LIPSCHITZ CONSTANTS TO CURVE COMPLEXES

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Abstract. We determine the asymptotic behavior of the optimal Lipschitz constant for the systole map from Teichmüller space to the curve complex.

1. Introduction

Let $S = S_g$ be a closed surface of genus $g \geq 2$. We equip the Teichmüller space $\mathcal{T}(S)$ of $S$ with the Teichmüller metric, and equip the 1-skeleton $\mathcal{C}(1)(S)$ of the complex of curves $\mathcal{C}(S)$ with its usual path metric $d_\mathcal{C}$.

In [8], Masur and Minsky study the systole map $\text{sys} : \mathcal{T}(S) \to \mathcal{C}(1)(S)$, which assigns a hyperbolic metric one of its shortest curves, called a systole. They prove that $\text{sys}$ is $(K,C)$-coarsely Lipschitz for some $K,C > 0$, meaning that, for all $X$ and $Y$ in $\mathcal{T}(S)$

$$d_\mathcal{C}(\text{sys}(X), \text{sys}(Y)) \leq K d_T(X,Y) + C.$$ 

This is the starting point of their proof that $\mathcal{C}(1)(S)$ is $\delta$-hyperbolic. (The constant $\delta$ has recently been shown to be independent of $g$, see [1, 4, 5] and [7].)

In this paper we consider the optimal Lipschitz constant

$$\kappa_g = \inf\{K \geq 0 \mid \text{sys is } (K,C) \text{-coarsely Lipschitz for some } C > 0\}.$$ 

We write $F(g) \asymp H(g)$ to mean that $F(g)/H(g)$ is bounded above and below by two positive constants, and prove the following theorem.

**Theorem 1.1.** We have

$$\kappa_g \asymp \frac{1}{\log(g)}.$$ 

This is a sharp version of the closed case of Theorem 1.4 of [1], which provides a Lipschitz constant that is independent of $\chi(S)$. An analogous result holds when hyperbolic length is replaced with extremal length; see Proposition 4.9.

The upper bound on $\kappa_g$ is established by a careful version of Masur and Minsky’s proof that $\text{sys}$ is coarsely Lipschitz. To establish the lower bound, we construct a sequence of pseudo-Anosov mapping classes whose translation lengths on $\mathcal{T}(S)$ and $\mathcal{C}(1)(S)$ behave like $\log(g)/g$ and $1/g$, respectively.
2. A Lipschitz constant

Given the isotopy class $[f : S \to X]$ of a marked hyperbolic surface and the homotopy class of a curve $\alpha$, we write $\ell_X(\alpha)$ for the hyperbolic length of $\alpha$ in $[f : S \to X]$. Let $\text{sys}(X)$ denote the set of $\alpha$ in $C^{(0)}(S)$ for which $\ell_X(\alpha)$ is minimal. If $\alpha, \beta$ are in $\text{sys}(X)$, then the geometric intersection number $i(\alpha, \beta)$ is at most 1, and so the diameter of $\text{sys}(X)$ in $C^{(1)}(S)$ is at most 2. We abuse notation and view $\text{sys}$ as a map from $\mathcal{T}(S)$ to $C^{(1)}(S)$, although the image of $X$ is actually a subset of diameter at most 2. One may obtain a bona fide map via the Axiom of Choice.

Given a hyperbolic surface $X$ and a geodesic $\alpha$ on $X$, a collar neighborhood of width $w$ about $\alpha$ is an $w/2$-neighborhood whose interior is homeomorphic to an open annulus. We denote this neighborhood $N_{w/2}(\alpha)$. We have the following lemma.

**Lemma 2.1.** Given a closed hyperbolic surface $X$, if $\alpha$ lies in $\text{sys}(X)$, then there is a collar neighborhood of $\alpha$ of width greater than $\ell_X(\alpha)/2$.

**Proof.** Consider a maximal–width collar neighborhood $N_{w/2}(\alpha)$ of width $w$. This has a self–tangency on its boundary. From this one can construct a (non-geodesic) curve $\gamma$ that runs a distance $w/2$ from one of the points of tangency to $\alpha$, then a distance at most $\ell_X(\alpha)/2$, and then a distance $w/2$ to the second point of tangency. Since $\alpha$ is a systole, we have

$$\ell_X(\alpha) \leq \ell_X(\gamma) < w + \ell_X(\alpha)/2.$$ 

So $w > \ell_X(\alpha)/2$ as required. □

Recall that a pair of isotopy classes of curves fills $S$ if, whenever the curves are realized transversally, the complement of their union is a set of topological disks.

**Lemma 2.2.** Given $\alpha$ and $\beta$ in $C^{(0)}(S)$ that fill the surface $S$, we have

$$i(\alpha, \beta) \geq 2g - 1.$$ 

**Proof.** The union $\alpha \cup \beta$ is a graph on $S$ with $i(\alpha, \beta)$ vertices and $2i(\alpha, \beta)$ edges. The complement is a union of $F \geq 1$ disks. Therefore

$$2g - 2 = -\chi(S) = -i(\alpha, \beta) + 2i(\alpha, \beta) - F = i(\alpha, \beta) - F \leq i(\alpha, \beta) - 1.$$ 

So $i(\alpha, \beta) \geq 2g - 1$ as required. □

We need Wolpert’s inequality [13] describing change in lengths in terms of the Teichmüller distance.

**Lemma 2.3** (Wolpert, Lemma 3.1 of [13]). Given $X, Y \in \mathcal{T}(S)$ and a curve $\alpha$ on $S$ we have

$$\ell_Y(\alpha) \leq e^{d_{\mathcal{T}}(X,Y)}\ell_X(\alpha).$$ 

Our upper bound on $\kappa_g$ now follows from the following proposition.

**Proposition 2.4.** For $g \geq 2$ and all $X, Y \in \mathcal{T}(S_g)$ we have

$$d_{\mathcal{C}}(\text{sys}(X), \text{sys}(Y)) \leq \frac{2}{\log(g - 1/2)}d_{\mathcal{T}}(X, Y) + 2.$$ 

We need the following lemma.

**Lemma 2.5.** If $d_{\mathcal{T}}(X, Y) \leq \log(g - 1/2)$, then $d_{\mathcal{C}}(\text{sys}(X), \text{sys}(Y)) \leq 2$. 
Proof. Suppose that \( d_T(X,Y) \leq \log(g - 1/2) \). Write \( \alpha = \text{sys}(X) \) and \( \beta = \text{sys}(Y) \), and, without loss of generality, assume that
\[
\ell_X(\alpha) \leq \ell_Y(\beta).
\]
According to Lemma 2.1, we have
\[
\frac{i(\alpha, \beta)\ell_Y(\beta)}{2} < \ell_Y(\alpha).
\]
On the other hand, Lemma 2.3 implies that
\[
\ell_Y(\alpha) \leq e^{\log(g^{-1/2})} \ell_X(\alpha) = (g - 1/2)\ell_X(\alpha) = \frac{(2g - 1)}{2} \ell_X(\alpha).
\]
Combining these two inequalities yields
\[
i(\alpha, \beta) < \frac{2\ell_Y(\alpha)}{\ell_Y(\beta)} \leq \frac{(2g - 1)\ell_X(\alpha)}{\ell_Y(\beta)} \leq 2g - 1.
\]
By Lemma 2.2, \( \alpha \) and \( \beta \) cannot fill the surface \( S \), and hence
\[
d_C(\text{sys}(X), \text{sys}(Y)) = d_C(\alpha, \beta) \leq 2.
\]
This proves the claim. \( \square \)

Proof of Proposition 2.4. Now, given any two points \( X \) and \( Y \) in \( \mathcal{F}(S) \), let \( n \) be the nonnegative integer such that
\[
n \log(g - 1/2) \leq d_T(X,Y) < (n + 1) \log(g - 1/2).
\]
Let \( X = X_0, \ldots, X_{n+1} = Y \) be a chain in \( \mathcal{F}(S) \) with
\[
d_T(X_{k-1}, X_k) \leq \log(g - 1/2)
\]
for each \( 1 \leq k \leq n + 1 \). By the triangle inequality and Lemma 2.5, we have
\[
d_C(\text{sys}(X), \text{sys}(Y)) \leq \sum_{k=1}^{n+1} d_C(\text{sys}(X_{k-1}), \text{sys}(X_k))
\leq 2(n + 1)
\leq \frac{2}{\log(g - 1/2)} d_T(X,Y) + 2
\]
as required. \( \square \)

3. Pseudo-Anosov maps

Given a pseudo-Anosov homeomorphism \( f : S \to S \), we let \( \lambda(f) \) denote the dilatation of \( f \). We recall a few facts about pseudo-Anosov homeomorphisms, and refer the reader to the listed references for more detailed discussions.
3.1. Asymptotic translation length. Given a homeomorphism \( f : S \to S \), the asymptotic translation length of \( f \) on \( \mathcal{C}(S) \) is defined by
\[
\ell_{\mathcal{C}}(f) = \liminf_{j \to \infty} \frac{d_{\mathcal{C}}(\alpha, f^j(\alpha))}{j},
\]
where \( \alpha \) is any simple closed curve. This is easily seen to be independent of \( \alpha \). When \( f \) is pseudo-Anosov, Masur and Minsky proved \( f \) has a quasi-invariant geodesic axis, and so this limit infimum is in fact a limit. Moreover, there is a \( C > 0 \) depending only on the genus of \( S \) such that \( \ell_{\mathcal{C}}(f) \geq C \), see [8] or Corollary 1.5 of [3]. It follows from the definition that \( \ell_{\mathcal{C}}(f^k) = k\ell_{\mathcal{C}}(f) \).

One can similarly define the asymptotic translation length of \( f : S \to S \) acting on \( \mathcal{T}(S) \). A pseudo-Anosov \( f \) has an axis in \( \mathcal{T}(S) \) (see [2]), and the asymptotic translation length is just the translation length \( \ell_{\mathcal{T}}(f) \). In fact, Bers’ proof of Thurston’s classification theorem shows that \( \ell_{\mathcal{T}}(f) = \log(\lambda(f)) \).

The following lemma allows us to use asymptotic translation lengths to bound optimal Lipschitz constants.

**Lemma 3.2.** For any pseudo-Anosov \( f : S_g \to S_g \) we have
\[
\kappa_g \geq \frac{\ell_{\mathcal{C}}(f)}{\log(\lambda(f))}.
\]

**Proof.** If \( K, C > 0 \) are such that \( \text{sys} \) is \((K,C)\)-coarsely Lipschitz, then, for any \( X \) in \( \mathcal{T}(S) \), we have
\[
\frac{\ell_{\mathcal{C}}(f)}{\log(\lambda(f))} = \lim_{j \to \infty} \frac{d_{\mathcal{C}}(\text{sys}(X), f^j(\text{sys}(X)))}{d_{\mathcal{T}}(X, f^j(X))} = \lim_{j \to \infty} \frac{d_{\mathcal{C}}(\text{sys}(X), \text{sys}(f^j(X)))}{d_{\mathcal{T}}(X, f^j(X))} \leq K.
\]
Since \( \kappa_g \) is the infimum of these \( K \), the lemma is proven. \( \square \)

3.3. Invariant train tracks for pseudo-Anosov maps. For more on train tracks, we refer the reader to [11], whose notation we adopt.

Given a pseudo-Anosov map \( f : S \to S \), let \( \tau \) denote an invariant train track. So \( \tau \) carries \( f(\tau) \), written \( f(\tau) \prec \tau \), and a carrying map sends vertices of \( f(\tau) \) to vertices of \( \tau \). Let \( P_{\tau} \) denote the polyhedron of measures on \( \tau \), viewed either as the space of weights on the branches \( B \) of \( \tau \) satisfying the switch conditions (a cone in \( \mathbb{R}^B_{\geq 0} \)), or a subset of the space \( \mathcal{ML}(S) \) of measured laminations on \( S \).

Although the carrying map is not unique, \( f \) induces a canonical linear inclusion \( f_* : P_{\tau} \to P_{\tau} \). There is a unique eigenray in \( P_{\tau} \) spanned by the stable lamination, and the corresponding eigenvalue is the dilatation \( \lambda(f) \). In fact, this is the unique eigenray in all of \( \mathbb{R}^B_{\geq 0} \) with eigenvalue greater than one.
Theorem 3.4. If \( \tau \) is an invariant train track for a pseudo-Anosov homeomorphism \( f : S \to S \) with transition matrix \( A \), then \( \lambda(f) \) is the spectral radius of \( A \).

The dilatation \( \lambda(f) \) is also the spectral radius of the matrix that defines the map \( \mathbb{R}^B_{\geq 0} \to \mathbb{R}^B_{\geq 0} \), induced by \( f \). Furthermore, given any \( f \)-invariant subspace \( V \) of \( P_\tau \), the dilatation is the spectral radius of the matrix (with respect to any basis) defining the map \( V \to V \) induced by \( f \). If the matrix is a nonnegative integral matrix \( A \), there is an associated directed graph, a digraph, with vertices the basis vectors, and \( A_{ij} \) edges from the \( i \)th basis vector to the \( j \)th basis vector.

3.5. Basic Nesting Lemma and lower bound for asymptotic translation length. A maximal train track \( \tau \) is recurrent if there is some \( \mu \) in \( P_\tau \) that has positive weights on every branch. The set of such \( \mu \) will be denoted \( \text{int}(P_\tau) \). A maximal train track \( \tau \) is transversely recurrent if every branch intersects some closed curve that intersects \( \tau \) efficiently. A train track that is both recurrent and transversely recurrent is called birecurrent.

For a maximal train track \( \tau \), Masur and Minsky observed that if \( \alpha \) is a curve in \( \text{int}(P_\tau) \) and a curve \( \beta \) is disjoint from \( \alpha \), then \( \beta \) is in \( P_\tau \), see Observation 4.1 of [8]. From this they deduce the following proposition.

Proposition 3.6. If \( \tau \) is a maximal birecurrent invariant train track for a pseudo-Anosov \( f : S \to S \) and \( r \geq 1 \) is such that \( f^r(P_\tau) \subset \text{int}(P_\tau) \), then

\[
\ell_C(f) \geq 1/r.
\]

We call an \( r \) satisfying the conditions of Proposition 3.6 a mixing number for \( f \) and \( \tau \). In the next section, we construct a family of pseudo-Anosov maps \( \phi_g : S_g \to S_g \) and maximal birecurrent invariant train tracks \( \tau_g \) with mixing numbers \( 2g - 1 \).

4. Lower bound on \( \kappa_g \).

We build a family of pseudo-Anosov maps \( \{\phi_g : S_g \to S_g\} \) for which the asymptotic translation lengths on \( \mathcal{T}(S_g) \) are on the order of \( \log g/g \), while the asymptotic translation lengths on \( \mathcal{C}^{(1)}(S_g) \) are bounded below by reciprocal of a linear function of \( g \). The lower bound on \( \kappa_g \) in Theorem 1.1 follows from this and Lemma 3.2. Our construction is similar to Penner’s [10], but the asymptotic behavior is different. In Penner’s construction the translation lengths on \( \mathcal{T}(S_g) \) are of the order \( 1/g \), while the asymptotic translation lengths on \( \mathcal{C}^{(1)}(S_g) \) are of the order \( 1/g^2 \) [6]. Consequently, Penner’s construction gives a lower bound \( 1/g \) for \( \kappa_g \), which is insufficient to prove Theorem 1.1.

Let \( g \geq 4 \) and consider the genus \( g \) surface \( S = S_g \) with curves

\[
\Omega = \Omega_g = \{a_0, \ldots, a_{g-2}, b_0, \ldots, b_{g-2}, c_0, \ldots, c_{g-2}, d_0, \ldots, d_{g-2}\}
\]
as indicated in figure 1 when \( g = 9 \). For a curve \( x \) in \( \Omega \), let \( T_x \) be the left-handed Dehn twist in \( x \). Let \( \rho = \rho_g \) be the symmetry of order \( g - 1 \) obtained by rotating \( S_g \) clockwise by \( 2\pi/(g - 1) \), and let

\[
\phi = \phi_g = \rho_g \circ T_{a_0} \circ T_{b_1} \circ T_{c_0} \circ T_{d_0}^{-1}.
\]
Observe that the only nonzero intersection numbers among curves in $\Omega$ are

$$i(d_j, a_j) = i(d_j, a_{j+1}) = i(d_j, b_j) = i(d_j, b_{j+1}) = 1 \text{ and } i(d_j, c_j) = 2$$

for $j \in \{0, \ldots, g - 2\}$, where indices are taken modulo $g - 1$. Smoothing intersection points as indicated in figure 2, we produce a maximal train track $\tau = \tau_g$. Each of the curves in $\Omega$ is carried by $\tau$, proving that $\tau$ is recurrent, and these curves are elements of $P_\tau$. Moreover, each of the curves can be pushed off $\tau$ to meet it efficiently, proving that $\tau$ is transversely recurrent. Let $P_\Omega \subset P_\tau$ be the subspace of measures carried by $\tau$ that lie in the span of $\Omega$. Because no two curves of $\Omega$ put nonzero weights on the same set of branches, the set $\Omega$ is a basis for $P_\Omega$. 

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**Figure 1.** The pseudo-Anosov $\phi_9$

**Figure 2.** Smoothing the intersection points. Here $x$ is some $a_i$, $b_i$, or $c_i$. 

- $i(d_j, a_j) = i(d_j, a_{j+1}) = i(d_j, b_j) = i(d_j, b_{j+1}) = 1$ and $i(d_j, c_j) = 2$

for $j \in \{0, \ldots, g - 2\}$, where indices are taken modulo $g - 1$. Smoothing intersection points as indicated in figure 2, we produce a maximal train track $\tau = \tau_g$. Each of the curves in $\Omega$ is carried by $\tau$, proving that $\tau$ is recurrent, and these curves are elements of $P_\tau$. Moreover, each of the curves can be pushed off $\tau$ to meet it efficiently, proving that $\tau$ is transversely recurrent. Let $P_\Omega \subset P_\tau$ be the subspace of measures carried by $\tau$ that lie in the span of $\Omega$. Because no two curves of $\Omega$ put nonzero weights on the same set of branches, the set $\Omega$ is a basis for $P_\Omega$. 

Figure 3. The digraph $G_9$.

Since $\Omega$ is $\rho$-invariant, we may assume that $\tau$ is. Furthermore, one has that $T_{a_j}(\tau)$, $T_{b_j}(\tau)$, $T_{c_j}(\tau)$, and $T_{d_j}^{-1}(\tau)$ are carried by $\tau$ for any $j$, as in [9]. In fact, we have $f(P_\Omega) \subset P_\Omega$ for any $f$ in $\{\rho, T_{d_j}^{-1}, T_{a_j}, T_{b_j}, T_{c_j} | 0 \leq j \leq g - 1\}$. It follows that $\phi(P_\Omega) \subset P_\Omega$ and, as in [10], $\phi$ is pseudo-Anosov. Let $A$ denote the matrix for the action of $\phi$ on $P_\Omega$ in terms of the basis $\Omega$. This is a Perron–Frobenius matrix whose associated digraph $G_g$ is shown in figure 3 in the case $g = 9$. The vertices are labeled by the corresponding elements of $\Omega$, and multiple edges are represented by an edge labeled with the multiplicity. An important feature is that $G$ has exactly one self-loop, at the vertex $a_1$.

First, we bound the translation length on $\mathcal{C}^{(1)}(S)$ from below.

**Proposition 4.4.** For every $g \geq 4$,

$$\ell_{\mathcal{C}}(\phi_g) \geq \frac{1}{2g - 1}.$$

**Proof.** By Proposition 3.6, it is enough to show that $r = 2g - 1$ is a mixing number for $\phi$ and $\tau$. We show this in two steps.

We first show that, for any $\mu \in P_\tau$, there is an $s \leq g$ so that $\phi^s(\mu) = t a_1 + \mu'$ for some $t > 0$ and $\mu' \in P_\tau$. Observe that $\mu$ has positive intersection number with some curve $a_j$ or $d_j$. Indeed, if we push all of the $a_j$ and $d_j$ off of $\tau$ in both directions so as to meet it efficiently, then the union of these curves intersects every branch. Next, set $s_0 = g - 1 - j$, so that $1 \leq s_0 \leq g - 1$. Then $\mu_{s_0} = \phi^{s_0}(\mu)$ has positive intersection...
number with either $a_0$ or $d_0$. From this we have
\[
T_{a_0}^{-1}(\mu_{s_0}) = \mu_{s_0} + i(\mu_{s_0}, d_0)\delta_0 + (\mu_{s_0} + i(\mu_{s_0}, d_0)\delta_0, a_0)\delta_0
\]
\[
= \mu_{s_0} + i(\mu_{s_0}, d_0)\delta_0 + (i(\mu_{s_0}, a_0) + i(\mu_{s_0}, d_0)i(\delta_0, a_0))\delta_0
\]
\[
= \mu_{s_0} + i(\mu_{s_0}, d_0)\delta_0 + (i(\mu_{s_0}, a_0) + i(\mu_{s_0}, d_0))\delta_0.
\]
Applying $\rho T_b T_{c_0}$ to this is the same as applying $\phi$ to $\mu_{s_0}$ since $T_{a_0}$ commutes with $T_b T_{c_0}$. Therefore
\[
\phi^{s_0+1}(\mu) = \phi(\mu_{s_0}) = ta_1 + \mu'
\]
where
\[
s = s_0 + 1,
\]
\[
t = i(\mu_{s_0}, a_0) + i(\mu_{s_0}, d_0) > 0, \quad \text{and}
\]
\[
\mu' = \rho T_b T_{c_0}(\mu_{s_0} + i(\mu_{s_0}, d_0)) \in P_{1}.
\]

The second step is to show that, for any $k \geq g - 1$, we have $\phi^k(a_1) \in \text{int}(P_{1})$. This follows from the fact that, for any $k \geq g - 1$, there is a path of length $k$ from $a_1$ to any other vertex $x \in \Omega$; see figure 3.

From these two steps, we have
\[
\phi^{2g-1}(\mu) = \phi^{2g-1-s}(\phi^s(\mu))
\]
\[
= \phi^{2g-1-s}(ta_1 + \mu')
\]
\[
= t\phi^{2g-1-s}(a_1) + \phi^{2g-1-s}(\mu').
\]
The iterate $s$ from step one satisfies $2g - 1 - s \geq g - 1$. By step two, we know that the right-hand side lies in $\text{int}(P_{1}) + P_{1} \subset \text{int}(P_{1})$. It follows that $\phi^{2g-1}(P_{1}) \subset \text{int}(P_{1})$ and so $2g - 1$ is a mixing number for $\phi$ and $\tau$.

4.5. Bounds on dilatations.

Lemma 4.6. For $g > 4$, the mapping classes $\phi_g$ satisfy
\[
\frac{\log(4g - 4)}{2g - 2} \leq \log(\lambda(\phi_g)) \leq \frac{\log(10g - 21)}{g - 2}.
\]

Proof. For any Perron–Frobenius digraph with $n$ vertices, a self-loop, and directed diameter $d$, the logarithm of the leading eigenvalue is bounded below by $(\log n)/2d$ (see the proof of Proposition 2.4 of [12]). The digraph $G_g$ that we consider has directed diameter $g - 1$, from which the lower bound follows.

For any $j \leq g - 2$, inspection reveals that the number of directed edge-paths in $G_g$ of length $j$ emanating from each of $a_0, a_1, b_0, b_1, c_0, d_{g-2},$ and $d_0$

to be
\[
(10j - 6), 5j, (10j - 1), 5j, (10j - 6), (10j - 11), \text{and } (5j - 1),
\]
respectively — see figure 3. For any other vertex $v$ of $G_g$, there is a unique edge-path starting at $v$ and ending at one of the vertices listed above, and every shorter edge-path is an initial segment of this one. It follows that the number of edge-paths of
length \( g - 2 \) starting at any vertex is maximized at one of the vertices listed above, and is hence at most \( 10g - 21 \).

Let \( A_g \) be the incidence matrix of \( G_g \). The maximum row sum of \( A_g^{g-2} \) is precisely the maximum number of edge-paths starting at any vertex, and is hence at most \( 10g - 21 \). But the maximum row sum of a Perron–Frobenius matrix is an upper bound for its spectral radius. Applying this to \( A_g^{g-2} \) we have

\[
\log(\lambda(\phi_g)) = \frac{\log(\lambda(\phi_g)^{g-2})}{g-2} = \frac{\log(\lambda(\phi_g^{g-2}))}{g-2} \leq \frac{\log(10g - 21)}{g-2}.
\]

4.7. The main theorem. We can now assemble the proof of the main theorem.

Proof of Theorem 1.1. Proposition 2.4 implies that

\[
\kappa_g \leq \frac{2}{\log(g - \frac{1}{2})} \times \frac{1}{\log(g)}.
\]

Lemma 3.2 applied to the sequence \( \phi_g : S_g \to S_g \) above, together with Proposition 4.4 and the upper bound in Lemma 4.6, implies

\[
\kappa_g \geq \frac{\ell_g(\phi_g)}{\log(\lambda(\phi_g))} \geq \frac{1/(2g - 1)}{\log(10g - 21)/(g - 2)} \times \frac{1}{\log(g)}.
\]

4.8. Extremal length. Masur and Minsky [8] use extremal length rather than hyperbolic length to define the map \( \mathcal{F}(S) \to \mathcal{C}^{(1)}(S) \). Recall that the extremal length of a curve \( \alpha \) with respect to \( X \) in \( \mathcal{F}(S) \) is \( \operatorname{Ext}_X(\alpha) = 1/\operatorname{mod}_X(\alpha) \), where \( \operatorname{mod}_X(\alpha) \) is the supremum of conformal moduli for embedded annuli with core curves homotopic to \( \alpha \). The set of curves with smallest extremal length,

\[
\text{sys}_{\operatorname{Ext}}(X) = \{ \alpha \in \mathcal{C}^{(1)}(S) \mid \operatorname{Ext}_X(\alpha) \leq \operatorname{Ext}_X(\beta) \text{ for all } \beta \in \mathcal{C}^{(0)}(S) \},
\]

is finite. As with hyperbolic length, the set \( \text{sys}_{\operatorname{Ext}}(X) \) has diameter bounded above by a constant \( c = c(S) \) (Lemma 2.4 of [8]), and again we view \( \text{sys}_{\operatorname{Ext}} \) as a map \( \mathcal{F}(S) \to \mathcal{C}^{(1)}(S) \). This map is also coarsely Lipschitz, and we let \( \kappa^\operatorname{Ext}_g \) denote the optimal Lipschitz constant for \( \text{sys}_{\operatorname{Ext}} : \mathcal{F}(S_g) \to \mathcal{C}^{(1)}(S_g) \).

Proposition 4.9. We have \( \kappa_g = \kappa^\operatorname{Ext}_g \) for all \( g \). In particular, \( \kappa^\operatorname{Ext}_g \asymp \frac{1}{\log(g)} \).

Proof. Suppose \( \alpha \) in \( \text{sys}(X) \). The collar neighborhood of width \( \ell_X(\alpha)/2 \) from Lemma 2.1 provides a conformal annulus of definite modulus (depending on \( \ell_X(\alpha) \)), and hence \( \operatorname{Ext}_X(\alpha) < L' \) for some \( L' = L'(S) \). Now let \( \beta \) lie in \( \text{sys}_{\operatorname{Ext}}(X) \), so that \( \operatorname{Ext}_X(\beta) \leq L' \). By Lemma 2.5 of [8], \( d(\alpha, \beta) \leq 2L' + 1 \). From this we deduce

\[
|\text{sys}(X) - \text{sys}_{\operatorname{Ext}}(X)| < 2L' + 1.
\]

Therefore, if one of \( \text{sys} \) or \( \text{sys}_{\operatorname{Ext}} \) is \((K, C)\)-coarsely Lipschitz, then, by the triangle inequality, the other is \((K, C + 2(2L' + 1))\)-coarsely Lipschitz. The proposition follows.

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