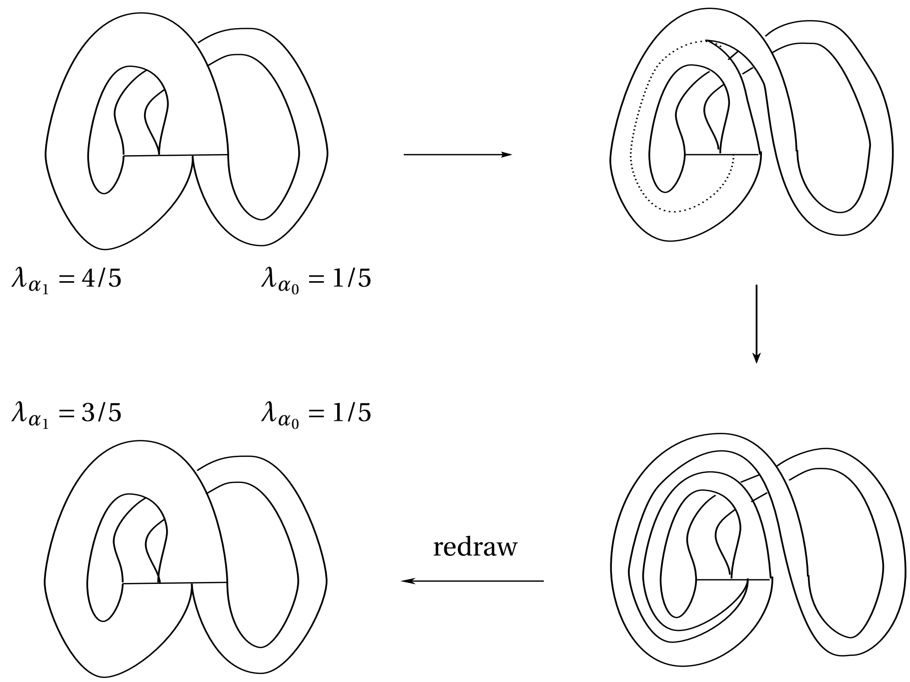

FIGURE 3. Rauzy Induction

component of $\mathcal{G}$ is an *attractor*: any pair of nodes can be connected by a directed path. This fact is crucial in the proof of the generalization of Proposition 5.8 for all classical interval exchanges.

5.2.3. *Encoding Rauzy induction on the parameter space.* Consider the vector space $\mathbb{R}^{\mathcal{A}}$ and let $\mathbb{R}^{\mathcal{A}}_{\geq 0}$ be the set of points with non-negative coordinates. Let Δ denote the standard $(d-1)$-simplex in $\mathbb{R}^{\mathcal{A}}$ given by the constrain the sum of the coordinates is 1. An assignment of widths to the bands is a point in $\mathbb{R}^{\mathcal{A}}$. Normalizing the widths so that their sum is 1 restricts us to Δ.

Let I denote the $d \times d$ identity matrix on $\mathbb{R}^{\mathcal{A}}$. Each instance of the Rauzy induction can be encoded as a non-negative matrix as follows: Let $M_{\alpha\beta}$ be the $d \times d$-matrix with the (α, β) entry 1 and all other entries 0. If the labels of the bands in the critical positions are α_0 and α_1, then the relationship between the old and new width data can be expressed by

$$\lambda = E\lambda', \tag{5.3}$$

where the matrix E has the form $E = I + M$. In the first instance of the split, when $\lambda_{\alpha_1} > \lambda_{\alpha_0}$, the matrix $M = M_{\alpha_1\alpha_0}$; in the second instance of the split, when $\lambda_{\alpha_0} > \lambda_{\alpha_1}$, the matrix $M = M_{\alpha_0\alpha_1}$. Thus, in either case the matrix E is an elementary matrix, in particular $E \in SL(d;\mathbb{Z})$.

Proceeding iteratively, we associate a matrix in $SL(d,\mathbb{Z})$ to any finite splitting sequence by requiring that the matrix Q at any stage in the sequence is obtained by multiplying the matrix Q' for the preceding stage on the right by the elementary matrix E associated to that particular split i.e., $Q = Q'E$. In this way, starting with an interval exchange determined by the width data (λ_α), we get an expansion by repeated splitting.

The Rauzy induction is undefined when $\lambda_{\alpha_0} = \lambda_{\alpha_1}$. However, this is always a co-dimension 1 subset of Δ and hence has measure zero. The set of widths whose Rauzy induction stops in finite number of steps is a countable union of such sets, and hence of measure zero. Thus for almost every width data, one can associate an infinite expansion. For an initial node $\phi \in \mathcal{G}$, expansions of the points (λ_α) in Δ by repeated Rauzy induction correspond to directed paths in $\mathcal{G}$ starting from ϕ.

Given a matrix Q with non-negative entries, we define the projectivization $\mathbb{P}Q : \Delta \to \Delta$ by

$$\mathbb{P}Q(\mathbf{y}) = \frac{Q\mathbf{y}}{|Q\mathbf{y}|},$$

where if $\mathbf{y} = (y_1, y_2, \cdots, y_d)$ in co-ordinates then $|\mathbf{y}| = \sum |y_i|$.

The subset of widths in Δ whose expansion begins with some finite splitting sequence $\jmath$ is given by $\mathbb{P}Q_\jmath(\Delta)$. To get an estimate of the relative probability of particular splitting sequence κ after $\jmath$ in terms of the probability that an expansion begins with κ it becomes essential to consider the Jacobian $\mathscr{J}_\Delta(\mathbb{P}Q_\jmath)$ of the projective linear map $\mathbb{P}Q_\jmath$ from Δ to itself. It is known that [4]

$$\mathscr{J}_\Delta(\mathbb{P}Q_\jmath)(\mathbf{y}) = \frac{1}{|Q_\jmath \mathbf{y}|^d}.$$

For a matrix $Q_\jmath$ of a stage $\jmath$, let $Q_\jmath(\alpha)$ be the column of $Q_\jmath$ corresponding to the band α.

Definition 5.4. For a constant $C > 1$, a stage $\jmath$ in the expansion is said C-distributed if the matrix $Q_\jmath$ of the stage has the property that for any pair of columns $Q_\jmath(\alpha)$ and $Q_\jmath(\beta)$

$$\frac{1}{C} < \frac{|Q_\jmath(\alpha)|}{|Q_\jmath(\beta)|} < C.$$

Suppose the splitting sequences $\jmath$ and κ, thought of as directed paths in $\mathscr{G}$, can be concatenated i.e., the sequence $\jmath$ can be followed by κ. We denote by $\jmath * \kappa$ the splitting sequence given by the concatenation. The main point of C-distribution is

Lemma 5.5. *If a stage $\jmath$ is C-distributed, then there exists a constant $c > 1$ that depends only on C and d, such that the relative probability that any sequence κ follows $\jmath$ satisfies*

$$\frac{1}{c}\ell(\mathbb{P}Q_\kappa(\Delta)) < \frac{\ell(\mathbb{P}Q_{\jmath*\kappa}(\Delta))}{\ell(\mathbb{P}Q_\jmath(\Delta))} < c \cdot \ell(\mathbb{P}Q_\kappa(\Delta)). \tag{5.6}$$

In other words, Lemma 5.5 says that if a stage $\jmath$ is C-distributed, the relative probability of a splitting sequence κ following $\jmath$ is up to a constant that depends on C and d alone, the same as the probability that an expansion begins with κ. See [10]

Proof. Let $Y_\jmath = \mathbb{P}Q_\jmath(\Delta)$, $Y_\kappa = \mathbb{P}Q_\kappa(\Delta)$ and $Y_{\jmath*\kappa} = \mathbb{P}Q_{\jmath*\kappa}(\Delta)$. Then

$$\frac{\ell(Y_{\jmath*\kappa})}{\ell(Y_\jmath)} = \frac{\int_{Y_\kappa} \mathscr{J}_\Delta(\mathbb{P}Q_\jmath)(\mathbf{x})\, d\ell(\mathbf{x})}{\int_\Delta \mathscr{J}_\Delta(\mathbb{P}Q_\jmath)(\mathbf{x})\, d\ell(\mathbf{x})}.$$

Because of the C-distribution of $\jmath$, the Jacobian $\mathscr{J}_\Delta(\mathbb{P}Q_\jmath)$ at any point inside Y_κ differs from the Jacobian $\mathscr{J}_\Delta(\mathbb{P}Q_\jmath)$ at any point in $\Delta \setminus Y_\kappa$ by a factor that lies in $(1/C, C)$. This finishes the proof. $\square$

5.7. **Classical interval exchange with 2 bands.** Since there is exactly a single permutation over two letters, there are only two combinatorial types for a classical interval exchange with 2 bands.

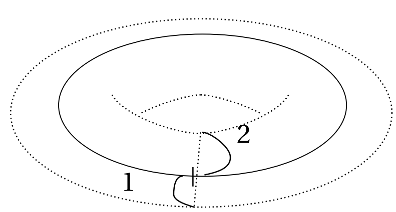

FIGURE 4. Interval exchange on the torus

One of the ways to embed this interval exchange on the torus is shown in Figure 4. The transverse arc represents the base interval. For this embedding, the widths (λ_1, λ_2) of the bands uniquely determines the foliation with slope λ_2/λ_1 in the standard homology basis of the torus. Thus the set $[0,1] \subset \mathbb{R} \cup \infty$ is equivalent to $\lambda_2 < \lambda_1$ initially.

Now we consider expansions by repeated Rauzy inductions. At any stage, there are exactly two types of splits that are possible. When band 2 splits band 1 we denote it by R, and if vice versa we denote it by L. Since initially $\lambda_2 < \lambda_1$, all our expansions begin with R. The elementary matrices associated to the splits are exactly the same as in Equation (5.1). Thus, the two moves possible on the dual tree correspond to the two possible splits of our classical interval exchange.

It is easy to check that the expansion stops in finite time if and only if the slope is rational. Thus an irrational slope gives an infinite splitting sequence. If we write down the infinite sequence as an infinite word $R^{a_1} L^{a_2} \cdots$, for positive integers a_n, then it is easy to check that the a_n are exactly the coefficients in the continued fraction expansion of the slope. Thus we have encoded the infinite sample paths $R^{a_1} L^{a_2} \cdots$ for the nearest neighbor non-backtracking random walk on the trivalent tree as splitting sequences of the interval exchange.

The main point is

Proposition 5.8. *In every infinite expansion $R^{a_1} L^{a_2} \cdots$, every stage at which one switches between the letters R and L or vice versa, is a 2-distributed stage i.e., every stage of the form $R^{a_1} \cdots R^{a_{2k-1}} L$ or $R^{a_1} \cdots L^{a_{2k}} R$ is 2-distributed. In particular, almost every expansion becomes 2-distributed infinitely often.*

The proof of the proposition is left to the reader.

5.9. **The singular set for $PSL(2,\mathbb{Z})$.** Let m be an odd integer. Consider the sets

$$X_{m,n} = \{\mathbf{x} \in (0,1) : a_m(\mathbf{x}) > n\}.$$

By direct computation, the Lebesgue measure of $X_{1,n}$ is

$$\ell(X_{1,n}) = \frac{1}{n+1}.$$

By Proposition 5.8, stages of the form $R^{a_1} \cdots L^{a_{2k}} R$ are 2-distributed. The expansions in which stages of this form occur finitely many times, must terminate in finite time, and hence are a subset of the rational numbers. This means that a stage of the form $R^{a_1} \cdots L^{a_{2k}} R$ occurs in almost every expansion. By Lemma 5.5, the relative probability that such a stage is followed by the sequence R^n, is up to a uniform constant the same as the probability that an expansion begins with R^n. Thus, for odd m,

$$\ell(X_{m,n}) \approx \ell(X_{1,n-1}). \tag{5.10}$$

Consider the set

$$X = \limsup_{n \text{ odd}} X_{n,n} = \bigcap_{k=1}^{\infty} \bigcup_{n \geqslant 2k-1} X_{n,n}$$

i.e., X is the set of elements $\mathbf{x} \in (0,1)$ such that $\mathbf{x} \in X_{n,n}$ for infinitely many odd $n \in \mathbb{N}$. The restriction that n be odd is not essential. To keep the explanations simple, we want the 2-distributed stages considered, to have the form $R^{a_1} L^{a_2} \cdots L^{a_{m-1}} R$, and not $R^{a_1} \cdots R^{a_{2k-1}} L$.

Let h be the harmonic measure on $(0,1)$ for the nearest neighbor non-backtracking random walk on the trivalent tree. Since there is no backtracking involved, every infinite sample path in the trivalent tree is convergent. Moreover, almost every infinite sample path necessarily passes through some vertex of the form $R^{a_1} L^{a_2} \cdots L^{a_{n-1}} R(\mathbf{b})$ exactly once. So to compute the proportion of sample paths that converge into $X_{n,n}$ we can condition on the set of vertices represented by the initial sample paths $R^{a_1} L^{a_2} \cdots L^{a_{n-1}} R$. From each such vertex, the probability of converging into $X_{n,n}$ is exactly $1/2^n$. Hence $h(X_{n,n}) = 1/2^n$

Proposition 5.11.

$$\ell(X) > 0 \quad , \quad h(X) = 0.$$

Proof. By Equation (5.10), $\ell(X_{n,n}) \approx \ell(X_{1,n-1}) = 1/n$. The sets $X_{n,n}$ are not independent for the Lebesgue measure, but as we shall see in Claim 5.13, they are pairwise almost independent (4.2). For the harmonic measure, the sets $X_{n,n}$ simply satisfy

$$\sum h(X_{n,n}) < \infty. \tag{5.12}$$

The proposition then follows directly from the quasi-independent Borel-Cantelli Lemma 4.1.

Claim 5.13. *There exists some constant $c > 1$ such that for any n, m odd and $n \neq m$*

$$\ell\big(X_{m,m} \cap X_{n,n}\big) < c \cdot \ell\big(X_{m,m}\big)\ell\big(X_{n,n}\big).$$

Proof. Let $U = X_{m,m} \cap X_{n,n}$ and $V = X_{1,m-1} \cap X_{n-m+1,n}$. For each set of positive integers $S = \{a_1, \cdots, a_{m-1}\}$, let $W(S)$ be the set of points whose expansion begins with the sequence $R^{a_1}L^{a_2}\cdots L^{a_{m-1}}R$. By Proposition 5.8, the sets $W(S)$ partition a set of full measure. So it is enough to show that for all S

$$\ell(W(S) \cap U) < c \cdot \ell(W(S) \cap X_{m,m})\ell(W(S) \cap X_{n,n}).$$

Taking union over all $W(S)$ gives us the claim. Because $R^{a_1}L^{a_2}\cdots L^{a_{m-1}}R$ is a 2-distributed stage, by Equation (5.10), up to a uniform constant, the relative probabilities satisfy

$$\begin{aligned}
\mathrm{Prob}(X_{m,m}|W(S)) &\approx \ell(X_{1,m-1}).\\
\mathrm{Prob}(X_{n,n}|W(S)) &\approx \ell(X_{n-m+1,n}).\\
\mathrm{Prob}(U|W(S)) &\approx \ell(V).
\end{aligned}$$

Thus it is enough to prove that there exists a constant $c > 1$ such that for any n, m odd and $n > m$

$$\ell(X_{1,m-1} \cap X_{n-m+1,n}) < c \cdot \ell(X_{1,m-1})\ell(X_{n-m+1,n}). \tag{5.14}$$

In fact, we shall show that there exists a constant $c > 1$ such that for any positive integers $m, k > 1$ and any odd integer n

$$\ell(X_{1,m} \cap X_{n,k}) < c \cdot \ell(X_{1,m})\ell(X_{n,k}). \tag{5.15}$$

Replacing m by $(m-1)$, n by $(n-m+1)$ and k by n in the above inequality implies Inequality (5.14). The proof of Inequality (5.15) is as follows: For each set of positive integers $S' = \{a_1, \cdots, a_{n-1}\}$, let $W(S')$ be the set of points whose expansion begins with the sequence $R^{a_1}L^{a_2}\cdots L^{a_{n-1}}R$. By Proposition 5.8, the stages $R^{a_1}L^{a_2}\cdots L^{a_{n-1}}R$ are 2-distributed and the sets $W(S')$ partition a set of full measure. Moreover each such stage $W(S')$ either belongs entirely to $X_{1,m}$ or belongs entirely to the complement of $X_{1,m}$ depending on whether $a_1 > m$ or not. Because of 2-distribution, we have $\mathrm{Prob}(X_{n,k}|W(S')) \approx \ell(X_{1,k-1})$. So

$$\begin{aligned}
\ell(X_{1,m} \cap X_{n,k}) &\approx \sum_{W(S')\subset X_{1,m}} \mathrm{Prob}(X_{n,k}|W(S'))\ell(W(S'))\\
&\approx \ell(X_{1,k-1}) \sum_{W(S')\subset X_{1,m}} \ell(W(S'))\\
&\approx \ell(X_{1,k-1})\ell(X_{1,m}).
\end{aligned}$$

Finally, by Equation (5.10), we have $\ell(X_{1,k-1}) \approx \ell(X_{n,k})$. Using this in the above equation finishes the proof of Inequality (5.15), and hence of the claim. □

□

Finally to get a singular set of full measure, we consider the union

$$Y = \bigcup_{g\in PSL(2,\mathbb{Z})} gX.$$

The set Y is a countable union of translates of X, so $h(Y) = 0$. On the other hand, Y is a set invariant under the action of $PSL(2,\mathbb{Z})$. By the ergodicity of the action of $PSL(2,\mathbb{Z})$ on $\partial\mathbb{H}$, invariant sets have zero or full measure. Hence $\ell(Y) = 1$ which proves that ℓ and h are mutually singular.

5.16. **General random walks on** $PSL(2,\mathbb{Z})$**.** The appropriate notion of a non-elementary subgroup of $PSL(2,\mathbb{Z})$ is that the subgroup contains a pair of hyperbolic isometries h_1 and h_2 of $\mathbb{H}$ with distinct attracting and repelling fixed points in $\partial\mathbb{H}$. By conjugating by a rotation of $\mathbb{H}$, we can assume that the attracting fixed point of h_1 is in $(0,1)$. For such a finitely supported initial distribution, the same construction produces a singular set provided Equation 5.12 still holds. So it is enough to show that for large n, the harmonic measure of $X_{n,n}$ decays uniformly exponentially i.e., there exists a constant $r<1$ and a positive integer N such that $h(X_{n,n}) < r^n$ for all $n > N$.

Since the group $PSL(2,\mathbb{Z})$ is quasi-isometric to the trivalent tree the random walk can be projected onto the trivalent tree by using the quasi-isometry. We assume that the quasi-isometry maps the identity $1 \in PSL(2,\mathbb{Z})$ to the base-point $\mathbf{b}$, so that the projected random walk starts from $\mathbf{b}$. Because of the quasi-isometry, the random walk on $PSL(2,\mathbb{Z})$ satisfies estimates similar to those satisfied by the projected random walk. The actual constants in the estimates for $PSL(2,\mathbb{Z})$ also depend on the quasi-isometry constants, but that does not affect the construction. Hence, it is enough to prove an exponential decay estimate holds for the projected random walk on the trivalent tree.

These estimates are established in [5]. For completeness, we present it as a consequence of a nesting argument by adapting the proof of Lemma 5.4 of [21]. For a vertex v distinct from $\mathbf{b}$, let C_v be the "cone" of vertices in the tree such that the unique geodesic connecting them to $\mathbf{b}$ passes through v. Similar to Proposition 5.3 of [21], one shows that there is a definite positive integer K and a constant $\epsilon > 0$ that depend only on the initial distribution, such that if $d(\mathbf{b}, v) = K$, then $h(\overline{C_v}) \leqslant 1-\epsilon$. To prove that $h(\overline{C_v})$ can be bounded away from 1, we construct a pair of Schottky cones for suitable powers of h_1 and h_2, similar to what Maher does at the beginning of Section 5 of [21]. The bound for $h(\overline{C_v})$ then follows by exploiting the δ-hyperbolicity of the tree. Finally, the set $X_{n,n}$ is the disjoint union of limit sets of cones for vertices of the form $v = R^{a_1}L^{a_2}\cdots L^{a_{n-1}}R^n$. So by using the estimate for the cones inductively we get the exponential decay for $h(X_{n,n})$.

The fact that the constants in the harmonic measure estimates are uniform over all n follows directly from the Markov property of the random walk. Hence, the issue of whether expansions that are generic in the harmonic measure get 2-distributed infinitely often, does not come up here.

5.17. **The general mapping class groups.** The element R as a split of the classical interval exchange on the torus in Fig 4, can be seen to be a Dehn twist in the vertex curve given by band 1. This vertex curve is the curve with slope 0 in the Farey graph of Figure 1, and R acts on $\partial\mathbb{H}$ as a parabolic element with fixed point 0.

For the general mapping class groups, the action of the reducible elements on $\mathscr{PMF}$, in particular Dehn twists, is similar to the action of R on $\partial\mathbb{H}$. We show that a suitable "half-open" neighborhood in $\mathscr{PMF}$ of the simple closed curve fixed by a Dehn twist scales down at the rate $1/\mathrm{poly}(n)$ for the Lebesgue measure and $\exp(-n)$ for the harmonic measure, under the repeated application of that Dehn twist. So the construction of the singular set is essentially similar. However, because of the issues outlined below, it becomes technically harder to construct the sequence of sets $Y_n^{(m)}$ to which Proposition 4.4 applies, and trickier to prove the above mentioned measure estimates for these sets.

First, projective classes of measured foliations carried by a complete train track can be encoded by splitting sequences of that track. However, this involves choosing a large edge to split at each stage of the expansion, which means that this coding is not unique. More importantly, the corresponding Rauzy diagram i.e., the directed graph with nodes the combinatorial types of complete train tracks and arrows given by splits, is not well understood. As a result, we do not know the analog of Proposition 5.8 in this general setup, and such an analog is essential (Lemma 5.5) to give *uniform* estimates in the Lebesgue measure necessary to apply Proposition 4.4. To circumvent these difficulties, we consider complete non-classical interval exchanges instead viz. complete train tracks with a single switch. Here, by the work of Boissy and Lanneau [2], the Rauzy diagram is better understood. In particular, they characterize the attractors of this directed graph (alternatively known as the Rauzy classes) in terms of suitable irreducibility criteria for the combinatorics of the interval exchanges. This allows us to construct explicit initial combinatorics for complete non-classical interval exchanges that belong to some attractor. The Rauzy induction generalizes directly to non-classical interval exchanges. Unlike the situation for general complete tracks, the expansions by repeated Rauzy induction uniquely encode the projective classes of measured foliations that are carried. Secondly, most of the techniques used to study expansions of classical interval exchanges, generalize to non-classical interval exchanges. In particular, in [10], we proved an analog of Proposition 5.8 for non-classical interval exchanges, namely Theorem 6.9.

The second issue is that for the general mapping class groups, the appropriate quasi-isometric models are the marking complexes. Unlike $PSL(2,\mathbb{Z})$ which is δ-hyperbolic, the geometry of the marking complex is intricate. Instead, we project the mapping class group random walk to the curve complex. The combination of the results of Kaimanovich-Masur and Klarreich shows that one does not lose any stochastic information about the random walk in the process. There are a number of advantages in considering the curve complex: foremost is that the curve complex is δ-hyperbolic; the coarse negative curvature proves to be very useful in analyzing the projected random walk. Secondly, the set of essential simple closed curves carried by a non-classical interval exchange is a quasi-convex set in the curve complex and it makes sense to consider "deeply carried" annuli. As we shall see, the techniques of efficient position [28] provide information about sub-surface projections to deeply carried annuli. This information enables us to apply the key tool, Theorem 11.6 of Maher, to get exponential decay estimates for the harmonic measure. Because of the Markov property of random walks, the harmonic measure estimates are uniform with constants that depend only on S and the initial distribution μ on G. Hence, the issue of whether expansions that are generic for the harmonic measure satisfy Theorem 6.9 does not come up.

6. Complete non-classical interval exchanges.

6.1. **Non-classical interval exchanges.** There is an abstract train track underlying our pictorial description of a classical interval exchange. In our picture, each band has a central branch which joins the midpoint of the subinterval of I_+ to the midpoint of the corresponding subinterval of I_-. Retract each band to its central branch and then horizontally retract $I_\pm$ to a vertical branch. This can be done in a way such that the branches associated to the bands share a vertical line of tangency on each side of the vertical branch. Finally, retract the vertical branch to a point while preserving the vertical tangency. The result is a train track with a single switch. The branches are in bijection with the bands and every outgoing branch on one side of the switch is incoming from the other. If we assign the width of the band as the weight on the corresponding branch then the weights satisfy the single switch condition.

The first step towards defining non-classical interval exchanges is to relax the constraint that every outgoing branch from one side of the switch is incoming from the other, i.e. to allow bands from I_+ to I_+ and I_- to I_-. We call such bands *orientation reversing* because the flow along such a band reverses the orientation of a subinterval of I_+ or I_-.

Definition 6.2. A *non-classical interval exchange* is the pair of intervals I_+ and I_- with bands such that there are orientation reversing bands on both I_+ and I_- i.e., the underlying train track is recurrent. The transformation $T: I_+ \sqcup I_- \to I_+ \sqcup I_-$ defined by it is the following composition: Except for the endpoints of the subintervals, every $x \in I_+ \sqcup I_-$ lies in exactly one band. Flow x along this band to its other end to get a point x'. Set $T(x)$ to be $\epsilon(x')$. It follows that the Lebesgue measure on $I_+ \sqcup I_-$ is invariant under T.

To relate this to Definition 2.1 in [2], the map $\tilde{T}$ in their notation is exactly the map given by the flow along the bands. The requirement that there are orientation reversing bands on I_+ and I_- is equivalent in their definition, to imposing that there are subintervals in I_+ and I_- that under $\tilde{T}$ are sent to the same I_+ and I_- respectively. As defined in [2], the labeling of the bands by $\mathscr{A}$ can be thought of as given by *a generalized permutation* ϕ: a 2-1 map from $\{1,\ldots,2d\}$ to $\mathscr{A}$. Thus, $\phi^{-1}\alpha$ gives the two ends of the band labelled α. The permutation is of type (l,m) where $l+m=2d$ if $\{1,\ldots,l\}$ enumerates the subintervals of I_+ from left to right and $(l+1,\ldots,l+m=2d\}$ enumerates the subintervals of I_- from left to right. A generalized permutation defines a fixed point free involution σ of $\{1,\dots,2d\}$ by:

$$\phi^{-1}(\phi(i)) = \{i,\sigma(i)\}.$$

Our definition implies that there is a positive integer i with $i,\sigma(i) \leqslant l$ and a positive integer j with $l+1 \leqslant j,\sigma(j)$. The equivalence classes under σ are indexed by the elements of $\mathscr{A}$ and correspond to the bands.

For reasons mentioned at the end of Section 5, we later restrict to *complete non-classical interval exchanges*: generalized permutations whose underlying train tracks admit embeddings as complete train tracks on an orientable surface S.

6.3. **Parameter space of a non-classical interval exchange.** For all non-classical interval exchanges sharing the same generalized permutation ϕ, the normalization $\sum \lambda_\alpha = 1$ restricts us to the standard simplex $\Delta \subset \mathbb{R}^{\mathscr{A}}$. The other constraint that the widths (λ_α) satisfy is the switch condition imposed by ϕ. Let S_+ and S_- be the labels of orientation reversing bands on I_+ and I_- respectively. The switch condition is equivalent to

$$\sum_{\alpha\in S_+} \lambda_\alpha = \sum_{\alpha\in S_-} \lambda_\alpha.$$

Let W_ϕ denote the subset of Δ cut out by the codimension 1 subspace of $\mathbb{R}^{\mathscr{A}}$ defined above. For $\alpha \in S_+$ and $\beta \in S_-$, let $e_{\alpha\beta}$ be the midpoint of the edge $[e_\alpha, e_\beta]$ of Δ joining the vertices e_α and e_β. Then W_ϕ is the convex hull of the points $e_{\alpha\beta}$ and e_γ for $\gamma \notin S_+ \cup S_-$. The vertex cycles of the train track underlying ϕ are in bijection with the extremal points of W_ϕ.

There are finitely many choices for the generalized permutations ϕ or alternatively for the pair (S_+, S_-) of disjoint subsets in $\mathscr{A}$. Hence, there is a finite set of convex codimension 1 subsets of Δ that arise this way. We call the different subsets of Δ coming from all possible pairs (S_+, S_-), *configuration spaces*, and denote the probability measure they inherit from the natural volume form on them by ℓ. When ϕ gives complete non-classical interval exchanges, W_ϕ defines a chart on $\mathscr{PMF}$ and ℓ is a Lebesgue measure in this chart. For the rest of this section, the measure referred to will be ℓ.

6.4. **Rauzy induction.** As before, we shall call the positions that are rightmost on the intervals I_+ and I_-, the *critical* positions. Similar to classical interval exchanges, Rauzy induction is defined for non-classical interval exchanges by comparing widths of the bands in critical positions on I_+ and I_-, and then splitting the broader band by the narrower one till we hit $I_+ \sqcup I_-$ again. Rauzy induction for non-classical interval exchanges preserves recurrence of the underlying train track. Splitting a non-classical interval exchange in this manner gives a new non-classical interval exchange with widths given exactly as in Equation (5.3).

For any choice of initial widths, repeated Rauzy induction associates a unique expansion to it. The induction is not defined when the widths of the critical bands match up. For geometrically irreducible generalized permutations defined in [2] and discussed below, this is always a co-dimension 1 condition and has measure zero in the associated configuration space. The set of foliations for which the expansion stops after finite steps is a countable union of such measure zero sets, and hence of measure zero. Thus a full measure set of measured foliations in the configuration space, has infinite expansion.

6.5. **Irreducibility and the Rauzy diagram.** Similar to the Rauzy diagram for classical interval exchanges, we can construct an oriented graph $\mathscr{G}$ for non-classical interval exchanges: the nodes of the graph are generalized permutations ϕ with an arrow from ϕ to ϕ', if ϕ' results from splitting ϕ. For each node ϕ, there are at most two arrows coming out of it. A splitting sequence gives us a directed path in $\mathscr{G}$.

However, there are some key differences in the non-classical case. Unlike classical interval exchanges, there can be transient nodes in the Rauzy diagram for non-classical interval exchanges: a node with a finite splitting sequence after which it is impossible to return to the node. In [2], Boissy and Lanneau formulate the notion of *geometric irreducibility* of generalized permutations and show that the sub-graph of $\mathscr{G}$ of geometrically irreducible generalized permutations is an attractor: any pair of nodes can be connected by a directed path. This fact is crucial in proving Theorem 6.9. See [10].

6.6. **Dynamics on the parameter space.** Assume that the initial generalized permutation ϕ_0 is in an attractor of $\mathscr{G}$. For brevity of notation, let W_0 be the configuration space at ϕ_0. Let $\mathbf{x} = (\lambda_\alpha) \in W_0$ be a non-classical interval exchange. Identical to classical interval exchanges, each step of Rauzy induction is encoded as an elementary matrix E. The expansion of $\mathbf{x}$ by repeated Rauzy induction gives a directed path starting from ϕ_0 in $\mathscr{G}$. Iteratively, to each stage $\phi_0 \to \phi_{\mathbf{x},1} \cdots \to \phi_{\mathbf{x},n}$ in the expansion of $\mathbf{x}$, there is an associated matrix $Q_{\mathbf{x},n}$ such that $Q_{\mathbf{x},n} = Q_{\mathbf{x},n-1}E$, where E is the elementary matrix for the split $\phi_{\mathbf{x},n-1} \to \phi_{\mathbf{x},n}$. Let W_n be the configuration space at $\phi_{\mathbf{x},n}$. One also gets a sequence of points $\mathbf{x}^{(n)} \in W_n$ such that $\mathbb{P}Q_{\mathbf{x},n}(\mathbf{x}^{(n)}) = \mathbf{x}$, where $\mathbb{P}Q_{\mathbf{x},n}$ is the projectivization of $Q_{\mathbf{x},n}$ as defined in Section 5.

Suppose $\jmath : \phi_0 \to \cdots \to \phi$ is a finite splitting sequence with associated matrix $Q_\jmath$. The set of $\mathbf{x}$ in W_0 whose expansion begins with $\jmath$ is $\mathbb{P}Q_\jmath(W)$, where W is the configuration space at ϕ. We call $\jmath$ a *stage* in the expansion.

As in the case of $PSL(2,\mathbb{Z})$, we have to understand the relative probability that a stage $\jmath$ is followed by a particular splitting sequence. The difference here is that we have to consider the Jacobian of the restriction of the projective linear map to the configuration spaces i.e., the Jacobian of $\mathbb{P}Q_\jmath : W \to W_0$. We denote this Jacobian by $\mathscr{J}(\mathbb{P}Q_\jmath)$. Because of this, the probability that a particular split follows ϕ can be quantitatively very different from the naive estimate using the full Jacobian $\mathscr{J}_\Delta$ of $\mathbb{P}Q_\jmath : \Delta \to \Delta$, which was used in the classical case. See Section 6 in [10]. Nevertheless, in [10], we prove an analog of Proposition 5.8 for non-classical interval exchanges. We state this below.

For a constant $C > 1$, we define a C-uniformly distorted stage to be a stage $\jmath$ such that for any pair of distinct points $\mathbf{y}, \mathbf{y}'$ in W,

$$\frac{1}{C} \leqslant \frac{\mathscr{J}(\mathbb{P}Q_\jmath)(\mathbf{y})}{\mathscr{J}(\mathbb{P}Q_\jmath)(\mathbf{y}')} \leqslant C.$$

Let κ be a finite splitting sequence $\phi \to \cdots \to \phi'$. Let W' denote the configuration space at ϕ', and let Q_κ be the matrix associated to κ. Then, $\mathbb{P}Q_\kappa(W') \subset W$. Exactly as Lemma 5.5, we have

Lemma 6.7. *If the stage $\jmath$ is C-uniformly distorted, then there exists a constant $c > 1$ that depends only on C and d, such that the relative probability that κ follows $\jmath$ satisfies*

$$\frac{1}{c}\ell(\mathbb{P}Q_\kappa(W_0)) < \frac{\ell(\mathbb{P}Q_{\jmath*\kappa}(W))}{\ell(\mathbb{P}Q_\jmath(W_0))} < c \cdot \ell(\mathbb{P}Q_\kappa(W)). \tag{6.8}$$

Thus, Lemma 6.7 gives us the same uniform control as Lemma 5.5 for estimating relative probabilities.

It is straightforward to check that if the matrix $Q_\jmath$ is C-distributed, then the stage $\jmath$ is C-uniformly distorted. The converse need not be true. However, it means that in the analog of Proposition 5.8, it suffices to show that starting from any stage, almost every expansion gets C-distributed. The precise statement proved as Theorem 1.3 in [10], is:

Theorem 6.9. *Suppose $\jmath : \phi_0 \to \ldots \to \phi$ is a stage in the expansion, with W the configuration space at ϕ and $Q_\jmath$ the associated matrix. There exists a constant $C > 1$, independent of $\jmath$, such that for almost every $\mathbf{x} \in \mathbb{P}Q_\jmath(W)$, there is a future stage $\phi_{\mathbf{x},m}$ after ϕ, such that the stage $\phi_{\mathbf{x},m}$ is C-distributed. Additionally, by choosing C large enough, we can assume that $\phi_{\mathbf{x},m}$ is combinatorially the same as ϕ_0 i.e., as generalized permutations $\phi_{\mathbf{x},m} = \phi_0$.*

From here on, we will restrict to complete non-classical interval exchanges.

7. Initial combinatorics.

When a splitting sequence of a generalized permutation giving complete non-classical interval exchanges returns to the same permutation, it corresponds to the action of a mapping class on S.

For $PSL(2,\mathbb{Z})$, the Dehn twist R was realized as a split of the interval exchange in Figure 4. Here, we show that it is possible to emulate this phenomena for any surface i.e., given a surface S with genus g and m punctures, we construct a generalized permutation giving complete non-classical interval exchanges on S with the property that a Dehn twist in one of its vertex cycles is realized by a splitting sequence. This is possible when the vertex cycle has a combing. We call this splitting sequence, the *Dehn twist* sequence.

7.1. **The construction.** The complementary regions to a complete train track on a surface of genus g and m punctures must consist of $(4g-4+m)$ ideal triangles and n once punctured monogons. If this exact number of ideal triangles and once punctured monogons can be glued in along the boundary of the pair of intervals with bands then the resulting surface is necessarily S and the associated generalized permutation gives complete non-classical interval exchanges on S. The maximality and recurrence of the underlying train track are immediate. Transverse recurrence follows from an easy application of Corollary 1.3.5 in [30].

We break down our construction into several cases. Throughout, the generalized permutation ϕ_0 that we construct contains a single orientation preserving band whose label in $\mathscr{A}$ we will denote by B, whose one end will be leftmost on I_+ and whose other end will be in the critical position rightmost on I_-. All other bands are orientation reversing. This means that if we split the end of B in the critical position on I_- twice by all bands on I_+, then we get back to the same generalized permutation ϕ_0. The associated mapping class is easily seen to be a Dehn twist in the simple closed curve on S which is a vertex cycle of ϕ_0 obtained by closing the spine of B with a segment that lies along a base interval. In the language of [28], if we orient this vertex cycle so that B is traversed from left to right, then both sides of the vertex cycle are *combed to the left.* Thus, we have a Dehn twist sequence of a complete non-classical interval exchange. The other details of the construction differ according to the case. The cases exclude some surfaces of low complexity, the combinatorics for which we shall construct separately at the end of the section.

Next, we describe the essential pictures we need. Only the spine is drawn in each figure.

7.1.1. *The basic block.* Consider a picture of a horizontal interval I_+ with orientation reversing bands whose spines are shown in Figure 5.

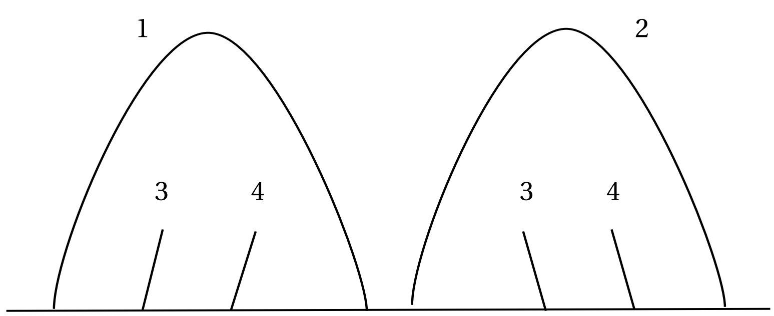

FIGURE 5. Basic Block

The right end of band 1 and the left end of band 2 are adjacent along I_+. The pair of segments marked 3 and 4 are ends of bands labeled 3 and 4 respectively. Only the ends are shown to keep the figure uncluttered. It is possible to glue two ideal triangles as complementary regions. The first triangle has sides made up of bands 1, 3 and 4 such that while traversing band 1 from left to right along I_+ the triangle is to the right (using the standard orientation of the plane in Figure 5). The second triangle is similar with sides made up of bands 2, 3 and 4. In other words, these triangles are on the "inside" of bands 1 and 2.

7.1.2. *The outer blocks.* We call the pictures in Figures 6 and 7 as *outer block 1* and *outer block 2* respectively. We shall use one or the other in the construction depending on the case in question.

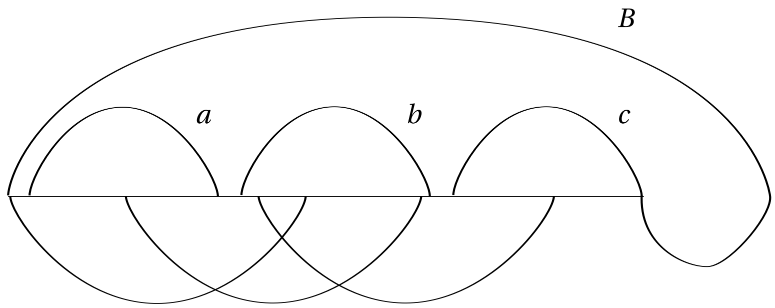

FIGURE 6. Outer Block 1

In outer block 1, we can identify 3 ideal triangles in the complement. The bands on I_- cut out two ideal triangles: the first contains the cusp between the bands leftmost on I_- and the second contains the cusp between the bands rightmost on I_-. Additionally, the bands a, b, c and B bound an ideal triangle.

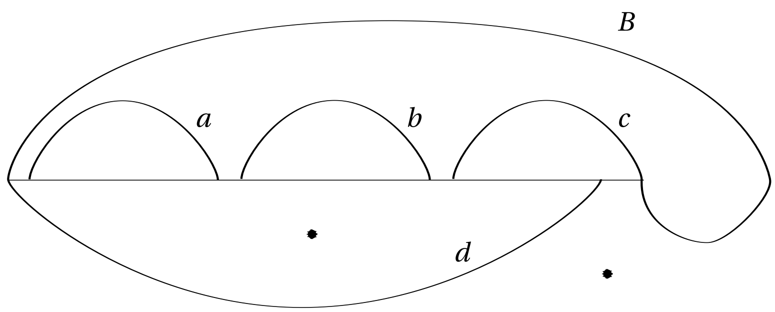

FIGURE 7. Outer Block 2

In outer block 2, it is necessary to have one puncture inside and one puncture outside band d as shown in Figure 7 to have those complementary regions as once punctured monogons.

Now we get to the various cases in the construction.

7.1.3. *Case 1:* $m = 0, g \geqslant 4$. Construct first the base intervals $I_\pm$ along with outer block 1. In a separate picture, concatenate $(g-1)$ copies of the basic block and rescale to get I_+. See Figure 8.

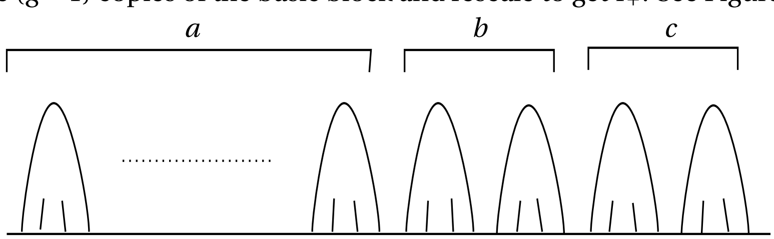

FIGURE 8. The stuffing

Superimpose Figure 8 over the outer block such that the basic blocks under the bracket marked a lie inside the orientation reversing band a in the outer block, and similarly for b and c.

Finally, notice that apart from the region which is on the inside of band a and outside the basic blocks inserted inside a, all the complementary regions are ideal triangles. The exceptional region is an ideal polygon with $2(g-3)+1$ cusps. This can be sub-divided into $2(g-3)-1$ ideal triangles by adding in bands a_i for $i=1,\cdots,2(g-3)-2$ as shown in Figure 9

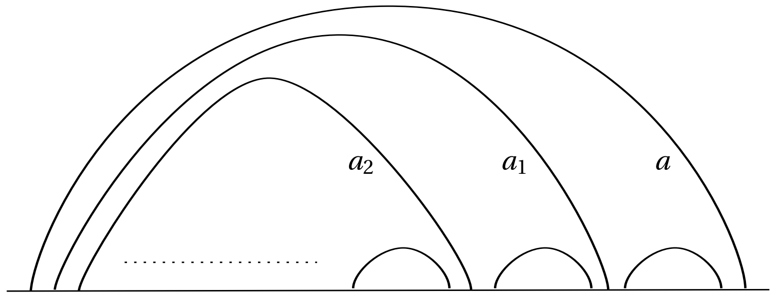

FIGURE 9. Dividing the polygon into ideal triangles

We claim that the resulting generalized permutation gives a complete non-classical interval exchange with a Dehn twist sequence, on a surface of genus g, with $g \geqslant 4$. First, it is easy to see that all complementary regions are ideal triangles. Second, the Dehn twist sequence is given by splitting B twice by all the bands on I_+. So it is enough to show that there are $(4g-4)$ ideal triangles. The number of ideal triangles inside the band a is: $2(g-3)$ ideal triangles coming from $(g-3)$ basic blocks inside a, and $2(g-3)-1$ ideal triangles coming from adding in bands a_i, so a total of $(4g-13)$ ideal triangles. The total number of ideal triangles inside bands b and c is 3 each. So the total number of ideal triangles is $(4g-13)+2(3)+(3)=4g-4$.

7.1.4. *Case 2:* $m=1, g \geqslant 3$. In outer block 1, we let the single puncture sit inside band c. Similar to Figure 8, in a separate picture, concatenate $(g-1)$ basic blocks and rescale to get I_+. Superimpose this picture over outer block 1 such that the last basic block on the right is inside band b and the rest of the basic blocks are inside band a. The region inside band a and outside the bands inserted inside a, is an ideal polygon with $2(g-2)+1$ cusps. Divide this ideal polygon into $2(g-2)-1$ ideal triangles by adding bands a_i for $i=1,\cdots,2(g-2)-2$, as in Figure 9.

The resulting generalized permutation gives a complete non-classical interval exchange with a Dehn twist sequence, on a surface with genus g with $g \geqslant 3$ and a single puncture. Except the inside of band c which is a once-punctured monogon, all complementary regions are ideal triangles. The Dehn twist sequence is similar to the previous case. A counting argument similar to the previous case shows that the complementary region consists of $(4g-3)$ ideal triangles and a single once punctured monogon.

7.1.5. *Case 3:* $m=2, g \geqslant 2$. In outer block 1, let the two punctures sit inside bands b and c. Superimpose a concatenation of $(g-1)$ basic blocks over outer block 1 such that all the basic blocks lie inside band a. The region inside band a and outside the basic blocks inserted inside a is an ideal polygon with $2(g-1)+1$ cusps. Divide this ideal polygon into $2(g-1)-1$ ideal triangles by adding bands a_i for $i=1,\cdots 2(g-1)-2$, as in Figure 9. The counting argument shows that the complementary regions consist of $(4g-2)$ ideal triangles and two once-punctured monogons. So the resulting generalized permutation gives a complete non-classical interval exchange with a Dehn twist sequence on a surface of genus g with $g \geqslant 2$ and two punctures.

7.1.6. *Case 4:* $m \geqslant 3, g \geqslant 1$. In outer block 1, let two of the punctures sit inside bands b and c. Concatenate $(g-1)$ basic blocks on the left with $(m-2)$ orientation reversing bands each containing a single puncture inside along I_+. Superimpose this figure over outer block 1 such that all the bands lie inside band a. The region inside band a and outside the bands inserted inside a, is an ideal polygon with $2(g-1)+(m-2)+1$ cusps. Divide this ideal polygon into $2(g-2)+(m-2)-1$ ideal triangles by adding bands a_i for $i = 1, \cdots, 2(g-2)+(m-2)-2$, as in Figure 9. The counting argument shows that the complementary regions consist of $(4g-4+m)$ ideal triangles and m once punctured monogons. So the resulting generalized permutation gives a complete non-classical interval exchange with a Dehn twist sequence on a surface of genus g with $g \geqslant 1$ and $m \geqslant 3$ punctures.

7.1.7. *Case 5: $m \geqslant 5, g = 0$.* In outer block 2, two of the punctures are already accounted for. Let two other punctures sit inside bands b and c. Concatenate $(m-4)$ orientation reversing bands each containing a single puncture along I_+. Superimpose this figure over outer block 2 such that all the bands lie inside band a. The region inside band a and outside the inserted bands is an ideal polygon with $(m-4)+1$ cusps. Divide this ideal polygon into $(m-4)-1$ ideal triangles by adding bands a_i for $i = 1, \cdots, (m-4)-2$, as in Figure 9. The counting argument shows that the complementary regions consist of $(m-4)$ ideal triangles and m once punctured monogons. So the resulting generalized permutation gives a complete non-classical interval exchange with a Dehn twist sequence on a sphere with m punctures where $m \geqslant 5$.

7.1.8. *The remaining low complexity surfaces.* The cases above exclude some low complexity examples. In each of these, we directly draw the combinatorics, and leave the details to the reader.

(1) *$m = 0, g = 2$*: Genus 2 in Figure 10.
(2) *$m = 0, g = 3$*: Genus 3 in Figure 11. Here the two instances of the same alphabet represent the two ends of the same band.
(3) *$m = 1, g = 2$*: Once-puncture genus 2 in Figure 12.
(4) *$m = 1, g = 1$*: In the once punctured torus case, the interval exchange is the same as the classical interval exchange with 2 bands that we considered for the torus.
(5) *$m = 2, g = 1$*: Twice punctured torus in Figure 13.
(6) *$m = 4, g = 0$*: 4-times punctured sphere in Figure 14.

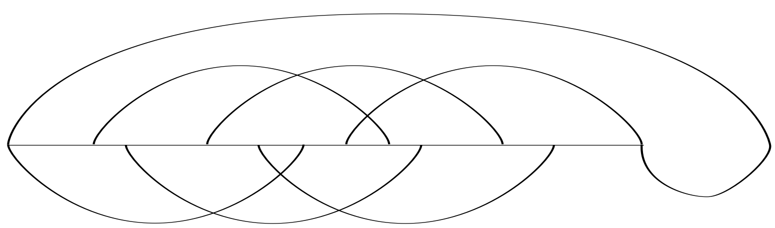

FIGURE 10. Genus 2

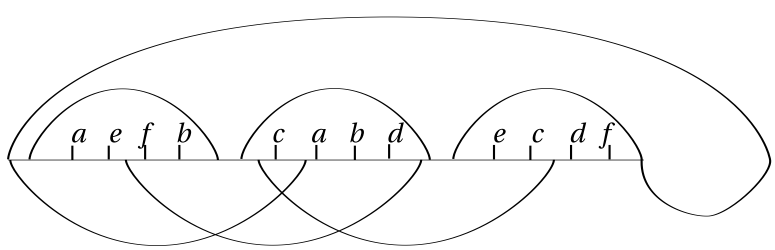

FIGURE 11. Genus 3

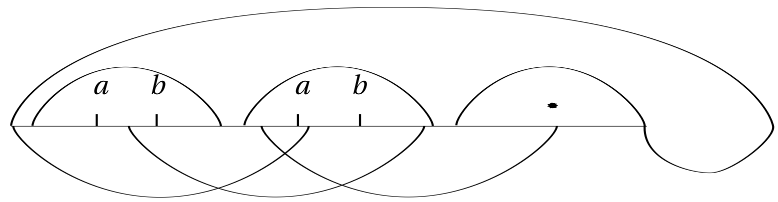

FIGURE 12. Once-punctured Genus 2

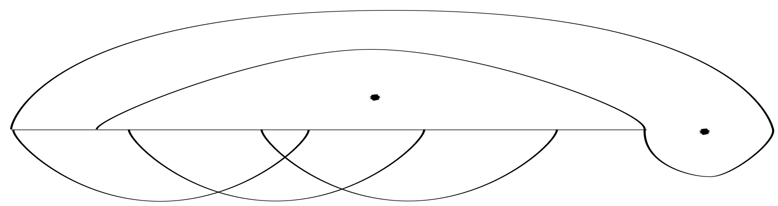

FIGURE 13. Twice-punctured Genus 1

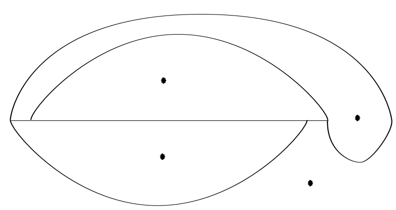

FIGURE 14. 4 punctured sphere

7.2. **Geometric Irreducibility.** Following Definition 3.1 of Boissy and Lanneau [2], it is easy to check directly in each case that the generalized permutation ϕ_0 constructed is geometrically irreducible. So by the results of [2], ϕ_0 lies in some attractor of the Rauzy diagram. Since the attractor is connected in the directed sense, the generalized permutations at any stage of the expansion, lie in the same attractor. Henceforth, instead of the full Rauzy diagram, we focus on the attractor containing ϕ_0, and call it $\mathscr{G}$.

7.3. **Remarks about ϕ_0.** In all cases, we shall call the vertex cycle about which we have a Dehn twist sequence the *stable vertex cycle* and denote it by v. We shall denote the Dehn twist sequence by $\jmath_0$. Under any embedding of ϕ_0 into the surface, the stable vertex cycle v gives a separating curve on the surface. Secondly, suppose we fix an orientation on v such that B is traversed from left to right. Let γ be any simple closed curve carried by the underlying track. Notice that γ can twist only to the left along v i.e., in the language of [28], the underlying track has a *left combing* on either side of v. Moreover, the Dehn twist sequence increases the twist to the left by 1. This property has important consequences for the sub-surface projection to the annulus with core curve v, as we shall see later.

7.4. **The Lebesgue measure estimate for iterations of the Dehn twist sequence.** Let W_0 denote the configuration space for ϕ_0. Thought of as a directed path in $\mathscr{G}$, the Dehn twist sequence $\jmath_0$ returns us to the same vertex ϕ_0. The matrix Q_0 associated to $\jmath_0$ has the effect that it adds the column e_B twice to all columns e_α associated to bands α on I_+. Let $n\jmath_0$ denote the splitting sequence $\jmath_0 * \jmath_0 * \cdots * \jmath_0$ i.e., the sequence given by iterating $\jmath_0$, n successive times. Let Q_n be the matrix associated to it i.e., $Q_n = (Q_0)^n$.

Proposition 7.5. *We have the estimate* $\ell(\mathbb{P}Q_n(W_0)) \approx 1/n^{6g-7+2m}$ *i.e., there is a constant* $a_0 > 1$ *such that*

$$\frac{1}{a_0 n^{6g-7+2m}} < \ell(\mathbb{P}Q_n(W_0)) < \frac{a_0}{n^{6g-7+2m}} \tag{7.6}$$

for all n.

Proof. The band B is the only orientation preserving band. This means that W_0 is a cone to e_B of a convex subset of the face F_B of Δ opposite e_B. For every pair of orientation reversing bands α on I_+ and β on I_- there is a vertex $e_{\alpha\beta}$ of W_0, where $e_{\alpha\beta}$ is the midpoint of the edge joining the vertices e_α and e_β of the standard simplex Δ. Thus W_0 is cone to e_B of the convex hull of the vertices $e_{\alpha\beta}$. A schematic picture of W_0 is shown in Figure 15.

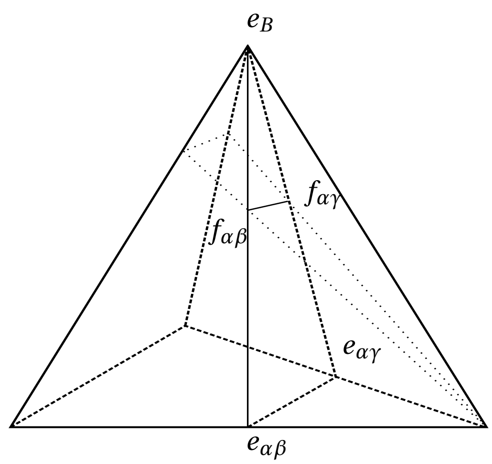

FIGURE 15

The subset $\mathbb{P}Q_n(W_0)$ of W_0 is a convex set. Hence, our goal is to identify the vertices of $\mathbb{P}Q_n(W_0)$. For α on I_+, the column $Q_n(\alpha) = e_\alpha + (2n)e_B$. All other columns of Q_n are the same as the corresponding columns of the identity matrix. This means, first, that e_B is a vertex of $\mathbb{P}Q_n(W_0)$ and second, as shown in Figure 15, on every side of W_0 that joins e_B to some $e_{\alpha\beta}$, there is a vertex $f_{\alpha\beta}$ of $\mathbb{P}Q_n(W_0)$, whose linear combination is

$$f_{\alpha\beta} = \frac{n}{n+1}e_B + \frac{1}{n+1}e_{\alpha\beta}.$$

The subset $\mathbb{P}Q_n(W_0)$ is the cone to e_B of the convex hull of the vertices $f_{\alpha\beta}$. From the linear combination there is a constant $a_0 > 1$ such that the $(d-2)$-volume of $\mathbb{P}Q_n(W_0)$ is related to the $(d-2)$-volume of W_0 by Estimate (7.6). Finally, by an easy Euler characteristic calculation, the number of bands in a complete non-classical interval exchanges on a surface of genus g and m punctures is given by $d = 6g-5+2m$. □

8. THE COMPLEX OF CURVES.

With the exception of the torus, once-punctured torus and the 4-punctured sphere, the curve complex $\mathscr{C}(S)$ of a surface S is a locally infinite simplicial complex whose vertices are the isotopy classes of essential, non-peripheral, simple closed curves on S. A collection of vertices span a simplex if there are representatives of the curves that can be realized disjointly on S. In the low complexity examples that are the exceptions the definition is modified: two vertices are connected by an edge if there are representatives of the simple closed curves intersecting minimally. The mapping class group acts on the curve complex in the obvious way. For detailed background on the curve complex, see the influential paper [25] by Masur and Minsky. Alternatively, see Bowditch [3].

Of primary interest is the coarse geometry of $\mathscr{C}(S)$. The curve complex $\mathscr{C}(S)$ is quasi-isometric to its 1-skeleton with the path metric on it. The 1-skeleton is a locally infinite graph with infinite diameter. The random walk on the mapping class group can be projected to this graph by using the group action.

Masur and Minsky [25] showed that the curve complex is δ-hyperbolic. For δ-hyperbolic spaces, it is possible to construct a natural boundary at infinity, called the Gromov boundary. Roughly speaking, points on the Gromov boundary correspond to equivalence classes of infinite geodesic rays under a suitable equivalence relation. It was shown by Klarreich [17] that the Gromov boundary $\partial\mathscr{C}(S)$ is the space $\mathscr{F}_{min}$ of *minimal* foliations on the surface i.e., foliations on S that have a non-zero intersection number with every simple closed curve. See also [12].

There is a coarse distance non-increasing map q from the Teichmüller space $T(S)$ to the curve complex $\mathscr{C}(S)$ defined as follows: A point in Teichmüller space gives a marked hyperbolic structure on S. The map q is defined by sending the point to the shortest curve or the *systole* in this hyperbolic metric. If there is more than one shortest curve, then it turns out that the set of shortest curves has bounded diameter in $\mathscr{C}(S)$, where the bound depends only on the topology of S. So picking one of the curves defines q coarsely. It is clear from the definition that q is coarsely equivariant with respect to the mapping class group action.

Here is another way to set up the definition of q: For a constant $\epsilon > 0$ and a simple closed curve α, define the ϵ-thin part of $T(S)$ corresponding to α, to be those points in $T(S)$ for which the geodesic representative of α has length less than ϵ. If ϵ is smaller than a universal constant, the intersection pattern of the thin parts corresponding to different simple closed curves, is modeled by the curve complex. The thin parts have unbounded diameter in the Teichmüller metric, but one can form the *electrified* Teichmüller space $T_{el}(S)$ by adding an extra point for each thin part, and connecting every point in that thin part to it's new point by an edge of length 1. This has the effect of collapsing the thin parts to bounded diameter. There is an obvious inclusion of $T(S)$ into $T_{el}(S)$. The electrified Teichmüller space $T_{el}(S)$ is quasi-isometric to the curve complex, giving us another definition of the map q. See [25], Lemma 3.1.

The precise statement of Klarreich's theorem viz. Theorems 1.2 and 1.4 from [17], we need is:

Theorem 8.1. *The inclusion map $q : T(S) \to T_{el}(S)$ extends continuously to the subset $\mathscr{PMF}_{min}$ of projective classes of measured minimal foliations, to give a map $\partial q : \mathscr{PMF}_{min} \to \partial T_{el}(S)$. The map ∂q is surjective and $\partial q(F) = \partial q(G)$ if and only if F and G are topologically equivalent minimal foliations. Moreover, the image under q of any sequence X_n in $T(S)$ that converges to a point in $\mathscr{PMF} \setminus \mathscr{PMF}_{min}$ cannot accumulate at any point of $\partial T_{el}(S)$.*

In fact, the image by q of a Teichmüller geodesic is an un-parameterized quasi-geodesic in $\mathscr{C}(S)$. The map ∂q is a quotient map in the sense that it takes a measured minimal foliation and forgets the measure. This quotient map is a bijection when restricted to the set $\mathscr{UE}$ of the uniquely ergodic foliations.

An immediate consequence of Klarreich's theorem and Theorem 3.1 in [14] is the theorem (See [20], Theorem 5.1)

Theorem 8.2. *Let μ be an initial distribution on the mapping class group G, such that the subgroup generated by the support of μ is non-elementary. Then, for any base point $\mathbf{b}$ in $\mathscr{C}(S)$ and almost every sample path ω, the sequence $\omega_n\mathbf{b}$ converges to a point in $\partial\mathscr{C}(S) = \mathscr{F}_{min}$. Let h denote the induced harmonic measure on $\mathscr{F}_{min}$. Then, the measure h is the push-forward under ∂q of the induced harmonic measure on $\mathscr{PMF}$.*

Proof. By Theorem 3.1 of Kaimanovich-Masur [14], for any base-point X in $T(S)$ and for almost every sample path $\omega = \{\omega_n\}$, the sequence $\omega_n X$ in $T(S)$ converges to a uniquely ergodic foliation in $\mathscr{PMF}$. Uniquely ergodic foliations are minimal, so by Klarreich's theorem the image in $T_{el}(S)$ of the sequence $\omega_n X$ converges to the same foliation in $\partial\mathscr{C}(S)$. Here, we have used the quasi-isometry between $T_{el}(S)$ and $\mathscr{C}(S)$ and chosen X such that the image of X in $\mathscr{C}(S)$ is $\mathbf{b}$. The random walk thus projected defines a harmonic measure on $\partial\mathscr{C}(S)$. The quotient map $\partial q : \mathscr{F}_{min} \to \partial\mathscr{C}(S)$ is injective on $\mathscr{UE}$. So the harmonic measure on $\mathscr{C}(S)$ is a push-forward of the harmonic measure h on $\mathscr{PMF}$, and we shall continue to call it h. $\square$

9. SUBSURFACE PROJECTIONS.

For details about subsurface projections, see Section 2 of [26] or [3]. Let Y be an essential connected subsurface of S, not a thrice punctured sphere or an annulus. Define a subsurface projection map π from $\mathscr{C}(S)$ to $\mathscr{C}(Y)$ as follows: intersect a simple closed curve x with the subsurface Y, and let $N(x,Y)$ be a regular neighborhood in Y of the union of $x \cap Y$ and ∂Y. Set $\pi(x)$ to be one of the components of $\partial N(x,Y)$, provided the component chosen is not parallel to a component of ∂Y. Strictly speaking, this is defined only when the curve intersects Y essentially, but the set of curves S_Y that do not, has diameter 2 in $\mathscr{C}(S)$. There is also a choice involved in selecting the component. However, the different choices result in a set of vertices, whose diameter in $\mathscr{C}(Y)$ is bounded above by a constant that depends only on the topology of S. This gives a coarse definition of the map π. Since π fails to be defined only on a bounded set S_Y, it extends to the boundary $\partial\mathscr{C}(S)$. The mapping class group of Y is a subgroup of the mapping class group of S in a natural way, and the map π is equivariant with respect to its action.

As explained in Section 2.4 of [26], it requires more care to set up the definition of the subsurface projection to an essential annulus A with core curve x. The goal is to define a complex $\mathscr{C}(x)$ associated to A, and then define the subsurface projection map $\pi : \mathscr{C}(S) \to \mathscr{C}(x)$ such that π records the twisting of a curve around x. For this, one would like $\mathscr{C}(x)$ to be simply $\mathbb{Z}$, but there is no natural way to do this. Instead, let $\widetilde{S}$ be the annulus cover of S corresponding to A. This cover is a quotient of $\mathbb{H}$ in a natural way. So one obtains a natural compactification $\widehat{S}$ of $\widetilde{S}$ from the compactification of $\mathbb{H}$ to the closed disk. Define $\mathscr{C}(x)$ as follows: Let the vertices of $\mathscr{C}(x)$ be paths connecting the two boundary components of $\widehat{S}$ modulo homotopies that fix the endpoints. Two vertices are connected by an edge if there are representatives of the vertices with no intersection in $\widetilde{S}$. Fixing an orientation of S and ordering the components of $\partial\widehat{S}$, we can define an algebraic intersection number of two vertices u and v, denoted by $u \cdot v$. For distinct vertices u and v of $\mathscr{C}(x)$, the distance in $\mathscr{C}(x)$ between u and v can be shown to be $1+|u\cdot v|$. Moreover, after fixing a vertex $u \in \mathscr{C}(S)$, it can be checked that the map $v \to v \cdot u$ gives a quasi-isometry between $\mathscr{C}(x)$ and $\mathbb{Z}$. In addition, the quasi-isometry constants are independent of the choice u. To define the subsurface projection $\pi(y)$ of a curve y intersecting A essentially, consider a lift $\widehat{y}$ in $\widehat{S}$ and choose a component of this lift running from one boundary component of $\widehat{S}$ to the other. The set of various components is of finite diameter in $\mathscr{C}(x)$ and so the subsurface projection map π to $\mathscr{C}(x)$ is coarsely well-defined. Finally the map π has the property that if D_x denotes the Dehn twist about x, then

$$d_{\mathscr{C}(x)}(\pi(D_x^n(y)), \pi(y)) = 2 + |n|. \tag{9.1}$$

Thus, defining π this way achieves the desired property of recording the twisting around x. There is a natural $\mathbb{Z}$ action on $\mathscr{C}(x)$ by Dehn twisting around the core curve of the annular cover $\widehat{S}$. The group $\mathbb{Z}$ also has an inclusion into the mapping class group of S as Dehn twists around x, and so it acts on $\mathscr{C}(S)$ through this inclusion. The subsurface projection map π is coarsely equivariant with respect to the $\mathbb{Z}$ action on $\mathscr{C}(S)$ and $\mathscr{C}(x)$.

9.2. **The relative space.** For technical reasons related to the proof of Theorem 11.6, we want to define sub-surface projections for the mapping class group itself. For this purpose, we introduce two spaces: the *marking complex* and the *relative space*. The marking complex is quasi-isometric to the mapping class group itself. The relative space is obtained by electrifying the mapping class group or alternatively, the marking complex because of the quasi-isometry between the two, and is quasi-isometric to the curve complex.

Suppose G is a group with a finite symmetric generating set A. For each group element g, one defines the word length of g with respect to A as the length of the shortest word in the generators representing g. This length function defines a left invariant metric on G which we call the *word metric*. The word metric can be recovered from a graph associated to (G, A) called the *Cayley graph*. The vertices of the Cayley graph are the group elements, and two elements g and h are connected by an edge of length 1 if $g^{-1}h$ is a generator. The word metric on G is simply the path metric on the Cayley graph. It is easy to check that different choices of finite symmetric generating sets result in Cayley graphs that are quasi-isometric to each other.

Given a group G with a set of generators A, and a collection of subgroups $\mathscr{H} = \{H_i\}$, the *relative* length of a group element g is defined to be the length of the shortest word representing g in the (typically infinite) set of generators $\mathscr{H} \cup A$. This defines a metric on G called the *relative metric*. We shall denote G with this metric by $\widehat{G}$, and call this the *relative space*.

Now let G be the mapping class group of S. By the classical theorem of Dehn-Lickorish [8] the mapping class group is finitely generated. We shall fix a favorite set of symmetric generators for G once and for all. The metric on G shall be implicitly assumed to be the word metric with respect to this chosen set.

We can consider the relative metric on G with respect to the following collection of subgroups: There are finitely many orbits of simple non-peripheral closed curves in S under the action of the mapping class group. Let $\{a_1, \dots, a_r\}$ be a list of representatives of these orbits, and let H_i be the subgroup of the mapping class group that fixes a_i.

By Theorem 1.3 of [25], the resulting relative space $\widehat{G}$ for the mapping class group is quasi-isometric to $\mathscr{C}(S)$.

9.3. **The marking complex.** Let $\{x_1, \dots, x_n\}$ be a simplex in $\mathscr{C}(S)$ i.e., the set $\{x_1 \dots, x_n\}$ is a set of disjoint simple closed curves in S. A *marking* in S is a set $\mathfrak{m} = \{p_1, \dots, p_n\}$, where either $p_i = x_i$, or p_i is a pair (x_i, t_i), where t_i is a diameter-one set of vertices of the complex $\mathscr{C}(x_i)$ associated to the annulus with core curve x_i. The x_i are called the *base curves* of the marking and the t_i, when defined, are called the transversals. A marking $\mathfrak{m}$ is *complete* if the simplex formed by the base is a maximal simplex in S and if every curve x_i has a transversal. In other words, a complete marking consists of a pants decomposition of S along with the choice of a transversal to each cuff in the pants decomposition.

Given $x \in \mathscr{C}(S)$, a *clean transverse curve* for x is a curve y such that a regular neighborhood F for $x \cup y$, is either a once punctured torus or a four times punctured sphere, and x and y are adjacent in the curve complex $\mathscr{C}(F)$ of F. A marking is *clean* if every p_i is a pair of the form $(x_i, \pi_{x_i}(y_i))$, where y_i is a clean transverse curve for x_i disjoint from all other curves in *base*(m).

Complete clean markings on S can be made into an infinite graph $M(S)$, by adding edges corresponding to the following *elementary moves*: Consider a marking $\mathfrak{m}$ and a base curve x of $\mathfrak{m}$. A marking $\mathfrak{m}'$ is obtained from $\mathfrak{m}$ by a *twist* move about x if *base*$(\mathfrak{m})$ = *base*$(\mathfrak{m}')$, the transversals for all base curves except x are the same, and if $t(x)$ and $t'(x)$ are the transversals to x in the markings $\mathfrak{m}$ and $\mathfrak{m}'$ respectively, then $d_{\mathscr{C}(x)}(t(x), t'(x)) \leqslant 2$. In other words, the transversal to x in $\mathfrak{m}'$ has a projection to $\mathscr{C}(x)$ that differs from the projection of the transversal to x in $\mathfrak{m}$, by at most two Dehn twists about x. A marking $\mathfrak{m}'$ is obtained from a marking $\mathfrak{m}$ by a *flip move* along x if there exists x' in *base*$(\mathfrak{m}')$ such that *base*$(\mathfrak{m}') \setminus \{x\}$ = *base*$(\mathfrak{m}) \setminus \{x'\}$, a regular neighborhood F of $x \cup x'$ is either a 1-holed torus or a 4-holed sphere, and $d_{\mathscr{C}(F)}(x, x') = 1$, and for the respective transversals $t(x)$ and $t(x')$ to x and x' in $\mathfrak{m}$ and $\mathfrak{m}'$, we have $d_{\mathscr{C}(x)}(t(x), \pi_x(x')) \leqslant 2$ and $d_{\mathscr{C}(x')}(t(x'), \pi_{x'}(x)) \leqslant 2$, where π_x and $\pi_{x'}$ are the subsurface projections to the respective annuli.

In Section 7 of [26], Masur and Minsky show that the space $M(S)$ of complete clean markings with the path metric defined above is quasi-isometric to the mapping class group G with the word metric. Let R_1 be the Lipshitz constant for this quasi-isometry i.e., the multiplicative constant in the quasi-isometry.

9.4. **Subsurface projections on the marking complex.** Here we summarize Section 2.5 of [26]. For a subsurface Y, we will write d_Y for distance in the complex of curves $\mathscr{C}(Y)$ of Y.

The subsurface projection map π on the curve complex can be extended to a subsurface projection map for complete clean markings. For a complete clean marking $\mathfrak{m}$, define $\pi(\mathfrak{m})$ as follows: if Y is an annulus with core curve $x \in$ *base*$(\mathfrak{m})$, then define $\pi(\mathfrak{m}) = \pi(t)$, where t is the transversal for x. Otherwise set $\pi(\mathfrak{m}) = \pi(base(\mathfrak{m}))$. As before, this gives a coarse definition for π. The map π is well defined on the entire space of complete clean markings $M(S)$, unlike the sub-surface projection map on $\mathscr{C}(S)$.

If distinct markings $\mathfrak{m}$ and $\mathfrak{m}'$ differ by a twist move about x, then $\pi(\mathfrak{m}) = \pi(\mathfrak{m}')$ unless Y is the annulus with core curve x, in which case $d_Y(\pi(\mathfrak{m}), \pi(\mathfrak{m}')) = d_Y(\pi(t), \pi(t')) \leqslant 2$. If the markings $\mathfrak{m}$ and $\mathfrak{m}'$ differ by a flip move along x, then in a similar way one can check that $d_Y(\pi(\mathfrak{m}), \pi(\mathfrak{m}')) \leqslant 2$. This implies that for any sub-surface Y, the projection map $\pi : M(S) \to \mathscr{C}(Y)$ is coarsely 2-Lipshitz. By pre-composing π on $M(S)$ by the quasi-isometry from G to $M(S)$ we get a projection π on G, which we continue to denote by π. The map π on G is then coarsely $2R_1$-Lipshitz.

In [26], Masur and Minsky proved a quasi-distance formula expressing the distance in the marking complex $M(S)$ in terms of subsurface projections. Here, we state a slightly simplified version of it. Given a number $A > 0$, for any $d \in \mathbb{N}$, let

$$\begin{aligned} [d]_A &= d \qquad \text{if } d \geq A \\ &= 0 \qquad \text{otherwise} \end{aligned}$$

Theorem 9.5 (Quasi-distance formula)**.** *There exists a constant $A > 0$ that depends only on the topology of S, such that for any pair of markings $\mathfrak{m}$ and $\mathfrak{m}'$ in $M(S)$ we have the estimate*

$$d_{M(S)}(\mathfrak{m}, \mathfrak{m}') \approx \sum_{Y \subseteq S} [d_Y(\pi_Y(\mathfrak{m}), \pi_Y(\mathfrak{m}'))]_A \tag{9.6}$$

where the sum runs over all sub-surfaces Y of S. The constants of approximation in the above formula depend only on the topology of S.

The map π can also be thought of as a subsurface projection map on the relative space $\widehat{G}$. Recall that the relative space is quasi-isometric to $\mathscr{C}(S)$. If we pull-back the subsurface projection on $\mathscr{C}(S)$ by this quasi-isometry, we get a map $\widehat{G} \to \mathscr{C}(Y)$ that is coarsely equivalent to the map π defined above. The reason for defining projection π on $\widehat{G}$ this way is that now π is defined everywhere on $\widehat{G}$. This feature is exploited in the proof of Theorem 11.6.

We will write $\widehat{d}$ for distance in the relative metric on G.

10. Useful facts about the relative space.

In this section, we state some facts due to Maher, about half-spaces in the relative space. These shall be used in the proof of Theorem 11.6 in the next section. Some results stated here are straightforward, and the proofs are left to the reader.

Since the relative space $\widehat{G}$ is δ-hyperbolic, nearest point projections are coarsely well defined. Denote the identity element in G by 1. Fixing 1 as the base point, we can define the Gromov product $(x \mid y)$ to be

$$(x \mid y) = \frac{1}{2}\left(\widehat{d}(1,x) + \widehat{d}(1,y) - \widehat{d}(x,y)\right).$$

It turns out that the points in the Gromov boundary $\mathscr{F}_{min}$ correspond to equivalence classes of sequences $\mathbf{x} = (x_i)$, where $\mathbf{x} \sim \mathbf{y}$ if and only if the Gromov product $(x_i \mid y_j) \to \infty$ as $i, j \to \infty$. It can be checked that the equivalence relation does not depend on the base point, and so this can be taken as a definition of the Gromov boundary.

Now we state the results about half-spaces.

Proposition 10.1. *Let p and q be nearest points to z on a geodesic $[x, y]$. Then $\widehat{d}(p,q) \leqslant 6\delta$.*

Proposition 10.2 (Maher [21] Proposition 3.4)**.** *Let $[x, y]$ be a geodesic and let p be a closest point on $[x, y]$ to a, and let q be a closest point on $[x, y]$ to b. If $\widehat{d}(p,q) > 14\delta$ then $\widehat{d}(a,b) \geqslant \widehat{d}(a,p) + \widehat{d}(p,q) + \widehat{d}(q,b) - 24\delta$.*

In particular, the above proposition implies that nearest point projection is coarsely distance non-increasing.

Corollary 10.3. *Let $[x,y]$ be a geodesic, and let p be a nearest point to a and q a nearest point to b on $[x,y]$. Then $\widehat{d}(p,q) \leqslant \widehat{d}(a,b) + 24\delta$.*

Let $H(x,y)$ be the half-space of points in $\widehat{G}$ that are closer to y than to x i.e.,

$$H(x,y) = \{a \in \widehat{G} \mid \widehat{d}(y,a) \leqslant \widehat{d}(x,a)\}.$$

Proposition 10.4 (Maher [21] Proposition 3.7)**.** *Let $z \in H(x,y)$, and let p be the nearest point to z on a geodesic $[x,y]$. Then $\widehat{d}(y,p) \leqslant (1/2)\widehat{d}(x,y) + 3\delta$. Conversely, if $\widehat{d}(y,p) \leqslant (1/2)\widehat{d}(x,y) - 3\delta$, then $z \in H(x,y)$.*

Henceforth, we consider half-spaces based at 1 i.e., half-spaces of the form $H(1,a)$.

Proposition 10.5. *There is a constant K_1, which only depends on δ, such that for any half-space $H(1,a)$, with $\widehat{d}(1,a) \geqslant K_1$, and for any set X of relative diameter at most D intersecting $H(1,a)$, the half-space $H(1,b)$ contains $H(1,a) \cup X$, where $b \in [1,a]$ with $\widehat{d}(1,b) = \widehat{d}(1,a) - 2D - K_1$.*

Proof. By Proposition 10.4, the nearest point projection of $H(1,a)$ to $[1,a]$ is distance at least $(1/2)\widehat{d}(1,a) - 3\delta$ from 1. By Corollary 10.3, the nearest point projection of $H(1,a) \cup X$ to $[1,a]$ is then relative distance at least $(1/2)\widehat{d}(1,a) - 27\delta - D$ from 1. So, if a point b on $[1,a]$ satisfies

$$\frac{\widehat{d}(1,b)}{2} + 3\delta \leqslant \frac{\widehat{d}(1,a)}{2} - 27\delta - D \tag{10.6}$$

then for any point in $H(1,a) \cup X$, the closest point p on $[1,b]$ satisfies $\widehat{d}(b,p) \leqslant (1/2)\widehat{d}(1,b) - 3\delta$. By the last line in Proposition 10.4, the set $H(1,a) \cup X$ has to lie in $H(1,b)$. Rewriting Inequality (10.6), we see that we may choose $K_1 = 30\delta$. □

Proposition 10.7. *There is a constant K_2, which depends only on δ, such that for any half-space $H(1,a)$, with $\widehat{d}(1,a) \geqslant K_2$, and for any large enough positive integer r, there is a point b with $\widehat{d}(1,b) = r - K_2$ such that every half-space $H(1,x)$, with $\widehat{d}(1,x) = r$, that intersects $H(1,a)$, is contained in $H(1,b)$.*

Proof. This follows immediately from the proof of Lemma 2.14 in [22]. □

Let $\overline{H(a,b)}$ be the limit set of the half space $H(a,b)$ in $\mathscr{F}_{min} = \partial\mathscr{C}(S)$.

Proposition 10.8. *Let μ be a probability distribution on G, whose support generates a non-elementary subgroup, and let h be the corresponding harmonic measure. Then $h(\overline{H(1,x)}) \to 0$ as $\widehat{d}(1,x) \to \infty$.*

Proof. Suppose not. Then for some $\epsilon > 0$, there is a sequence x_i with $\widehat{d}(1,x_i) \to \infty$ such that $h(\overline{H(1,x_i)}) \geqslant \epsilon$. Set

$$U = \limsup \overline{H(1,x_i)} = \bigcap_n \bigcup_{i \geqslant n} \overline{H(1,x_i)}$$

i.e., the set U consists of all points in $\mathscr{F}_{min}$ which lie in infinitely many $\overline{H(1,x_i)}$. The sets $U_n = \cup_{i\geqslant n}\overline{H(1,x_i)}$ form a decreasing sequence i.e., $U_n \supseteq U_{n+1}$. Moreover, $h(U_n) \geqslant \epsilon$ for all n. So $h(U) \geqslant \epsilon$, which implies that U is non-empty.

Let $\lambda \in U$, and pass to a subsequence x_i such that $\lambda \in \overline{H(1,x_i)}$ for all i in the subsequence. We claim that $\cap\overline{H(1,x_i)} = \{\lambda\}$. Suppose ξ is a minimal foliation that also lies in $\cap\overline{H(1,x_i)}$.

Let y and z be any points in $H(1,x_i)$. Let p and q be the points on $[1,x_i]$ closest to y and z respectively. By the triangle inequality, we have $\widehat{d}(y,z) \leqslant \widehat{d}(y,p) + \widehat{d}(p,q) + \widehat{d}(z,q)$. Hence, the Gromov product $(y \mid z)$ satisfies

$$(y \mid z) \geqslant \frac{1}{2}\left(\widehat{d}(1,y) - \widehat{d}(y,p) + \widehat{d}(1,z) - \widehat{d}(z,q) + \widehat{d}(p,q)\right).$$

By Proposition 3.2 from [21], we have $\widehat{d}(1,y) - \widehat{d}(y,p) \geqslant \widehat{d}(1,p) - 6\delta$ and $\widehat{d}(1,z) - \widehat{d}(z,q) \geqslant \widehat{d}(1,q) - 6\delta$. Hence

$$(y \mid z) \geqslant \frac{1}{2}\left(\widehat{d}(1,p) + \widehat{d}(1,q) - \widehat{d}(p,q) - 12\delta\right).$$

Now, either $\widehat{d}(1,q) - \widehat{d}(p,q) = \widehat{d}(1,p)$ or $\widehat{d}(1,p) - \widehat{d}(p,q) = \widehat{d}(1,q)$. Without loss of generality, assuming that the former is true we get

$$(y \mid z) \geqslant \frac{1}{2}\left(2\widehat{d}(1,p) - 12\delta\right).$$

By Proposition 10.4, we have $\widehat{d}(1,p) \geqslant (1/2)\widehat{d}(1,x_i) - 3\delta$. So

$$(y \mid z) \geqslant \frac{1}{2}\widehat{d}(1,x_i) - 9\delta.$$

Thus the Gromov product tends to infinity as i goes to infinity implying $\xi = \lambda$.

By assumption, the harmonic measure $h(\overline{H(1,x_i)}) \geqslant \epsilon$ for all i in the subsequence. This implies that $h(\lambda) \geqslant \epsilon$. But then, the measure h has atoms, which contradicts Theorem 3.1. Therefore $h(\overline{H(1,x_i)}) \to 0$ as $\widehat{d}(1,x_i) \to \infty$. □

In fact, Maher proves that the harmonic measure $h(\overline{H(1,x)})$ decays exponentially in $\widehat{d}(1,x)$. To be precise,

Proposition 10.9 (Maher [22] Lemma 4.3)**.** *Let μ be a finitely supported probability distribution on G whose support generates a non-elementary subgroup, and let h be the corresponding harmonic measure. There are positive constants K_3 and $L < 1$ that depend only on the topology of the surface and μ, such that if $\widehat{d}(1,x) \geqslant K_3$ then $h(\overline{H(1,x_i)}) \leqslant L^{\widehat{d}(1,x)}$.*

11. Maher's theorem.

Roughly speaking, Maher's theorem states that for any sub-surface projection, the probability that the random walk nests distance n in the sub-surface projection away from the base point is exponentially small in n.

11.1. **Application of the bounded geodesic image theorem.** The important tool is the following bounded geodesic image theorem of Masur and Minsky (Theorem 3.1 in [26]), which says that a geodesic in $\mathscr{C}(S)$ for which sub-surface projection to Y is defined for every vertex projects to a set of bounded diameter in $\mathscr{C}(Y)$, where the bound depends only on the topology.

Theorem 11.2 (Bounded geodesic image)**.** *Let Y be an essential connected subsurface of S, not a three-punctured sphere, and let γ be a geodesic segment in $\mathscr{C}(S)$, such that $\pi(x) \neq \varnothing$, for every vertex $x \in \gamma$. Then, there is a constant M_Y, which depends only on the topological type of Y, such that the diameter of $\pi(\gamma)$ is at most M_Y.*

In particular, since there are only finitely many topological types of subsurfaces Y in S, we can choose M to be the maximum over all M_Y. The constant M is called the Masur-Minsky constant. Since the relative space $\widehat{G}$ is quasi-isometric to $\mathscr{C}(Y)$, the bounded geodesic image theorem holds in $\widehat{G}$ also. Since, for the remainder of this section, we work in $\widehat{G}$, we continue to denote the Masur-Minsky constant in $\widehat{G}$ by M. We will think of a geodesic γ in G as a function $\gamma : \mathbb{Z} \to G$, and we will write γ_n for $\gamma(n)$.

The set of simple closed curves in S that are distance at most one from the set of boundary curves ∂Y of Y, is a set of diameter at most 3 in $\mathscr{C}(S)$. Let $\widehat{N}_Y$ be the pre-image of this set in $\widehat{G}$, under the quasi-isometry between $\widehat{G}$ and $\mathscr{C}(Y)$. Then there is a positive constant R_2 such that the set $\widehat{N}_Y$ has diameter at most $3R_2$ in $\widehat{G}$. Recall from 9.4 that by defining the projection π on $\widehat{G}$ using the marking complex, we ensure that π is defined at all points of $\widehat{N}_Y$. However, the bounded geodesic image theorem 11.2 fails to hold for geodesics in $\widehat{G}$ that pass through $\widehat{N}_Y$.

Let $y_0 \in \mathscr{C}(Y)$ be the image of the identity element 1 in G, under the sub-surface projection π i.e., $y_0 = \pi(1)$.

Lemma 11.3. *There is a constant K_4, which depends only on δ, such that there is a finite collection of half-spaces $H(1,x_i)$ with $\widehat{d}(1,x_i) = K_4$, such that for any subsurface Y of S, the union of the half-spaces disjoint from $\widehat{N}_Y$ has harmonic measure at least* 1/2.

Proof. First, we want $K_4 > K_3$, so that Proposition 10.9 can be applied. Second, by choosing K_4 sufficiently large such that $L^{K_4-3R_2-K_1-K_2} < 1/4$, we can bound the harmonic measure of any half-space $H(1,a)$ with $\widehat{d}(1,a) = K_4$, from above by $1/4$. The collection of endpoints of geodesic rays based at 1 is dense in $\mathscr{F}_{min}$. This implies that for any $K_4 > 0$,

$$h(\cup_{\widehat{d}(1,x)=K_4} \overline{H(1,x)}) = 1.$$

Therefore, for any $\epsilon > 0$, there is a finite collection $\{H(1,x_i)\}_{1\leqslant i\leqslant N}$ of half-spaces, with $\widehat{d}(1,x_i) = K_4$, such that $h(\cup_i \overline{H(1,x_i)} \geqslant 1-\epsilon$. It will be convenient for us to choose $\epsilon = 1/4$. Suppose $\widehat{N}_Y$ hits some half-space $H(1,x_i)$ in the finite collection. Since $K_4 \geqslant 3R_2 + K_1$, we can apply Proposition 10.5 to conclude there is a $y \in [1,x_i]$ with $\widehat{d}(1,y) = K_4 - 3R_2 - K_1$ such that the union $H(1,x_i) \cup \widehat{N}_Y$ belongs to the half-space $H(1,y)$. By Proposition 10.7, there is a half-space $H(1,z)$ with $\widehat{d}(1,z) = K_4 - 3R_2 - K_1 - K_2$ such that any half-space $H(1,x_j)$ with $\widehat{d}(1,x_j) \geqslant K_4$, intersecting $H(1,y)$, is contained in $H(1,z)$. By Proposition 10.9, the harmonic measure of half-spaces decays exponentially in the relative distance. So $h(\overline{H(1,z)} \leqslant L^{K_4-3R_2-K_1-K_2} < 1/4$. So, the measure of the half-spaces disjoint from $\widehat{N}_Y$ is at least $1/2$. □

Lemma 11.4. *Given the finite collection of half-spaces as in Lemma 11.3, there is a constant K_5, depending on the collection, such that for any subsurface Y, the projection of the union of the half-spaces disjoint from $\widehat{N}_Y$ is contained in the K_5-neighborhood of $y_0 = \pi(1)$ in $\mathscr{C}(Y)$.*

Proof. Every point in a half-space $H(1,x_i)$ can be connected to 1 by a piecewise geodesic path with at most two pieces: the geodesic $[1,x_i]$ followed by a geodesic connecting x_i to the point. Consider half-spaces in the collection that are disjoint from $\widehat{N}_Y$. If $\widehat{N}_Y$ does not hit the geodesic $[1,x_i]$, then by Theorem 11.2, the projection of the union $H(1,x_i) \cup [1,x_i]$ has bounded image in $\mathscr{C}(Y)$ with diameter at most $2M$. So suppose that $\widehat{N}_Y$ hits the geodesic $[1,x_i]$. Since there is a fixed collection of finitely many such geodesics, we can set

$$M' = \max\left(\max_i(d(1,x_i)), M, A\right).$$

where d is the actual distance in G and A is the cutoff in the quasi-distance formula 9.6. By the quasi-distance formula, up to a universal constant that depends only on the topology of S, for any sub-surface Y, the projection of all geodesic segments $[1,x_i]$ lies in a M'-neighborhood of y_0. So we may choose $K_5 = M + M'$ to conclude the proof of the lemma. □

11.5. **Exponential decay.** Recall that we have chosen a base-point y_0 in $\mathscr{C}(Y)$ to be the image under π of the identity element $1 \in G$. Let d_Y denote the metric in $\mathscr{C}(Y)$.

Given a subset A in $\mathscr{C}(Y)$, we will write A^c to be the complement of A in $\mathscr{C}(Y)$. We say a pair of sets $A_1 \supset A_2$ in $\mathscr{C}(Y)$ is K-*nested* if $d_Y(A_1^c, A_2) \geqslant K$. We say a nested collection of sets $A_1 \supset A_2 \supset \cdots$ is K-nested, if each adjacent pair $A_i \supset A_{i+1}$ is K-nested in $\mathscr{C}(Y)$.

A pair of sets $B_1 \supset B_2$ in $\widehat{G}$ is K-nested for π if the image pair $A_1 = \pi(B_1)$ containing $A_2 = \pi(B_2)$ is K-nested in $\mathscr{C}(Y)$. It should be pointed out that such a pair $B_1 \supset B_2$ is not necessarily K-nested for the relative metric $\widehat{d}$ on $\widehat{G}$.

The main theorem due to Maher [23] is

Theorem 11.6 (Exponential Decay)**.** *Let μ be a probability distribution on G, with finite support, such that the sub-group generated by the support is non-elementary. Let Y be a sub-surface of S. Then there is a constant K_6, which depends on μ but is independent of the sub-surface Y, such that if $\pi(1) \notin A_1 \supset A_2 \supset \cdots$ is a collection of K_6-nested subsets in $\mathscr{C}(Y)$, then $h(\overline{\pi^{-1}(A_k)}) \leqslant (1/2)^k$.*

Let $H(1,x_i)$ be a collection of half-spaces as in Lemmas 11.3 and 11.4 above, and let K_5 be the corresponding constant from Lemma 11.4. Let h_g for the harmonic measure for starting at base-point g instead of 1, so $h_g(X) = h(g^{-1}X)$.

Lemma 11.7. *For any sub-subsurface Y,*

$$h_g\left(\overline{\pi^{-1}(B_{K_5}(\pi(g)))}\right) > \frac{1}{2},$$

where $B_{K_5}(\pi(g))$ is the ball in $\mathscr{C}(Y)$ of radius K_5, centered at $\pi(g)$.

Proof. Consider the projection into $\mathscr{C}(Y)$ of the half-spaces $H(g, gx_i)$ disjoint from $\widehat{N}_Y$. By Lemma 11.4, the projection of the union of these half-spaces lies in a K_5-neighborhood of $\pi(g)$ in $\mathscr{C}(Y)$. By Lemma 11.3, $\pi^{-1}(B_{K_5}(\pi(g)))$ contains all the half-spaces $H(g, gx_i)$ disjoint from $\widehat{N}_Y$, and so has measure at least 1/2. $\square$

Let D be the diameter of the support of μ and let $K_6 > K_5 + 2D$. We can now do conditional measure on $B_i = \pi^{-1}(A_i)$ of K_6-nested collections of sets for π. Given such a nested collection B_i for π, one can define a "midpoint set" to be the set $\{g \in B_i \mid d_Y(\pi(g), A_{i+1}) = d_Y(\pi(g), A_i^c)\}$. However, for the subsequent argument, we need a neighborhood E_i of this set so that $\pi_Y(E_i)$ covers a $2R_1D$-neighborhood of the projection of the "midpoint set", where recall from the previous section that the map π on G is coarsely $2R_1$-Lipshitz. So we make the following definition.

Definition 11.8. Let $\{B_i\}$ be a nested collection of sets for π with a large enough nesting distance. Define the *midpoint sets* E_i to be

$$E_i = \{g \in B_i : |d_Y(\pi(g), A_{i+1}) - d_Y(\pi(g), A_i^c)| \leqslant 2R_1D\},$$

where recall from 9.3 that π is coarsely $2R_1$-Lipshitz on G.

Proposition 11.9. *Choose $K_6 > 2R_1D + M$ also, and let (B_i) be a K_6-nested collection of sets for π. If a sample path ω converges to $\overline{B_{i+1}}$, then for some n, the point ω_n lies in a midpoint set E_i.*

Proof. The main idea is as follows: Along a sample path ω_n that converges into $\overline{B_{i+1}}$ the quantity $f(n) = d_Y(\pi_Y(\omega_n), A_i^c) - d_Y(\pi(\omega_n), A_{i+1})$ satisfies $|f(n) - f(n+1)| \leqslant 4R_1D$. If $K_6 > 2R_1D + M$ then for small values of n, we have $f(n) \leqslant -2R_1D - M < -2R_1D$. Next, if we show that for n large, $\pi_Y(\omega_n)$ gets within d_Y-distance M of A_{i+1}, then $f(n) > 2R_1D$. The increment bound on $f(n)$ now forces that for some n in between we must have $\omega_n \in E_i$.

As μ has finite support, the sample path satisfies $d(\omega_n, \omega_{n+1}) \leqslant D$, where d is the actual distance in G. Since π is coarsely $2R_1$-Lipshitz, $d_Y(\pi(\omega_n), \pi(\omega_{n+1})) \leqslant 2R_1D$ which implies $|f(n) - f(n+1)| \leqslant 4R_1D$.

Now suppose ω_n converges to $\lambda \in \overline{B_{i+1}}$. Then, there is a sequence $l_n \in B_{i+1}$ such that l_n converges to λ, and so the Gromov product $(\omega_n | l_n)$ converges to ∞. The foliation λ is minimal, so for n large enough, a geodesic joining ω_n to l_n misses $\widehat{N}_Y$. Theorem 11.2 then implies that $d_Y(\pi(\omega_n), \pi(l_n)) \leqslant M$. In particular, for all n sufficiently large, $d_Y(\pi(\omega_n), A_{i+1}) \leqslant M$ i.e., the projection of ω_n has to get within distance M of A_{i+1}. Therefore, if the nesting distance $K_6 > 2R_1D + M$, then for some n we have

$$|d_Y(\pi(\omega_n), A_{i+1}) - d_Y(\pi(\omega_n), A_i^c)| \leqslant 2R_1D$$

i.e., there is an $\omega_n \in E_i$. $\square$

Proof. (of Theorem 11.6) We will compute the probability a sample path converges into $\overline{B_{k+1}}$, given that it converges into $\overline{B_k}$. By Proposition 11.9, any sample path ω that converges into $\overline{B_{k+1}}$ has to hit the midpoint set E_k. So we can condition on an element g in E_k.

Consider the collection of all random walks starting at g. Since $\mathscr{F}_{min}$ is the disjoint union of the sets $\overline{B_k^c} \cup (\overline{B_k} \setminus \overline{B_{k+1}}) \cup \overline{B_{k+1}}$, a random walk converges into precisely one of these sets. Set $p_1 = h_g(\overline{B_k^c}), p_2 = h_g(\overline{B_k} \setminus \overline{B_{k+1}}), p_3 = h_g(\overline{B_{k+1}})$. Then $p_1 + p_2 + p_3 = 1$. We want to give an upper bound for the relative probability

$$\frac{h_g(\overline{B_{k+1}})}{h_g(\overline{B_k})} = \frac{p_3}{p_2 + p_3}$$

i.e., the probability that a sample path starting from g converges into $\overline{B_{k+1}}$, given that it converges into $\overline{B_k}$. The nesting distance K_6 can be chosen to be larger than K_5+D. Since the sets B_i are K_6-nested for π, the ball $B_{K_5}(\pi(g))$ in $\mathscr{C}(Y)$ of radius K_5 centered at $\pi(g)$, is contained in $A_k \setminus A_{k+1}$. By Lemma 11.7, the measure $p_2 = h_g(\overline{B_k} \setminus \overline{B_{k+1}}) > 1/2$. This implies that for all k

$$\frac{h_g(\overline{B_{k+1}})}{h_g(\overline{B_k})} = \frac{p_3}{p_2+p_3} < \frac{p_3}{1/2+p_3} = 1 - \frac{1}{2}\left(\frac{1}{1/2+p_3}\right) < \frac{1}{2} \tag{11.10}$$

A sample path converging to $\overline{B_k}$ either hits E_k or not. Those that hit E_k define a distribution ν on E_k for the first hit i.e., $\nu(g)$ is the probability that a path that converges to $\overline{B_k}$ and hits E_k does so first at g. To simplify notation, we have suppressed the dependence of ν on k. Then

$$h(\overline{B_k}) \geqslant \sum_{g \in E_k} \nu(g) h_g(\overline{B_k})$$

and

$$h(\overline{B_{k+1}}) = \sum_{g \in E_k} \nu(g) h_g(\overline{B_{k+1}}).$$

Thus,

$$\frac{h(\overline{B_{k+1}})}{h(\overline{B_k})} \leqslant \frac{\sum_{g\in E_k} \nu(g) h_g(\overline{B_{k+1}})}{\sum_{g\in E_k} \nu(g) h_g(\overline{B_k})} < \frac{1}{2} \tag{11.11}$$

where the second inequality follows from Estimate (11.10). Using Estimate (11.11) inductively we have $h(\overline{B_k}) = h(\overline{\pi^{-1}(A_k)}) < (1/2)^k$ as required. □

12. Efficient position and the push-in sequence.

We switch back to the curve complex $\mathscr{C}(S)$. For any train track τ, let $P(\tau)$ be the set of simple closed curves carried by τ.

At this point, the key ideas for the construction of a singular set are in place: Start with an embedding of ϕ_0 in S. Then W_0 gives a chart on $\mathscr{PMF}$. Denote the underlying complete train track with a single switch on S by τ_0. Choose the embedding such that the base-point $\mathbf{b}$ is outside $P(\tau_0)$. By Theorem 6.9, the expansion of almost every point in W_0 becomes C-distributed infinitely often with each instance of C-distribution having combinatorics ϕ_0. In particular, the union of stages that are the m-th instances of C-distribution has full Lebesgue measure in W_0. Follow each such C-distributed stage by n successive Dehn twist sequences, to get a collection of stages. The union of stages in this collection should be the set $Y_n^{(m)}$, which appears in Proposition 4.4. The estimate (7.6) implies that for the Lebesgue measure, each stage in the collection has proportion $\approx 1/n^j$ (where $j = 6g-7+2m$) in the corresponding C-distributed stage. Taking union over the C-distributed stages implies $\ell(Y_n^{(m)}) \approx 1/n^j$, as required in Proposition 4.4. On the other hand, for each C-distributed stage in question, consider the sub-surface projection to its stable vertex cycle (which is defined in Section 7.3). Following a C-distributed stage by its Dehn twist sequence n times increasing the projections to its stable vertex cycle by n. Then by Theorem 11.6, we expect that in the harmonic measure, the stage obtained after the twists has proportion $\leqslant \exp(-n)$ in the C-distributed stage. Taking union over all the C-distributed stages would then imply $h(Y_n^{(m)}) \leqslant \exp(-n)$.

Except that this runs into the following problem: Theorem 11.6 applies to pre-images under sub-surface projections and under projection to its stable vertex cycle a C-distributed stage (while being a part) is not the whole pre-image of its projection. The whole pre-image will typically intersect nearby C-distributed stages inside which altogether different projections are being considered. Hence, Theorem 11.6 cannot be directly applied. This issue would be resolved if we knew that in the harmonic measure every C-distributed stage has a definite proportion in the corresponding pre-image. However, we do not know if this is true. So we have to finesse a little to achieve a setup to which Theorem 11.6

applies. The finesse involves projecting to the stable vertex cycle of a nested stage with combinatorics ϕ_0 rather than the C-distributed stage itself. In this section, we give the technical details for this trick.

12.1. **Sub-surface projections to carried curves.** A train track is said to be *generic* when every switch in it is trivalent. Let τ be a generic complete train track. The following proposition proves that the sub-surface projection of the complement $\mathscr{C}(S) \setminus P(\tau)$ to an annulus with a core curve "deeply carried" by τ, has a universally bounded diameter.

Proposition 12.2. *Let $x \in P(\tau)$ be carried by τ such that it passes over every branch of τ at least thrice. Let π be the sub-surface projection to the annulus A with core curve x. Then*

$$diam(\pi(\mathscr{C}(S) \setminus P(\tau))) \leqslant 3.$$

Proof. The main ingredient of the proof is the apparatus of *efficient position* developed by Masur, Mosher and Schleimer [28].

12.3. **Efficient position.** Given a train track τ, let $N(\tau)$ be a neighborhood of a train track foliated by ties. For non-classical interval exchanges, this is just our picture of the interval with bands. For a generic track τ, a curve c is said to be in efficient position with respect to τ, if

(1) every component of $c \cap N(\tau)$ is either a tie or carried by τ,
(2) every region in $S \setminus (c \cup N)$ has negative index or is a rectangle. (See [28] for the precise definition of index).

For our purposes, all we need is the implication that if c is in efficient position with respect to τ, then there is no embedded bigon in the complement $S \setminus (c \cup \tau)$. The main theorem of Masur, Mosher and Schleimer is

Theorem 12.4. *(Theorem 4.1 in* [28]*:) Let τ be a birecurrent generic train track and suppose c is a non-peripheral simple closed curve. Then efficient position of c with respect to τ exists and is unique up to rectangle swaps, annulus swaps, and isotopies of S preserving the foliation of $N(\tau)$ by ties.*

Going back to the proof of Proposition 12.2, let y and z be curves in $\mathscr{C}(S) \setminus P(\tau)$. By the theorem of Masur-Mosher-Schleimer, the curves y and z can be put in efficient position with respect to τ. Since τ is complete, every efficient position of y meets some branch of τ dually i.e., the intersection with the branch is a tie. Similarly for every efficient position of z. Let $\widehat{S}$ be the compactified annulus cover of S corresponding to A. There are countably many lifts of x to $\widehat{S}$; exactly one of these is the core curve of $\widehat{S}$. Call this lift X_0. All other lifts of x are inessential arcs with both endpoints on the same boundary component of $\widehat{S}$.

Let s be the tie that is the intersection of an efficient position of y with a branch of τ. Choose a lift S of the tie s meeting X_0 such that the point of intersection $S \cap X_0$ is sandwiched in between intersections of S with other lifts of x on either side of X_0. See Figure 16.

On either side of X_0, let X_+ and X_- be the "outermost" lifts of x that S intersects i.e., the lifts X_+ and X_- are the farthest to the point $S \cap X_0$ along S. For instance, the dotted lines in Figure 16 show a portion about s of the tie neighborhood $N(\tau)$ lifted up to $\widehat{S}$. The condition that x passes over each branch at least thrice implies that there is at least one lift of x that passes through the branch marked b and intersects S. But as is clear from the picture, a lift through b is not the outermost one in the above sense.

Since the lifts $X_\pm$ are inessential arcs, they cut off discs $D_\pm$ in $\widehat{S}$. In Figure 16, the discs $D_\pm$ can be seen to be bounded by $X_\pm$ and the boundary components marked $\pm$. The lift Y of y extending S, cannot intersect $X_\pm$ again; otherwise, there is a bigon in the complement $S \setminus (y \cup \tau)$ violating efficient position. This implies that the lift Y is contained in $D_+ \cup S \cup D_-$.

We can similarly consider an efficient position of z, and assume that it intersects τ dually in a tie t different from s. Lifting t up to $\widehat{S}$ to get a suitable arc T, we repeat the argument above to show that there exists lifts $X'_\pm$ of x cutting off discs $D'_\pm$ in $\widehat{S}$, such that a lift Z of z extending T is contained in $D'_+ \cup T \cup D'_-$.

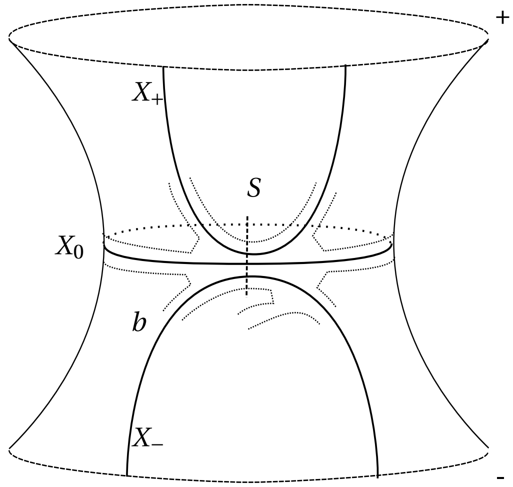

FIGURE 16. Lifts to $\widehat{S}$

Fixing the endpoints of Y and Z, it follows from the above containments that the algebraic intersection number $Y \cdot Z$ is at most 2. Recall from the initial part of Section 9, that the distance between Y and Z in the complex $\mathscr{C}(x)$ is given by $1 + |Y \cdot Z|$, thus finishing the proof of Proposition 12.2. □

Even though Proposition 12.2 is stated for generic complete train tracks, the result is also true for complete non-classical interval exchanges. We can *comb* (See Section 1.4 of [30]) a complete non-classical interval exchange moving left to right along the base interval to get a generic train track. For example, see the first picture in Figure 21 of [7]. It can be directly checked that the resulting track is transversely recurrent. Then, by Proposition 1.4.1 of [30], the resulting generic train track is complete, and so Proposition 12.2 applies to it. The operation of combing is isotopic to identity, and hence the set of carried curves remains unchanged.

12.5. **The push-in sequence.** Fixing the quasi-isometry defined in Section 9, henceforth we think of all annular projections as maps to $\mathbb{Z}$. Starting from τ_0, we focus on complete train tracks on S underlying splitting sequences $\jmath$ that terminate at ϕ_0. Let $P(\jmath)$ denote the set of simple closed curves carried by the track underlying $\jmath$. Let κ be a directed loop in $\mathscr{G}$ starting and ending at ϕ_0, and consider the splitting sequence $\jmath * \kappa$. Let $P(\jmath * \kappa)$ be the set of simple closed curves carried by the track underlying $\jmath * \kappa$. and let $v(\jmath * \kappa)$ be the stable vertex cycle of $\jmath * \kappa$.

Lemma 12.6. *There are sequences κ in $\mathscr{G}$ starting and ending at ϕ_0 such that for any stage $\jmath$ with combinatorics ϕ_0*

$$\pi^{-1}((-\infty,-3] \cup [3,\infty)) \subset P(\jmath),$$

where $\pi : \mathscr{C}(S) \to \mathbb{Z}$ is the projection to the annulus with core curve $v(\jmath * \kappa)$ post-composed with a translation such that $\pi(\mathbf{b}) = 0$.

Proof. Let κ be any sequence starting and ending at ϕ_0 such that every entry in the associated matrix Q_κ is at least 3. This implies that the stable vertex cycle $v(\jmath * \kappa)$ passes over every band in $\jmath$ at least thrice.

Now consider the subsurface projection π to the annulus with core curve $v(\jmath * \kappa)$. By post-composing with a translation of $\mathbb{Z}$, we may assume that $\pi(\mathbf{b}) = 0$. By Proposition 12.2, the projection π of every point in $\mathscr{C}(S) \setminus P(\jmath)$ is within distance 3 of $\pi(\mathbf{b})$. This implies that the pre-image $\pi^{-1}((-\infty,-3) \cup (3,\infty))$ is contained in $P(\jmath)$. □

In other words, when the vertex cycle $v(\jmath * \kappa)$ gets nested deep enough in $P(\jmath)$ a fixed pre-image of the projection to $v(\jmath * \kappa)$ is contained in $P(\jmath)$. It is in this sense that such a κ is a *push-in sequence*. We fix such a sequence κ once and for all. This achieves a setup to which Theorem 11.6 can be applied.

13. A SINGULAR SET FOR HARMONIC MEASURE.

13.1. **Choosing the initial chart.** We choose the initial embedding of ϕ_0 in S with the underlying track τ_0 such that the base-point $\mathbf{b} \in \mathscr{C}(S)$ does not belong to $P(\tau_0)$. The embedding identifies the configuration space W_0 with a chart in $\mathscr{PMF}$. The construction of a singular set will proceed inside this chart.

13.2. **Relative probability that the sequence $\kappa * n\jmath_0$ follows a C-distributed stage.** By Lemma 6.7, the relative probability that the push-in sequence κ follows a C-distributed stage $\jmath$ is up to a uniform constant $c > 1$, the same as the probability $\ell(\mathbb{P}Q_\kappa(W_0))$ that an expansion begins with κ i.e.,

$$\frac{1}{c}\ell(\mathbb{P}Q_\kappa(W_0)) < \frac{\ell(\mathbb{P}Q_{\jmath*\kappa}(W_0))}{\ell(\mathbb{P}Q_\jmath(W_0))} < c \cdot \ell(\mathbb{P}Q_\kappa(W_0)).$$

Since κ is a priori fixed, whenever the sequence κ follows a C-distributed stage $\jmath$, the resulting stage $\jmath * \kappa$ is C'-distributed, for some C' that depends only on C and d. Recall from Section 7 that $\jmath_0$ denotes the Dehn twist sequence. Denote the matrix associated to the sequence $\jmath * \kappa * n\jmath_0$ by $Q_{\jmath,n}$. Again by Lemma 6.7, there exists a constant a_1 that depends only on a_0 and C', such that

$$\frac{1}{a_1 n^j} < \frac{\ell(\mathbb{P}Q_{\jmath,n}(W_0))}{\ell(\mathbb{P}Q_{\jmath*\kappa}(W_0))} < \frac{a_1}{n^j}.$$

Hence the relative probability that the sequence $\kappa * n\jmath_0$ follows a C-distributed stage $\jmath$ satisfies

$$\frac{1}{ca_1 n^j} < \frac{\ell(\mathbb{P}Q_{\jmath,n}(W_0))}{\ell(\mathbb{P}Q_\jmath(W_0))} < \frac{ca_1}{n^j} \tag{13.3}$$

i.e., the relative probability $\ell(\mathbb{P}Q_{\jmath,n}(W_0))/\ell(\mathbb{P}Q_\jmath(W_0)) \approx 1/n^j$.

13.4. **Construction of the doubly indexed sequence of sets $Y_n^{(m)}$.** By Theorem 6.9, almost every expansion becomes C-distributed infinitely often. Hence, for every non-negative integer m, almost every expansion has a stage that is the m-th instance of C-distribution. When $m = 0$, we mean the initial stage itself with no splitting whatsoever.

Let S_m be the set of stages $\jmath$ that are the m-th instances of C-distribution. For $\jmath \in S_m$, let $Y_\jmath$ be the set of $\mathbf{x}$ in W_0 whose expansion begins with $\jmath$ i.e., the subset $Y_\jmath = \mathbb{P}Q_\jmath(W_0)$. For distinct $\jmath, \bar{\jmath}$ in S_m, the sets $Y_\jmath, Y_{\bar{\jmath}}$ have disjoint interiors. By Theorem 6.9, the union over all $\jmath \in S_m$ of the sets $Y_\jmath$ is a set of full measure i.e.,

$$\sum_{\jmath \in S_m} \ell(Y_\jmath) = 1 \tag{13.5}$$

Follow each $\jmath$ in S_m by the sequence $\kappa * n\jmath_0$, and let $Q_{\jmath,n}$ be the matrix associated to $\jmath * \kappa * n\jmath_0$. Let $Y_{\jmath,n} = \mathbb{P}Q_{\jmath,n}(W_0)$. Then, the ratio $\ell(Y_{\jmath,n})/\ell(Y_\jmath)$ satisfies Estimate (13.3). Let $Y_n^{(m)}$ be the union

$$Y_n^{(m)} = \bigcup_{\jmath \in S_m} Y_{\jmath,n}.$$

For $m = 0$, the set $Y_n^{(0)}$ is just $\mathbb{P}Q_{n\jmath_0}(W_0)$ i.e., the set of $\mathbf{x}$ whose expansion begins with $n\jmath_0$. First, we estimate the Lebesgue measure of $Y_n^{(m)}$.

Lemma 13.6.

$$\frac{1}{ca_1 n^j} < \ell(Y_n^{(m)}) < \frac{ca_1}{n^j} \tag{13.7}$$

Proof. Write

$$\ell(Y_n^{(m)}) = \sum_{J \in S_m} \ell(Y_{J,n}) = \sum_{J \in S_m} \ell(Y_J)\frac{\ell(Y_{J,n})}{\ell(Y_J)}.$$

Estimate (13.3) for the ratio $\ell(Y_{J,n})/\ell(Y_J)$, and Equation (13.5) finishes the proof. ☐

In particular, the above lemma shows that for any m_1, m_2,

$$\ell(Y_n^{(m_1)}) \approx \ell(Y_n^{(m_2)}) \tag{13.8}$$

13.9. **Almost independence of $Y_n^{(m)}$ for the Lebesgue measure.** Let (m_1, n_1) and (m_2, n_2) be a pair of indices with $m_1 < m_2$. Since the sets $Y_\iota : \iota \in S_{m_1}$ is a partition of a set of full measure, it is enough to check that almost independence holds in each Y_ι. For any $\iota \in S_{m_1}$ and $J \in S_{m_2}$, either Y_J is contained in Y_{ι,n_1} or has interior disjoint from it. By Lemma 6.7, given $\iota \in S_{m_1}$, the relative probabilities satisfy

$$\begin{aligned}
\mathrm{Prob}(Y_{n_1}^{(m_1)}|\iota) &\approx \ell(Y_{n_1}^{(0)}) \\
\mathrm{Prob}(Y_{n_2}^{(m_2)}|\iota) &\approx \ell(Y_{n_2}^{(m_2-m_1)}) \\
\mathrm{Prob}(Y_{n_1}^{(m_1)} \cap Y_{n_2}^{(m_2)}|\iota) &\approx \ell(Y_{n_1}^{(0)} \cap Y_{n_2}^{(m_2-m_1)})
\end{aligned}$$

So it is enough to check that for any $m > 0$, the sets $Y_{n_1}^{(0)}$ and $Y_{n_2}^{(m)}$ are almost independent. Again, the main point is that for all J in S_m, the set Y_J is either contained in $Y_{n_1}^{(0)}$ or has interior disjoint from $Y_{n_1}^{(0)}$. Let T_m be the subset of S_m consisting of those J for which Y_J is contained in $Y_{n_1}^{(0)}$. The union over all $J \in T_m$ of the sets Y_J is a set of full measure in $Y_{n_1}^{(0)}$. Hence, using Estimate (13.3) and Equation (13.8), we get

$$\begin{aligned}
\ell\left(Y_{n_1}^{(0)} \cap Y_{n_2}^{(m)}\right) &\approx \sum_{J \in T_m} \ell(Y_{J,n_2}) \\
&\approx \sum_{J \in T_m} \ell(Y_{n_2}^{(0)})\ell(Y_J) \\
&\approx \ell(Y_{n_2}^{(m)}) \sum_{J \in J_m} \ell(Y_J) \\
&\approx \ell(Y_{n_2}^{(m)})\ell(Y_{n_1}^{(0)})
\end{aligned}$$

showing almost independence.

13.10. **Harmonic measure estimate for $Y_n^{(m)}$.** Recall from Remark 7.3 that ϕ_0 has the property that simple closed curves carried by the underlying train track twist only to the left around the stable vertex cycle v. Let π be the projection to the annulus with core curve v. Since the twisting is in one direction only, the projection of the set of carried curves is a one-sided interval of the form $[M_1, \infty)$ or $(-\infty, -M_1]$. Reversing the order on $\mathbb{Z}$ if necessary, we fix the convention that it is of the form $[M_1, \infty)$.

For the initial track τ_0, let π be the projection to the annulus with core curve $v(\kappa)$. Post-composing by a translation of $\mathbb{Z}$ assume that $\pi(\mathbf{b}) = 0$. By our chosen convention, there is a positive integer $M_1 \geqslant 3$ such that $\pi(P(\kappa)) = [M_1, \infty)$.

For $J \in S_m$, let $v(J * \kappa)$ be the stable vertex cycle of $J * \kappa$. Denote by $\pi_{J*\kappa}$, the sub-surface projection to the annulus with core curve $v(J * \kappa)$. Post-composing by a translation, we assume that $\pi_{J*\kappa}(\mathbf{b}) = 0$.

By Lemma 12.6, the push-in sequence κ ensures that the pre-image $\pi_{J*\kappa}^{-1}([3,\infty) \cup (-\infty, -3])$ sits entirely inside the set $P(J)$. By our chosen convention, $\pi_{J*\kappa}(P(J * \kappa))$ is a one sided interval. We show below that this interval is contained in $[M_1 - 3, \infty)$

Lemma 13.11. *For all positive integers m and for all $J \in S_m$*

$$\pi_{J*\kappa}(P(J * \kappa)) \subset [M_1 - 3, \infty),$$

where $\pi_{J*\kappa}$ is the sub-surface projection to the annulus with core curve $v(J * \kappa)$, and $\pi(\mathbf{b}) = 0$.

Proof. Since $\jmath$ returns to the combinatorics ϕ_0, there is a mapping class g such that $g^{-1}(P(\jmath)) = P(\tau_0)$. It follows that $g^{-1}(P(\jmath * \kappa)) = P(\kappa)$. For the sub-surface projection $\pi_{\jmath*\kappa}$ to $v(\jmath * \kappa)$, set the origin in $\mathbb{Z}$ by $\pi_{\jmath*\kappa}(g\mathbf{b})$ instead of $\pi_{\jmath*\kappa}(\mathbf{b})$. With this choice of origin, we have $\pi_{\jmath*\kappa}(P(\jmath*\kappa)) = [M_1, \infty)$ because of equivariance. Finally, notice that since $\mathbf{b}$ is not in $P(\tau_0)$ both $\mathbf{b}$ and $g\mathbf{b}$ are in $\mathscr{C}(S) \setminus P(\jmath)$. By Proposition 12.2, we must have $|\pi_{\jmath*\kappa}(\mathbf{b}) - \pi_{\jmath*\kappa}(g\mathbf{b})| \leqslant 3$, and we are done. $\square$

For any interval exchange $\jmath \in S_m$, to define the set $Y_{\jmath,n}$, we followed the sequence $\jmath * \kappa$ by $n\jmath_0$. This means that the set $P(\jmath * \kappa * n\jmath_0)$ is obtained from the set $P(\jmath * \kappa)$ by n positive Dehn twists in the stable vertex cycle $v(\jmath * \kappa)$. By the equivariance property (9.1), this increases the projection under $\pi_{\jmath*\kappa}$ by n. Let $a(n)$ denote the greatest integer less than or equal to $(n + M_1 - 6)/K_6$, where K_6 is the nesting distance required in Theorem 11.6. For n large enough, Theorem 11.6, Lemma 13.11 and the push-in property of κ, in particular, the fact that $Y_\jmath \supset \pi^{-1}_{\jmath*\kappa}[3, \infty)$ imply that the harmonic measures satisfy the estimate

$$\frac{h(Y_{\jmath,n})}{h(Y_\jmath)} \leqslant \frac{h(\overline{\pi^{-1}_{\jmath*\kappa}[M_1 - 3 + n, \infty)})}{h(\overline{\pi^{-1}_{\jmath*\kappa}[3, \infty)})} < \left(\frac{1}{2}\right)^{a(n)}$$

Taking union over all $\jmath \in S_m$ we get

$$h(Y_n^{(m)}) = \sum_{\jmath \in S_m} h(Y_{\jmath,n}) < \left(\frac{1}{2}\right)^{a(n)} \cdot \sum_{\jmath \in S_m} h(Y_\jmath) = \left(\frac{1}{2}\right)^{a(n)} \tag{13.12}$$

The number $a(n)$ increases linearly in n. Thus we get the exponential decay we want.

13.13. **A singular set.** We have shown that the doubly indexed sequence of sets $Y_m^{(n)}$ satisfy almost independence and the polynomial decay estimate (13.7) for the Lebesgue measure, and the exponential decay estimate (13.12) for the harmonic measure. Hence, Proposition 4.4 constructs a set X that has positive Lebesgue measure and zero harmonic measure. The set

$$Z = \bigcup_{g \in G} gX$$

is a measurable G-invariant subset of $\mathscr{PMF}$. By the ergodicity of the action of the mapping class group G on $\mathscr{PMF}$ [24] [16], the set Z has full Lebesgue measure. On the other hand, since Z is a countable union of sets with zero harmonic measure, it has zero harmonic measure. This proves Theorem 3.4.

13.14. **Concluding Remarks.** Theorem 3.4 is true for all initial distributions for which the estimate of Theorem 11.6 holds. This should be a larger set of initial distributions than just the finitely supported ones. We expect that if the initial distribution decays at a fast enough exponential rate then Theorem 3.4 holds. The precise description is still unclear to us.

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