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LIPSCHITZ CONSTANTS TO CURVE COMPLEXES

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Abstract. We determine the asymptotic behavior of the optimal Lipschitz constant for the systole map from Teichmüller space to the curve complex.

1. Introduction

Let $S = S_g$ be a closed surface of genus $g \geqslant 2$. We equip the Teichmüller space $\mathscr{T}(S)$ of S with the Teichmüller metric, and equip the 1-skeleton $\mathscr{C}^{(1)}(S)$ of the complex of curves $\mathscr{C}(S)$ with its usual path metric $d_{\mathscr{C}}$.

In [8], Masur and Minsky study the *systole map*

$$\mathrm{sys} : \mathscr{T}(S) \to \mathscr{C}^{(1)}(S),$$

which assigns a hyperbolic metric one of its shortest curves, called a *systole*. They prove that sys is (K, C)-*coarsely Lipschitz* for some $K, C > 0$, meaning that, for all X and Y in $\mathscr{T}(S)$

$$d_{\mathscr{C}}(\mathrm{sys}(X), \mathrm{sys}(Y)) \leqslant K d_T(X, Y) + C.$$

This is the starting point of their proof that $\mathscr{C}^{(1)}(S)$ is δ-hyperbolic. (The constant δ has recently been shown to be independent of g, see [1, 4, 5] and [7].)

In this paper we consider the *optimal Lipschitz constant*

$$\kappa_g = \inf\{K \geqslant 0 \mid \mathrm{sys} \text{ is } (K, C)\text{–coarsely Lipschitz for some } C > 0\}.$$

We write $F(g) \asymp H(g)$ to mean that $F(g)/H(g)$ is bounded above and below by two positive constants, and prove the following theorem.

Theorem 1.1. *We have*

$$\kappa_g \asymp \frac{1}{\log(g)}.$$

This is a sharp version of the closed case of Theorem 1.4 of [1], which provides a Lipschitz constant that is independent of $\chi(S)$. An analogous result holds when hyperbolic length is replaced with extremal length; see Proposition 4.9.

The upper bound on κ_g is established by a careful version of Masur and Minsky's proof that sys is coarsely Lipschitz. To establish the lower bound, we construct a sequence of pseudo-Anosov mapping classes whose translation lengths on $\mathscr{T}(S)$ and $\mathscr{C}^{(1)}(S)$ behave like $\log(g)/g$ and $1/g$, respectively.

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2. A Lipschitz constant

Given the isotopy class $[f : S \to X]$ of a marked hyperbolic surface and the homotopy class of a curve α, we write $\ell_X(\alpha)$ for the hyperbolic length of α in $[f : S \to X]$. Let $\mathrm{sys}(X)$ denote the set of α in $\mathscr{C}^{(0)}(S)$ for which $\ell_X(\alpha)$ is minimal. If α, β are in $\mathrm{sys}(X)$, then the geometric intersection number $i(\alpha, \beta)$ is at most 1, and so the diameter of $\mathrm{sys}(X)$ in $\mathscr{C}^{(1)}(S)$ is at most 2. We abuse notation and view sys as a map from $\mathscr{T}(S)$ to $\mathscr{C}^{(1)}(S)$, although the image of X is actually a subset of diameter at most 2. One may obtain a *bona fide* map via the Axiom of Choice.

Given a hyperbolic surface X and a geodesic α on X, a *collar neighborhood of width* w about α is an $w/2$-neighborhood whose interior is homeomorphic to an open annulus. We denote this neighborhood $N_{w/2}(\alpha)$. We have the following lemma.

Lemma 2.1. *Given a closed hyperbolic surface X, if α lies in* $\mathrm{sys}(X)$*, then there is a collar neighborhood of α of width greater than $\ell_X(\alpha)/2$.*

Proof. Consider a maximal–width collar neighborhood $N_{w/2}(\alpha)$ of width w. This has a self–tangency on its boundary. From this one can construct a (non-geodesic) curve γ that runs a distance $w/2$ from one of the points of tangency to α, then at most half-way around α a distance at most $\ell_X(\alpha)/2$, and then a distance $w/2$ to the second point of tangency. Since α is a systole, we have

$$\ell_X(\alpha) \leqslant \ell_X(\gamma) < w + \ell_X(\alpha)/2.$$

So $w > \ell_X(\alpha)/2$ as required. □

Recall that a pair of isotopy classes of curves *fills* S if, whenever the curves are realized transversally, the complement of their union is a set of topological disks.

Lemma 2.2. *Given α and β in $\mathscr{C}^{(0)}(S)$ that fill the surface S, we have*

$$i(\alpha, \beta) \geqslant 2g - 1.$$

Proof. The union $\alpha \cup \beta$ is a graph on S with $i(\alpha, \beta)$ vertices and $2i(\alpha, \beta)$ edges. The complement is a union of $F \geqslant 1$ disks. Therefore

$$2g - 2 = -\chi(S) = -i(\alpha, \beta) + 2i(\alpha, \beta) - F = i(\alpha, \beta) - F \leqslant i(\alpha, \beta) - 1.$$

So $i(\alpha, \beta) \geqslant 2g - 1$ as required. □

We need Wolpert's inequality [13] describing change in lengths in terms of the Teichmüller distance.

Lemma 2.3 (Wolpert, Lemma 3.1 of [13])**.** *Given $X, Y \in \mathscr{T}(S)$ and a curve α on S we have*

$$\ell_Y(\alpha) \leqslant e^{d_{\mathscr{T}}(X,Y)} \ell_X(\alpha).$$

Our upper bound on κ_g now follows from the following proposition.

Proposition 2.4. *For $g \geqslant 2$ and all $X, Y \in \mathscr{T}(S_g)$ we have*

$$d_{\mathscr{C}}(\mathrm{sys}(X), \mathrm{sys}(Y)) \leqslant \frac{2}{\log(g - \frac{1}{2})} d_{\mathscr{T}}(X, Y) + 2.$$

We need the following lemma.

Lemma 2.5. *If $d_{\mathscr{T}}(X, Y) \leqslant \log(g - 1/2)$, then $d_{\mathscr{C}}(\mathrm{sys}(X), \mathrm{sys}(Y)) \leqslant 2$.*

Proof. Suppose that $d_{\mathscr{T}}(X,Y) \leqslant \log(g-1/2)$. Write $\alpha = \mathrm{sys}(X)$ and $\beta = \mathrm{sys}(Y)$, and, without loss of generality, assume that

$$\ell_X(\alpha) \leqslant \ell_Y(\beta).$$

According to Lemma 2.1, we have

$$\frac{i(\alpha,\beta)\ell_Y(\beta)}{2} < \ell_Y(\alpha).$$

On the other hand, Lemma 2.3 implies that

$$\ell_Y(\alpha) \leqslant e^{\log(g-1/2)}\ell_X(\alpha) = (g-1/2)\ell_X(\alpha) = \frac{(2g-1)}{2}\ell_X(\alpha).$$

Combining these two inequalities yields

$$i(\alpha,\beta) < \frac{2\ell_Y(\alpha)}{\ell_Y(\beta)} \leqslant \frac{(2g-1)\ell_X(\alpha)}{\ell_Y(\beta)} \leqslant 2g-1.$$

By Lemma 2.2, α and β cannot fill the surface S, and hence

$$d_{\mathscr{C}}(\mathrm{sys}(X),\mathrm{sys}(Y)) = d_{\mathscr{C}}(\alpha,\beta) \leqslant 2.$$

This proves the claim. □

Proof of Proposition 2.4. Now, given any two points X and Y in $\mathscr{T}(S)$, let n be the nonnegative integer such that

$$n\log(g-1/2) \leqslant d_{\mathscr{T}}(X,Y) < (n+1)\log(g-1/2).$$

Let $X = X_0,\dots,X_{n+1} = Y$ be a chain in $\mathscr{T}(S)$ with

$$d_{\mathscr{T}}(X_{k-1},X_k) \leqslant \log(g-1/2)$$

for each $1 \leqslant k \leqslant n+1$. By the triangle inequality and Lemma 2.5, we have

$$\begin{aligned} d_{\mathscr{C}}(\mathrm{sys}(X),\mathrm{sys}(Y)) &\leqslant \sum_{k=1}^{n+1} d_{\mathscr{C}}(\mathrm{sys}(X_{k-1}),\mathrm{sys}(X_k)) \\ &\leqslant 2(n+1) \\ &\leqslant \frac{2}{\log(g-1/2)} d_{\mathscr{T}}(X,Y) + 2 \end{aligned}$$

as required. □

3. Pseudo-Anosov maps

Given a pseudo-Anosov homeomorphism $f: S \to S$, we let $\lambda(f)$ denote the dilatation of f. We recall a few facts about pseudo-Anosov homeomorphisms, and refer the reader to the listed references for more detailed discussions.

3.1. Asymptotic translation length. Given a homeomorphism $f : S \to S$, the asymptotic translation length of f on $\mathscr{C}^{(1)}(S)$ is defined by

$$\ell_{\mathscr{C}}(f) = \liminf_{j\to\infty} \frac{d_{\mathscr{C}}(\alpha, f^j(\alpha))}{j},$$

where α is any simple closed curve. This is easily seen to be independent of α. When f is pseudo-Anosov, Masur and Minsky proved f has a quasi-invariant geodesic axis, and so this limit infimum is in fact a limit. Moreover, there is a $C > 0$ depending only on the genus of S such that $\ell_{\mathscr{C}}(f) \geqslant C$, see [8] or Corollary 1.5 of [3]. It follows from the definition that $\ell_{\mathscr{C}}(f^k) = k\ell_{\mathscr{C}}(f)$.

One can similarly define the asymptotic translation length of $f : S \to S$ acting on $\mathscr{T}(S)$. A pseudo-Anosov f has an axis in $\mathscr{T}(S)$ (see [2]), and the asymptotic translation length is just the translation length $\ell_{\mathscr{T}}(f)$. In fact, Bers' proof of Thurston's classification theorem shows that

$$\ell_{\mathscr{T}}(f) = \log(\lambda(f)).$$

The following lemma allows us to use asymptotic translation lengths to bound optimal Lipschitz constants.

Lemma 3.2. *For any pseudo-Anosov $f : S_g \to S_g$ we have*

$$\kappa_g \geqslant \frac{\ell_{\mathscr{C}}(f)}{\log(\lambda(f))}.$$

Proof. If $K, C > 0$ are such that sys is (K, C)–coarsely Lipschitz, then, for any X in $\mathscr{T}(S)$, we have

$$\begin{aligned}
\frac{\ell_{\mathscr{C}}(f)}{\log(\lambda(f))} &= \lim_{j\to\infty} \frac{d_{\mathscr{C}}(\mathrm{sys}(X), f^j(\mathrm{sys}(X)))}{d_{\mathscr{T}}(X, f^j(X))} \\
&= \lim_{j\to\infty} \frac{d_{\mathscr{C}}(\mathrm{sys}(X), \mathrm{sys}(f^j(X)))}{d_{\mathscr{T}}(X, f^j(X))} \\
&\leqslant \lim_{j\to\infty} \frac{K d_{\mathscr{T}}(X, f^j(X)) + C}{d_{\mathscr{T}}(X, f^j(X))} \\
&\leqslant K.
\end{aligned}$$

Since κ_g is the infimum of these K, the lemma is proven. □

3.3. Invariant train tracks for pseudo-Anosov maps. For more on train tracks, we refer the reader to [11], whose notation we adopt.

Given a pseudo-Anosov map $f : S \to S$, let τ denote an invariant train track. So τ carries $f(\tau)$, written $f(\tau) \prec \tau$, and a carrying map sends vertices of $f(\tau)$ to vertices of τ. Let P_τ denote the polyhedron of measures on τ, viewed either as the space of weights on the branches B of τ satisfying the switch conditions (a cone in $\mathbb{R}^B_{\geqslant 0}$), or a subset of the space $\mathscr{ML}(S)$ of measured laminations on S.

Although the carrying map is not unique, f induces a canonical linear inclusion $f_* : P_\tau \to P_\tau$. There is a unique eigenray in P_τ spanned by the stable lamination, and the corresponding eigenvalue is the dilatation $\lambda(f)$. In fact, this is the unique eigenray in all of $\mathbb{R}^B_{\geqslant 0}$ with eigenvalue greater than one.

Theorem 3.4. *If τ is an invariant train track for a pseudo-Anosov homeomorphism $f : S \to S$ with transition matrix A, then $\lambda(f)$ is the spectral radius of A.*

The dilatation $\lambda(f)$ is also the spectral radius of the matrix that defines the map

$$\mathbb{R}^B_{\geqslant 0} \to \mathbb{R}^B_{\geqslant 0},$$

induced by f. Furthermore, given any f-invariant subspace V of P_τ, the dilatation is the spectral radius of the matrix (with respect to any basis) defining the map $V \to V$ induced by f. If the matrix is a nonnegative integral matrix A, there is an associated directed graph, a *digraph*, with vertices the basis vectors, and A_{ij} edges from the i^{th} basis vector to the j^{th} basis vector.

3.5. Basic Nesting Lemma and lower bound for asymptotic translation length. A maximal train track τ is *recurrent* if there is some μ in P_τ that has positive weights on every branch. The set of such μ will be denoted $\mathrm{int}(P_\tau)$. A maximal train track τ is *transversely recurrent* if every branch intersects some closed curve that intersects τ efficiently. A train track that is both recurrent and transversely recurrent is called birecurrent.

For a maximal train track τ, Masur and Minsky observed that if α is a curve in $\mathrm{int}(P_\tau)$ and a curve β is disjoint from α, then β is in P_τ, see Observation 4.1 of [8]. From this they deduce the following proposition.

Proposition 3.6. *If τ is a maximal birecurrent invariant train track for a pseudo-Anosov $f : S \to S$ and $r \geqslant 1$ is such that $f^r(P_\tau) \subset int(P_\tau)$, then*

$$\ell_{\mathscr{C}}(f) \geqslant 1/r.$$

We call an r satisfying the conditions of Proposition 3.6 a *mixing number* for f and τ. In the next section, we construct a family of pseudo-Anosov maps $\phi_g : S_g \to S_g$ and maximal birecurrent invariant train tracks τ_g with mixing numbers $2g-1$.

4. Lower bound on κ_g.

We build a family of pseudo-Anosov maps $\{\phi_g : S_g \to S_g\}$ for which the asymptotic translation lengths on $\mathscr{T}(S_g)$ are on the order of $\log g/g$, while the asymptotic translation lengths on $\mathscr{C}^{(1)}(S_g)$ are bounded below by reciprocal of a linear function of g. The lower bound on κ_g in Theorem 1.1 follows from this and Lemma 3.2. Our construction is similar to Penner's [10], but the asymptotic behavior is different. In Penner's construction the translation lengths on $\mathscr{T}(S_g)$ are of the order $1/g$, while the asymptotic translation lengths on $\mathscr{C}^{(1)}(S_g)$ are of the order $1/g^2$ [6]. Consequently, Penner's construction gives a lower bound $1/g$ for κ_g, which is insufficient to prove Theorem 1.1.

Let $g \geqslant 4$ and consider the genus g surface $S = S_g$ with curves

$$\Omega = \Omega_g = \{a_0, \ldots, a_{g-2}, b_0, \ldots, b_{g-2}, c_0, \ldots, c_{g-2}, d_0, \ldots, d_{g-2}\}$$

as indicated in figure 1 when $g = 9$. For a curve x in Ω, let T_x be the left-handed Dehn twist in x. Let $\rho = \rho_g$ be the symmetry of order $g-1$ obtained by rotating S_g clockwise by $2\pi/(g-1)$, and let

$$\phi = \phi_g = \rho_g \circ T_{a_0} \circ T_{b_1} \circ T_{c_0} \circ T_{d_0}^{-1}.$$

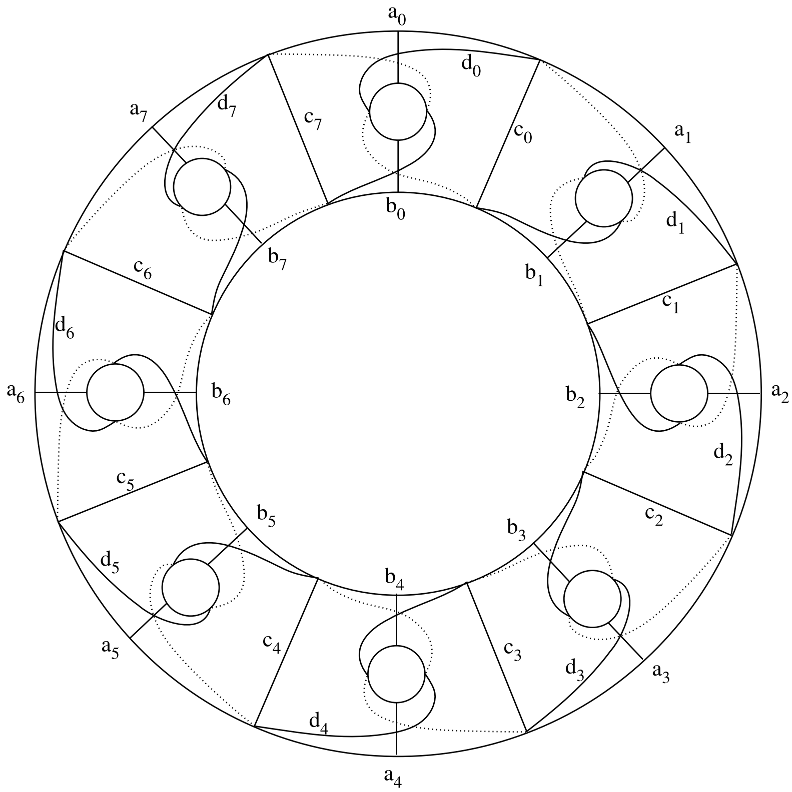

FIGURE 1. The pseudo-Anosov ϕ_9

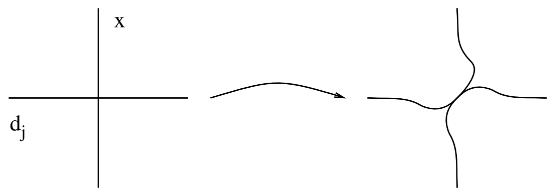

FIGURE 2. Smoothing the intersection points. Here x is some a_i, b_i, or c_i.

Observe that the only nonzero intersection numbers among curves in Ω are

$$i(d_j, a_j) = i(d_j, a_{j+1}) = i(d_j, b_j) = i(d_j, b_{j+1}) = 1 \text{ and } \; i(d_j, c_j) = 2$$

for $j \in \{0, \ldots, g-2\}$, where indices are taken modulo $g-1$. Smoothing intersection points as indicated in figure 2, we produce a maximal train track $\tau = \tau_g$. Each of the curves in Ω is carried by τ, proving that τ is recurrent, and these curves are elements of P_τ. Moreover, each of the curves can be pushed off τ to meet it efficiently, proving that τ is transversely recurrent. Let $P_\Omega \subset P_\tau$ be the subspace of measures carried by τ that lie in the span of Ω. Because no two curves of Ω put nonzero weights on the same set of branches, the set Ω is a basis for P_Ω.

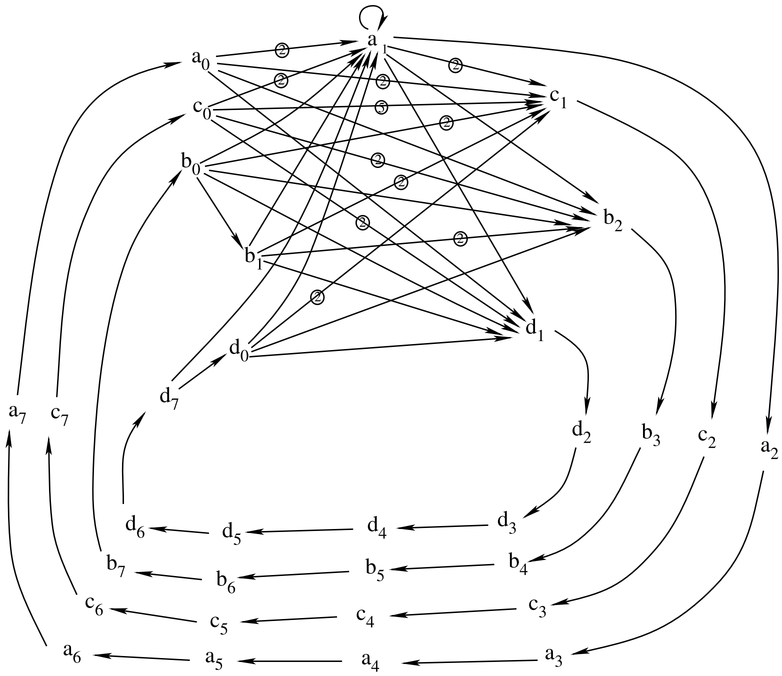

FIGURE 3. The digraph G_9.

Since Ω is ρ-invariant, we may assume that τ is. Furthermore, one has that $T_{a_j}(\tau)$, $T_{b_j}(\tau)$, $T_{c_j}(\tau)$, and $T_{d_j}^{-1}(\tau)$ are carried by τ for any j, as in [9]. In fact, we have $f(P_\Omega) \subset P_\Omega$ for any f in $\{\rho, T_{d_j}^{-1}, T_{a_j}, T_{b_j}, T_{c_j} \mid 0 \leqslant j \leqslant g-1\}$. It follows that $\phi(P_\Omega) \subset P_\Omega$ and, as in [10], ϕ is pseudo-Anosov. Let A denote the matrix for the action of ϕ on P_Ω in terms of the basis Ω. This is a Perron–Frobenius matrix whose associated digraph G_g is shown in figure 3 in the case $g = 9$. The vertices are labeled by the corresponding elements of Ω, and multiple edges are represented by an edge labeled with the multiplicity. An important feature is that G has exactly one self-loop, at the vertex a_1.

First, we bound the translation length on $\mathscr{C}^{(1)}(S)$ from below.

Proposition 4.4. *For every $g \geqslant 4$,*

$$\ell_{\mathscr{C}}(\phi_g) \geqslant \frac{1}{2g-1}.$$

Proof. By Proposition 3.6, it is enough to show that $r = 2g-1$ is a mixing number for ϕ and τ. We show this in two steps.

We first show that, for any $\mu \in P_\tau$, there is an $s \leqslant g$ so that $\phi^s(\mu) = ta_1 + \mu'$ for some $t > 0$ and $\mu' \in P_\tau$. Observe that μ has positive intersection number with some curve a_j or d_j. Indeed, if we push all of the a_j and d_j off of τ in both directions so as to meet it efficiently, then the union of these curves intersects every branch. Next, set $s_0 = g-1-j$, so that $1 \leqslant s_0 \leqslant g-1$. Then $\mu_{s_0} = \phi^{s_0}(\mu)$ has positive intersection number with either a_0 or d_0. From this we have

$$\begin{aligned}T_{a_0}T_{d_0}^{-1}(\mu_{s_0}) &= \mu_{s_0} + i(\mu_{s_0}, d_0)d_0 + i(\mu_{s_0} + i(\mu_{s_0}, d_0)d_0, a_0)a_0 \\ &= \mu_{s_0} + i(\mu_{s_0}, d_0)d_0 + (i(\mu_{s_0}, a_0) + i(\mu_{s_0}, d_0)i(d_0, a_0))\, a_0 \\ &= \mu_{s_0} + i(\mu_{s_0}, d_0)d_0 + (i(\mu_{s_0}, a_0) + i(\mu_{s_0}, d_0))\, a_0.\end{aligned}$$

Applying $\rho T_{b_1} T_{c_0}$ to this is the same as applying ϕ to μ_{s_0} since T_{a_0} commutes with $T_{b_1} T_{c_0}$. Therefore

$$\phi^{s_0+1}(\mu) = \phi(\mu_{s_0}) = ta_1 + \mu'$$

where

$$\begin{aligned}s &= s_0 + 1, \\ t &= i(\mu_{s_0}, a_0) + i(\mu_{s_0}, d_0) > 0, \quad \text{and} \\ \mu' &= \rho T_{b_1} T_{c_0}(\mu_{s_0} + i(\mu_{s_0}, d_0)d_0) \in P_\tau.\end{aligned}$$

The second step is to show that, for any $k \geqslant g-1$, we have $\phi^k(a_1) \in \mathrm{int}(P_\tau)$. This follows from the fact that, for any $k \geqslant g-1$, there is a path of length k from a_1 to any other vertex $x \in \Omega$; see figure 3.

From these two steps, we have

$$\begin{aligned}\phi^{2g-1}(\mu) &= \phi^{2g-1-s}(\phi^s(\mu)) \\ &= \phi^{2g-1-s}(ta_1 + \mu') \\ &= t\phi^{2g-1-s}(a_1) + \phi^{2g-1-s}(\mu').\end{aligned}$$

The iterate s from step one satisfies $2g - 1 - s \geqslant g - 1$. By step two, we know that the right–hand side lies in $\mathrm{int}(P_\tau) + P_\tau \subset \mathrm{int}(P_\tau)$. It follows that $\phi^{2g-1}(P_\tau) \subset \mathrm{int}(P_\tau)$ and so $2g - 1$ is a mixing number for ϕ and τ. □

4.5. Bounds on dilatations.

Lemma 4.6. *For $g > 4$, the mapping classes ϕ_g satisfy*

$$\frac{\log(4g-4)}{2g-2} \leqslant \log(\lambda(\phi_g)) \leqslant \frac{\log(10g-21)}{g-2}.$$

Proof. For any Perron–Frobenius digraph with n vertices, a self-loop, and directed diameter d, the logarithm of the leading eigenvalue is bounded below by $(\log n)/2d$ (see the proof of Proposition 2.4 of [12]). The digraph G_g that we consider has directed diameter $g - 1$, from which the lower bound follows.

For any $j \leqslant g-2$, inspection reveals that the number of directed edge-paths in G_g of length j emanating from each of

$$a_0,\ a_1,\ b_0,\ b_1,\ c_0,\ d_{g-2},\ \text{and } d_0$$

to be

$$(10j-6),\ 5j,\ (10j-1),\ 5j,\ (10j-6),\ (10j-11),\ \text{and } (5j-1),$$

respectively — see figure 3. For any other vertex v of G_g, there is a unique edge-path starting at v and ending at one of the vertices listed above, and every shorter edge-path is an initial segment of this one. It follows that the number of edge-paths of length $g-2$ starting at any vertex is maximized at one of the vertices listed above, and is hence at most $10g-21$.

Let A_g be the incidence matrix of G_g. The maximum row sum of A_g^{g-2} is precisely the maximum number of edge-paths starting at any vertex, and is hence at most $10g-21$. But the maximum row sum of a Perron–Frobenius matrix is an upper bound for its spectral radius. Applying this to A_g^{g-2} we have

$$\log(\lambda(\phi_g)) = \frac{\log(\lambda(\phi_g)^{g-2})}{g-2} = \frac{\log(\lambda(\phi_g^{g-2}))}{g-2} \leqslant \frac{\log(10g-21)}{g-2}. \qquad \square$$

4.7. The main theorem. We can now assemble the proof of the main theorem.

Proof of Theorem 1.1. Proposition 2.4 implies that

$$\kappa_g \leqslant \frac{2}{\log(g-\frac{1}{2})} \asymp \frac{1}{\log(g)}.$$

Lemma 3.2 applied to the sequence $\phi_g : S_g \to S_g$ above, together with Proposition 4.4 and the upper bound in Lemma 4.6, implies

$$\kappa_g \geqslant \frac{\ell_{\mathscr{C}}(\phi_g)}{\log(\lambda(\phi_g))} \geqslant \frac{1/(2g-1)}{\log(10g-21)/(g-2)} \asymp \frac{1}{\log(g)}. \qquad \square$$

4.8. Extremal length. Masur and Minsky [8] use extremal length rather than hyperbolic length to define the map $\mathscr{T}(S) \to \mathscr{C}^{(1)}(S)$. Recall that the extremal length of a curve α with respect to X in $\mathscr{T}(S)$ is $\mathrm{Ext}_X(\alpha) = 1/\mathrm{mod}_X(\alpha)$, where $\mathrm{mod}_X(\alpha)$ is the supremum of conformal moduli for embedded annuli with core curves homotopic to α. The set of curves with smallest extremal length,

$$\mathrm{sys}_{\mathrm{Ext}}(X) = \{\alpha \text{ in } \mathscr{C}^{(1)}(S) \mid \mathrm{Ext}_X(\alpha) \leqslant \mathrm{Ext}_X(\beta) \text{ for all } \beta \in \mathscr{C}^{(0)}(S)\},$$

is finite. As with hyperbolic length, the set $\mathrm{sys}_{\mathrm{Ext}}(X)$ has diameter bounded above by a constant $c = c(S)$ (Lemma 2.4 of [8]), and again we view $\mathrm{sys}_{\mathrm{Ext}}$ as a map $\mathscr{T}(S) \to \mathscr{C}^{(1)}(S)$. This map is also coarsely Lipschitz, and we let κ_g^{Ext} denote the optimal Lipschitz constant for $\mathrm{sys}_{\mathrm{Ext}} : \mathscr{T}(S_g) \to \mathscr{C}^{(1)}(S_g)$.

Proposition 4.9. *We have $\kappa_g = \kappa_g^{\mathrm{Ext}}$ for all g. In particular, $\kappa_g^{\mathrm{Ext}} \asymp \frac{1}{\log(g)}$.*

Proof. Suppose α in $\mathrm{sys}(X)$. The collar neighborhood of width $\ell_X(\alpha)/2$ from Lemma 2.1 provides a conformal annulus of definite modulus (depending on $\ell_X(\alpha)$), and hence $\mathrm{Ext}_X(\alpha) < L'$ for some $L' = L'(S)$. Now let β lie in $\mathrm{sys}_{\mathrm{Ext}}(X)$, so that $\mathrm{Ext}_X(\beta) \leqslant L'$. By Lemma 2.5 of [8], $d(\alpha, \beta) \leqslant 2L' + 1$. From this we deduce

$$|\mathrm{sys}(X) - \mathrm{sys}_{\mathrm{Ext}}(X)| < 2L' + 1.$$

Therefore, if one of sys or $\mathrm{sys}_{\mathrm{Ext}}$ is (K, C)-coarsely Lipschitz, then, by the triangle inequality, the other is $(K, C+2(2L'+1))$-coarsely Lipschitz. The proposition follows. $\square$

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