STATISTICAL HYPERBOLICITY FOR HARMONIC MEASURE

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ABSTRACT. We consider harmonic measures that arise from a finitely supported random walk on the mapping class group whose support generates a non-elementary subgroup. We prove that Teichmüller space with the Teichmüller metric is statistically hyperbolic for such a harmonic measure.

1. INTRODUCTION

The notion of *statistical hyperbolicity*, introduced by Duchin-Lelièvre-Mooney [5], encapsulates whether a space is on average hyperbolic at large scales, that is, whether as $r \to \infty$ the average distance between pairs of points on a sphere of radius r based at any point in the space is $2r$. To make sense of the average distance, one requires a reasonable measure on spheres.

For many Lebesgue-class measures on Teichmüller space, Dowdall-Duchin-Masur showed that Teichmüller space with the Teichmüller metric is statistically hyperbolic. See [4, Theorems B, C and D]. See also [14]. Here, we consider the same question for harmonic measures that arise from finitely supported random walks on the mapping class group.

Kaimanovich-Masur showed that a random walk on the mapping class group, whose initial support generates a non-elementary subgroup, converges to the Thurston boundary of Teichmüller space with probability one. This defines a harmonic measure on the Thurston boundary and Kaimanovich-Masur showed that this measure is supported on the set of uniquely ergodic measured foliations. See [11, Theorem 2.2.4] for both statements. Since Teichmüller rays with uniquely ergodic vertical foliations asymptotically converge to this vertical foliation, it is possible to pull back the harmonic measure to the unit cotangent space at a base-point. This allows us to equip spheres in Teichmüller space with a harmonic measure. We can then consider the question of whether Teichmüller space is statistically hyperbolic with respect to these measures.

Our main theorem is:

Theorem 1.1. *Let S be a surface of finite type. Let μ be a finitely supported probability distribution on the mapping class group $\mathrm{Mod}(S)$ such that the support generates a non-elementary subgroup. Then $\mathcal{T}(S)$ is statistically hyperbolic with respect to the harmonic measure defined by the μ-random walk on $\mathrm{Mod}(S)$.*

When S is a torus or a torus with one marked point or a sphere with four marked points, the Teichmüller space $\mathcal{T}(S)$ with the Teichmüller metric is isometric to $\mathbb{H}$. By a theorem of Guivarch-LeJan [10], the harmonic measure from the μ-random walk is singular with respect to the Lebesgue measure class. See also [1], [3] and [8] for other proofs. When the complex dimension is greater than one, the harmonic measure from the μ-random walk is also singular with respect to the Lebesgue measure class. See [7, Theorem 1.1]. As a result, Thereom 1.1 is distinct from the main results of Dowdall-Duchin-Masur [4] and, as we outline below, requires different tools.

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We will first present the proof of Theorem 1.1 when the complex dimension of $\mathcal{T}(S)$ is greater than one. This is the harder case. For the exceptional surfaces, that is when $\mathcal{T}(S) = \mathbb{H}$, the proof of Theorem 1.1 is obviously easier because the ambient geometry is already hyperbolic. However, as mentioned above, the harmonic measure is singular. So there is something to prove. The argument is straightforward and uses the geodesic separation property for harmonic measure that is already formulated in the proof of the harder case of Theorem 1.1.

In fact, we present the exceptional case as a special case of a more general theorem when the ambient geometry is hyperbolic. We prove:

Theorem 1.2. *Let Γ be a non-uniform lattice in* $\text{Isom}(\mathbb{H}^n)$ *for $n \geqslant 2$. Let μ be a finitely supported probability distribution on Γ such that the support of μ generates a subgroup that contains a pair of loxodromic elements with distinct axes. Then with respect to the harmonic measure defined by the μ-random walk on Γ the space $\mathbb{H}^n$ with the hyperbolic metric is statistically hyperbolic.*

We will present the proof of Theorem 1.2 after the proof of the harder case of Theorem 1.1. This lets us use the geodesic separation property for the harmonic measure formulated in the earlier proof. We note that when $n > 2$, Randecker-Tiozzo proved that the harmonic measure is singular when the support of μ generates Γ. See [19, Theorem 2]. So Theorem 1.2 has some new content.

From now on we assume that the complex dimension of $\mathcal{T}(S)$ is greater than one and present Theorem 1.1 with that assumption.

1.3. **Strategy of the proof.** To derive statistical hyperbolicity, Dowdall-Duchin-Masur set up two properties to check. The first property is called the thickness property. See [4, Definition 5.4]. It states that as the radius of a sphere goes to infinity a typical radial geodesic segment spends a definite proportion of its time in the thick part of Teichmüller space. The second property is called the separation property. See [4, Definition 6.1]. It states that as the radius of a sphere goes to infinity a typical pair of radial geodesic segments exhibit good separation. For Lebesgue-class visual measures, the ergodicity of the Teichmüller geodesic flow is the key tool for the thickness property. For rotationally invariant Lebesgue measures, they verify the separation property by disintegrating the measure along and transverse to Teichmüller discs and then use the hyperbolic geometry of these discs.

For random walks, different tools are needed. The main tool is the ergodicity of the shift map on the space of bi-infinite sample paths. The ergodicity can be leveraged to prove that a typical bi-infinite sample path recurs to a neighbourhood of its tracked geodesic with a positive asymptotic frequency. Since sample paths lie in a thick part, the recurrence implies that tracked geodesics spend a positive proportion of their time in a thick part. By tweaking the size of the neighbourhood, and hence the thick part, we show that any positive proportion for the time spent in it by the tracked geodesic can be achieved. While a positive proportion of thickness is suggested directly by the main theorem in [9], the precise quantitative version that we need here requires some work.

For the separation property, we project two fellow travelling radial geodesic segments to the curve complex. By a theorem of Maher, a typical sample path makes linear progress in the curve complex. Combining this theorem with the recurrence, we show that the projections of fellow travelling radial geodesic segments must nest in to a shadow. Maher also shows that the harmonic measure of nested shadows decays exponentially to zero. This then enables us to conclude the required separation property.

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2. PRELIMINARIES

2.1. **Statistical hyperbolicity:** Let (X, d) be a metric space. Let $x \in X$. Let $r > 0$ We call the set $S_r(x) = \{x' \in X \text{ such that } d(x, x') = r\}$ the *sphere* of radius r centred at x. Suppose ν_r is a family of probability measures supported on $S_r(x)$. Provided the limit exists, one defines a numerical index $E(X) := E(X, x, d, \{\nu_r\})$ by

$$E(X) = \lim_{r \to \infty} \frac{1}{r} \int_{S_r(x) \times S_r(x)} d(x', x'')\, d\nu_r(x') d\nu_r(x'')$$

A space is said to be *statistically hyperbolic* if $E(X) = 2$. This is motivated by the fact that $E(\mathbb{H}^n) = 2$ for any dimension equipped with the natural measures on spheres. Moreover, it was demonstrated by Duchin-Lelièvre-Mooney [5, Theorem 4] that $E(G) = 2$ for any non-elementary hyperbolic group G with any choice of generating set. We direct the reader to [5] for further discussion on the sensitivity of E. Indeed, it is not quasi-isometrically invariant, and has dependence on the base-point x and the choice of measures v_r. Furthermore, δ-hyperbolicity and exponential volume growth are not sufficient to guarantee statistical hyperbolicity.

2.2. **Background on Teichmüller spaces.** Let S be a surface of finite type, that is, S is an oriented surface with finite genus and finitely many marked points. The *Teichmüller space* $\mathcal{T}(S)$ of S is the space of marked conformal structures on S. By the Uniformisation Theorem, if S has negative Euler characteristic then there is a unique hyperbolic metric in each marked conformal class. The *mapping class group* $\operatorname{Mod}(S)$ is the group of orientation preserving diffeomorphisms of S modulo isotopy. The mapping class group $\operatorname{Mod}(S)$ acts on $\mathcal{T}(S)$ by changing the marking. The quotient space $\mathcal{M}(S) = \mathcal{T}(S)/\operatorname{Mod}(S)$ is the *moduli space of Riemann surfaces*. Given $\epsilon > 0$, a marked hyperbolic surface $x \in \mathcal{T}(S)$ is *ϵ-thick* if the hyperbolic length of every closed geodesic on x is at least ϵ. We let $\mathcal{T}_\epsilon(S)$ be the subset of $\mathcal{T}(S)$ consisting of all ϵ-thick marked hyperbolic surfaces. We note that there exists an $\epsilon > 0$ that depends only on the complexity of the surface such that $\mathcal{T}_\epsilon(S)$ is non-empty. See the discussion on the Bers constant in [6, Chapter 12 Section 4.2]. Observe that if x is ϵ-thick then so is gx for any mapping class g. Hence, we deduce that we get a thick-thin decomposition of the moduli space $\mathcal{M}(S)$. The Mumford compactness theorem says that for any $\epsilon > 0$, the thick part $\mathcal{M}_\epsilon(S)$ is compact.

Given a marked conformal surface x, let $\mathcal{Q}(x)$ be the set of meromorphic quadratic differentials on x with simple poles at and only at the marked points. This gives a bundle $\mathcal{Q}$ over $\mathcal{T}(S)$. This bundle is stratified by the orders of the zeroes of the quadratic differentials. By contour integration and a choice of square root, each $q \in \mathcal{Q}(X)$ defines a half-translation structure on S. That is, it defines charts to $\mathbb{C} = \mathbb{R}^2$ with half-translation transition functions of the form $z \to \pm z + c$. It then makes senses to impose the condition that the half-translation surfaces that we consider have unit area. Given a stratum of quadratic differentials, one may fix a basis for the homology of S relative to the marked points and zeroes. One can associate a period to each basis element of the homology by integrating a square root of the quadratic differential over a contour representing it. These periods give local co-ordinates on the stratum and can be used to define the Lebesgue measure class on it. The principal stratum is the stratum of quadratic differentials whose zeroes are all simple. The Lebesgue measure class on the principal stratum can be pushed down to define a Lebesgue measure class on the Teichmüller space $\mathcal{T}(S)$.

The $SL(2, \mathbb{R})$-action on $\mathbb{R}^2$ preserves the area and also the form of the transition functions. Hence, it descends to an action on $\mathcal{Q}$. The compact part $SO(2, \mathbb{R})$ acts by rotations and preserves the conformal structure. The diagonal part of the action given by

$$\begin{bmatrix} e^{t/2} & 0 \\ 0 & e^{-t/2} \end{bmatrix}$$

is called the *Teichmüller flow* and we will denote it by ϕ_t. Given a pair x, y of marked hyperbolic surfaces, Teichmüller's theorem states that there is a unit area quadratic differential q on x and a time t such that $\phi_t q$ projects to y. The time t is called the *Teichmüller distance* between x and y.

The Teichmüller distance gives a Finsler metric on $\mathcal{T}(S)$ which we will call the *Teichmüller metric* and denote by d_{Teich}. The mapping class group acts by isometries and hence the Teichmüller distance descends to $\mathcal{M}$. Masur-Wolf [16] showed that Teichmüller space with the Teichmüller metric is not a δ-hyperbolic space. This adds value to the question of whether $\mathcal{T}(S)$ is statistically hyperbolic. Dowdall-Duchin-Masur showed that for measures in the Lebesgue-class, $\mathcal{T}(S)$ is statistically hyperbolic. See [4, Theorems B, C and D].

2.3. **Random walks on the mapping class group.** Let G be a finitely generated group. Let μ be a probability measure on G. A sample path w_n of length n for the μ-random walk on G is a random product $w_n = g_1 g_2 \cdots g_n$ where each g_i is sampled by μ. The n-fold convolution $\mu^{(n)}$ of μ gives the distribution of w_n. If G has an action on a space X, one can use the orbit of a base-point to project the random walk onto X. We are interested in limiting behaviour of sample paths as $n \to \infty$. For this reason, we will consider the shift on $G^{\mathbb{N}}$. It is convenient to consider both forward and backward random walks. The backward random walk is simply the random walk with respect to the reflected measure $\hat{\mu}$ defined by $\hat{\mu}(g) = \mu(g^{-1})$. We then consider bi-infinite sequences as elements of $G^{\mathbb{Z}}$ with the shift acting as a step of the random walk. There is a measure h on $G^{\mathbb{Z}}$ such that the conditional measure for the shift is given by μ. We can separate the forward and backward directions to write h as the product $\nu \times \hat{\nu}$. We call the measure ν the harmonic measure.

By the Nielsen-Thurston classification, a mapping class is finite order, reducible or pseudo-Anosov. A finite order mapping class is an automorphism of some Riemann surface. A reducible mapping class has some power that fixes a multi-curve on the surface. A pseudo-Anosov mapping class f has a Teichmüller axis: an f-invariant bi-infinite Teichmüller geodesic along which the map translates realising the infimum of $d_{\text{Teich}}(x, f(x))$ over $\mathcal{T}(S)$ by this translation. This description of a pseudo-Anosov map implies that the Teichmüller axis is unique and that its vertical and horizontal measured foliations are uniquely ergodic.

A subgroup of $\text{Mod}(S)$ is non-elementary if it contains a pair of pseudo-Anosov mapping classes with distinct Teichmüller axes. Let $x \in \mathcal{T}(S)$ be a base-point. Kaimanovich-Masur showed that if the support of a probability distribution μ on $\text{Mod}(S)$ generates a non-elementary subgroup then for h-almost every sample path $\omega = (w_n)$ the sequence $w_n x$ converges to a projective class of a measure foliation on S. See [11, Theorem 2.2.4]. Thurston showed that there is a natural way in which the space $\text{PMF}(S)$ of projective classes of measured foliations serves as a boundary of $\mathcal{T}(S)$. So the theorem of Kaimanovich-Masur can be rephrased as the convergence to the boundary $\text{PMF}(S)$ for h-almost every sample path. In particular, the measure ν on $\text{Mod}(S)^{\mathbb{N}}$ pushes forward to a measure on $\text{PMF}(S)$. We call this the harmonic measure on $\text{PMF}(S)$. In fact, if μ has finite entropy and finite first logarithmic moment with respect to the Teichmüller metric then the push-forward measure is measurably isomorphic to ν. See [11, Theorem 2.3.1]. For this reason, and to keep the notation simple, we denote the measure on $\text{PMF}(S)$ by ν.

2.4. **Statistical hyperbolicity for a harmonic measure.** Let $\mathcal{Q}^1(x)$ be the set of unit area quadratic differentials for the marked Riemann surface x. When $q \in \mathcal{Q}^1(x)$ has a uniquely ergodic vertical foliation, the Teichmüller ray $\phi_t q$ converges as $t \to \infty$ to the projective class of the vertical foliation. Since ν is supported on the set of uniquely ergodic foliations, ν can be pulled back to a measure on $\mathcal{Q}^1(x)$. This gives us a measure on every sphere $S_r(x)$. Thus, it makes sense to consider whether $\mathcal{T}(S)$ with the Teichmüller metric is statistically hyperbolic with respect to harmonic measure.

2.5. **Statistical hyperbolicity in Teichmüller space.** Dowdall-Duchin-Masur [4] reduce statistical hyperbolicity of Teichmüller space with the Teichmüller metric for a family of measures $\{\nu_r\}$ to the verification of two properties: the thickness property [4, Definition 5.2] and the separation property [4, Definition 6.1]. We will now state these properties and in Section 4, we will give a quick sketch of how these properties imply statistical hyperbolicity.

For a choice of $\epsilon > 0$ and a geodesic segment $[x, x'] \subset \mathcal{T}(S)$, we denote the proportion of time $[x, x']$ spends in $\mathcal{T}_\epsilon(S)$ by

$$\mathrm{Thick}^{\%}_{\epsilon}([x,x']) := \frac{|\{0 \leqslant t \leqslant d_{\mathrm{Teich}}(x,x') \ : \ x'_t \in \mathcal{T}_\epsilon(S)\}|}{d_{\mathrm{Teich}}(x,x')},$$

where x'_t is the point at distance t from x along $[x, x']$.

The thickness property is the following.

Definition 2.6 (Thickness property)**.** *A family of measures $\{\nu_r\}$ on spheres in $\mathcal{T}(S)$ has the thickness property if for all $0 < \theta, \eta < 1$ there exists an $\epsilon > 0$ such that*

$$\lim_{r\to\infty} \frac{\nu_r\left(\{x' \in S_r(x) \mid \mathrm{Thick}^{\%}_{\epsilon}([x,x'_t]) \geqslant \theta \textit{ for all } t \in [\eta r, r]\}\right)}{\nu_r(S_r(x))} = 1,$$

for all $x \in \mathcal{T}(S)$.

The separation property is the following.

Definition 2.7 (Separation property)**.** *A family of measures $\{\nu_r\}$ on spheres in $\mathcal{T}(S)$ has the separation property if for all $M > 0$ and $0 < \eta < 1$, we have*

$$\lim_{r\to\infty} \frac{\nu_r \times \nu_r\left(\{(x',x'') \in S_r(x) \times S_r(x) \mid d_{\mathrm{Teich}}(x'_t, x''_t) \geqslant M \textit{ for all } t \in [\eta r, r]\}\right)}{\nu_r \times \nu_r(S_r(x) \times S_r(x))} = 1,$$

for all $x \in \mathcal{T}(S)$.

In the next section, we derive these properties for a harmonic measure that arises from a finitely supported random walk on the mapping class group.

3. Derivation of the thickness and separation properties

3.1. **Recurrence.** Let x be a base-point in Teichmüller space. Let ω be a bi-infinite sample path. As a convenient notation, we let $x_n = w_n x$ for any $n \in \mathbb{Z}$. For almost every ω, the sequences x_n and x_{-n} as $n \to \infty$ converge projectively to distinct uniquely ergodic measured foliations λ^+ and λ^- respectively. For such sample paths, let γ_ω be the bi-infinite Teichmüller geodesic between λ^+ and λ^-. As convenient notation, let $\gamma = \gamma_\omega$ and let γ_n be a point of γ that is closest to x_n.

Let

$$\Lambda_R = \{\omega \text{ such that } d_{\mathrm{Teich}}(x, \gamma_\omega) < R/3\}$$

By [11, Lemma 1.4.4], the function $\omega \to d_{\mathrm{Teich}}(x, \gamma_\omega)$ is measurable. So if R is large enough then $h(\Lambda_R) > 0$ and $h(\Lambda_R) \to 1$ as $R \to \infty$.

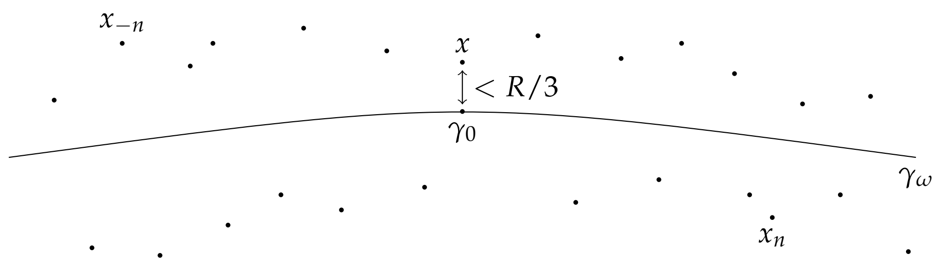

FIGURE 3.2. A sample path ω in Λ_R.

An integer k will be called an *R-recurrence time* for ω if $\sigma^k\omega \in \Lambda_R$. Suppose $j < k$ are R-recurrence times for ω and suppose $d(x_j, x_k) = 2d$ for some $d \geqslant 0$. Let

$$D = \max_{g \in \mathrm{supp}(\mu)} d_{\mathrm{Teich}}(x, gx).$$

We deduce the bound $2d \leqslant (k-j)D$.

We let $[\gamma_j, \gamma_k]$ be the segment of γ_ω connecting γ_j and γ_k. We note that

$$\mathrm{Length}[\gamma_j, \gamma_k] \leqslant 2R + 2d.$$

If $2d \leqslant 2R$, then

$$[\gamma_j, \gamma_k] \subset B(x_j, 3R) \cup B(x_k, 3R).$$

So suppose $2d > 2R$. We consider the sub-segments of $[\gamma_j, \gamma_k]$ that might be outside of the union $B(x_j, 3R) \cup B(x_k, 3R)$. We call the union of these sub-segments C_{jk}. Note that the total length $L(j,k)$ of the sub-segments in C_{jk} is at most $2d - 2R$.

Let $0 < \rho < p < 1$. We choose R large enough such that $h(\Lambda_R) \geqslant p$. Let $n \in \mathbb{N}$ and set

$$E_n^{(1)} = \left\{ \omega \text{ such that } \frac{1}{m} \sum_{0 \leqslant k \leqslant m} \chi_R(\sigma^k \omega) < p - \rho \text{ for some } m \geqslant n \right\}.$$

By ergodicity of the shift map σ it follows that $h(E_n^{(1)}) \to 0$ as $n \to \infty$.

Suppose ω is in the complement of $E_n^{(1)}$. Then the number of times $i \in \{0, \cdots, n\}$ such that $\sigma^i\omega \notin \Lambda_R$ is at most $(1 + \rho - p)n$. Let $j_{\min}$ and $j_{\max}$ be the smallest and largest R-recurrence times in $\{0, \ldots, n\}$. Then we note that

$$j_{\min} \leqslant (1 + \rho - p)n \quad \text{and} \quad j_{\max} \geqslant n - (1 + \rho - p)n.$$

Setting $x_0 = x$ we get the estimates

$$d_{\mathrm{Teich}}(x_0, x_{j_{\min}}) \leqslant (1 + \rho - p)nD \quad \text{and} \quad d_{\mathrm{Teich}}(x_n, x_{j_{\max}}) \leqslant (1 + \rho - p)nD.$$

This implies

$$d_{\mathrm{Teich}}(\gamma_0, \gamma_{j_{\min}}) \leqslant 2(1 + \rho - p)nD + 2R \quad \text{and} \quad d_{\mathrm{Teich}}(\gamma_n, \gamma_{j_{\max}}) \leqslant 2(1 + \rho - p)nD + 2R.$$

A pair $j < k$ of recurrence times is *consecutive* if every J satisfying $j < J < k$ is not a recurrence time. We deduce that

$$\sum_{\substack{j<k \\ \text{consecutive}}} L(j,k) \leqslant \sum_{\substack{j<k \\ \text{consecutive}}} (k-j)D - 2R \leqslant 2(1 + \rho - p)nD.$$

3.3. **Linear progress.** The Teichmüller metric is sub-additive along sample paths. By Kingman's sub-additive ergodic theorem, there exists a constant $A \geqslant 0$ such that for almost every sample path ω we have

$$\lim_{n\to\infty} \frac{d_{\text{Teich}}(x_0, x_n)}{n} = A.$$

By [12, Theorem 1.1], $A > 0$.

Let $0 < a < 1$ be a constant smaller than A. Let $n \in \mathbb{N}$. Consider the set of sample paths

$$\Omega_n^{(2)} = \{\omega \text{ such that } (A-a)m < d_{\text{Teich}}(x_0, x_m) < (A+a)m \text{ for all } m \geqslant n\}.$$

Let $E_n^{(2)}$ be the complement $\Omega \setminus \Omega_n$. It follows that $h(E_n^{(1)}) \to 0$ as $n \to \infty$.

3.4. **Thickness along tracked geodesics.** Let $0 < \theta' < 1$.

We parameterise the tracked geodesic $\gamma = \gamma_\omega$ by unit speed such that at time zero we are at γ_0, the closest point to $x_0 = x$, and $\gamma(t) \to \lambda^+$ as $t \to \infty$.

Let $\Lambda(r, \theta', \epsilon')$ be the set of sample paths ω such that for all $s > r$ we have

$$\text{Thick}^{\%}_{\epsilon'}[\gamma(0), \gamma(s)] \geqslant \theta'.$$

Proposition 3.5. *Given* $0 < \theta' < 1$ *there exists an* $\epsilon' > 0$ *such that*

$$\lim_{r\to\infty} h(\Lambda(r, \theta', \epsilon')) = 1.$$

Proof. Given $R > 0$ there exists $\epsilon(R) > 0$ such that $B(x_0, 3R) \subset \mathcal{T}_{\epsilon(R)}(S)$. By equivariance, $B(x_n, 3R) \subset \mathcal{T}_{\epsilon(R)}(S)$ for all $n \in \mathbb{Z}$.

Suppose that ω is in the complement of $E_n^{(1)}$. We first prove the proposition along the discrete set of times γ_n along γ_ω. By the triangle inequality

$$d_{\text{Teich}}(\gamma_0, \gamma_n) \geqslant d_{\text{Teich}}(x_0, x_n) - d_{\text{Teich}}(x_0, \gamma_0) - d_{\text{Teich}}(x_n, \gamma_n).$$

Since γ_0 is the closest point in γ_ω to x_0

$$d_{\text{Teich}}(x_0, \gamma_0) \leqslant d_{\text{Teich}}(x_0, x_{j_{\min}}) + R \leqslant (1 + \rho - p)nD + R.$$

Similarly

$$d_{\text{Teich}}(x_n, \gamma_n) \leqslant d_{\text{Teich}}(x_n, x_{j_{\max}}) + R \leqslant (1 + \rho - p)nD + R.$$

So we get

$$d_{\text{Teich}}(\gamma_0, \gamma_n) \geqslant d_{\text{Teich}}(x_0, x_n) - 2(1 + \rho - p)nD - 2R.$$

Further assume that ω is also in the complement of $E_n^{(2)}$. We deduce from the above estimates that

$$d_{\text{Teich}}(\gamma_0, \gamma_n) \geqslant (A - a)n - 2(1 + \rho - p)nD - 2R. \tag{3.6}$$

We note that the points in the segment $[\gamma_0, \gamma_n]$ that are not in $\mathcal{T}_{\epsilon(R)}(S)$ are in the union of the sets $[\gamma_0, \gamma_{j_{\min}}]$, $[\gamma_{j_{\max}}, \gamma_n]$ and the sets C_{jk} for all consecutive recurrence pairs $j < k$. The individual upper bounds on the lengths of each set in the union gives us the bound on the thick proportion for a choice of $\epsilon' \leqslant \epsilon(R)$

$$1 - \text{Thick}^{\%}_{\epsilon'}[\gamma_0, \gamma_n] \leqslant \frac{4(1 + \rho - p)D}{(A - a) - 2(1 + \rho - p)D - 2R/n}.$$

Now we make explicit choices. By choosing

- p sufficiently close to 1 and hence R sufficiently large,
- ρ sufficiently close to 0, and
- a sufficiently small compared to A

we get that the right hand side can be made smaller than $1-\theta'$, for n sufficiently large.

Now we conclude the proposition as the time $s \to \infty$ along γ_ω. By an argument identical to the derivation of (3.6), we get the upper bound

$$d_{\mathrm{Teich}}(\gamma_0, \gamma_n) \leqslant (A+a)n + 2(1+\rho-p)nD + 2R.$$

Given a time $s > 0$, we may choose n to satisfy

$$(A-a)n - 2(1+\rho-p)nD - 2R < s < (A+a)n + 2(1+\rho-p)nD + 2R.$$

When s is large enough, such a choice always exist. Since we are only interested in the limit as $s \to \infty$, we may make this choice. Further tweaking p, ρ and a we may also arrange that the ratio of the upper bound to the lower bound is as close to one as we want. This implies that as $s \to \infty$ the thick proportion of $[\gamma_0, \gamma(s)]$ is the same as the thick proportion of $[\gamma_0, \gamma_n]$.

Finally, we note that the set of exceptions is the union $E_n^{(1)} \cup E_n^{(2)}$ whose measure tends to zero as $n \to \infty$. In particular, this implies $h(\Lambda(r, \theta', \epsilon')) \to 1$ as $r \to \infty$, and we are done. $\square$

As a direct consequence of Proposition 3.5, we get the following conclusion.

Proposition 3.7. *Let $0 < \theta' < 1$. Then there exists an $\epsilon' > 0$ such that for almost every bi-infinite sample path ω there exists t_ω such that for all $t > t_\omega$*

$$\mathrm{Thick}^{\%}_{\epsilon'}[\gamma(0), \gamma(t)] \geqslant \theta'.$$

3.8. **Thickness along rays.** Now let y be some base-point in $\mathcal{T}(S)$. Masur proved that Teichmüller rays with the same vertical foliation are asymptotic if the foliation is uniquely ergodic. See [15, Theorem 2]. We now use this result to transfer the thickness estimates from tracked geodesics to corresponding rays from y. Suppose that ω is a typical bi-infinite sample path with the tracked geodesic γ_ω. Let λ^+_ω be the projective measured foliation that γ_ω converges to in the forward direction. Let ξ_ω be the geodesic ray from y that converges to λ^+_ω. We may parameterise ξ_ω with unit speed so that $\xi_\omega(0) = y$ and $\xi_\omega(t) \to \lambda^+_\omega$ as $t \to \infty$.

Proposition 3.9. *Let $0 < \theta < 1$. Then there exists $\epsilon > 0$ such that for almost every bi-infinite sample path ω there is a time $T_\omega > 0$ such that*

$$\mathrm{Thick}^{\%}_{\epsilon}[\xi_\omega(0), \xi_\omega(t)] \geqslant \theta$$

for all $t \geqslant T_\omega$.

Proof. By [11, Theorem 2.2.4], for almost every ω the foliation λ^+_ω is uniquely ergodic. By Masur's theorem, there is a time $s > 0$ such that for all $t \geqslant s$ we have $d_{\mathrm{Teich}}(\xi_\omega(t), \gamma_\omega) < 1/2$. We may choose $\epsilon \leqslant \epsilon'$ such that the 1-neighbourhood of $\mathcal{T}_{\epsilon'}(S)$ is contained in $\mathcal{T}_\epsilon(S)$. This means after the time s along ξ_ω any ϵ'-thick segment of γ_ω gives an ϵ-thick segment of ξ_ω of at least the same length. We now set $\theta' > \theta$ and use Proposition 3.7. Let $t > s$. The total length of ϵ'-thick segments of γ_ω that are inside a 1-neighbourhood of $[\xi_\omega(s), \xi_\omega(t)]$ is at least $\theta' t - s - d_{\mathrm{Teich}}(\gamma(0), y)$. If t is large enough then $\theta' t - s - d_{\mathrm{Teich}}(\gamma(0), y) > \theta t$, and we are done. $\square$

As an immediate corollary, we get

Corollary 3.10. *The measures ν_r on spheres arising from the harmonic measure satisfy the thickness property 2.6.*

3.11. **Separation properties.** To every marked hyperbolic surface $x \in \mathcal{T}(S)$, one can consider a systole on x, that is, a shortest closed hyperbolic geodesic on x. A systole is always a simple closed curve and hence can be thought of as a vertex in the curve complex $\mathcal{C}(S)$. This defines a coarse projection $\mathrm{sys} : \mathcal{T}(S) \to \mathcal{C}(S)$ from Teichmüller space to the curve complex. By [17, Theorems 2.3 and 2.6], the projection $\mathrm{sys}(\gamma)$ of a Teichmüller geodesic γ is an unparameterised quasi-geodesic in $\mathcal{C}(S)$.

Let $M > 0$. Suppose $\xi = \xi_\omega$ and $\xi' = \xi_{\omega'}$ are geodesic rays from y chosen with respect to the harmonic measure, where ω and ω' are the associated bi-infinite sample paths. Let $T = T_\omega$ and $T' = T_{\omega'}$ be the thresholds given by Proposition 3.9 for ω and ω', respectively. Pick S larger than T and T'. Suppose that $d_{\mathrm{Teich}}(\xi(S), \xi'(S)) < M$.

We consider the projections $\mathrm{sys}([y, \xi(S)])$ and $\mathrm{sys}([y, \xi'(S)])$. Since ξ and ξ' spend at least θ proportion of their time in the thick part, there exists a constant $\kappa > 0$ such that

$$d_{\mathcal{C}}(\mathrm{sys}(y), \mathrm{sys}(\xi(S))) \geqslant \kappa S \text{ and } d_{\mathcal{C}}(\mathrm{sys}(y), \mathrm{sys}(\xi'(S))) \geqslant \kappa S.$$

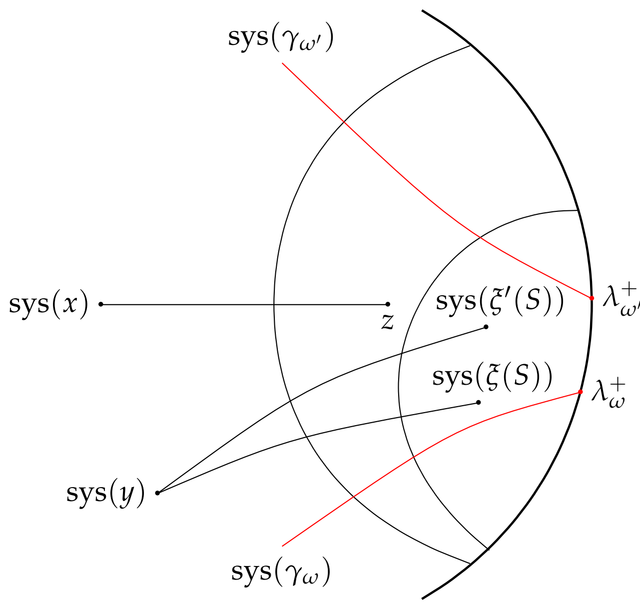

FIGURE 3.12. The nested shadows in $\mathcal{C}(S)$.

Recall the definition from [13, Section 2.3] of a shadow in a δ-hyperbolic space. By hyperbolicity of $\mathcal{C}(S)$ there is a constant $r > 0$ such that ω and ω' both converge to a point of the Gromov boundary of $\mathcal{C}(S)$ that is at infinity for the shadow $\mathrm{Shad}_{\mathrm{sys}(y)}(\mathrm{sys}(\xi(S)), r)$. Note that $d_{\mathcal{C}}(\mathrm{sys}(x), \mathrm{sys}(\xi(S))) \geqslant d_{\mathcal{C}}(\mathrm{sys}(y), \mathrm{sys}(\xi(S))) - d_{\mathcal{C}}(\mathrm{sys}(x), \mathrm{sys}(y))$. After S is large enough the right hand side grows linearly in S. In particular, $\mathrm{Shad}_{\mathrm{sys}(y)}(\mathrm{sys}(\xi(S)), r)$ is contained in a shadow from $\mathrm{sys}(x)$ of a point z that is linearly in S far away from $\mathrm{sys}(x)$. See Figure 3.12.

Hence, by [13, Lemma 2.10], the harmonic measure of the limit set at infinity of the shadow $\mathrm{Shad}_{\mathrm{sys}(y)}(\mathrm{sys}(\xi(S)), r)$ decays exponentially in S.

As an immediate corollary we get

Corollary 3.13. *The measures ν_r on spheres arising from the harmonic measure satisfy the separation property 2.7.*

4. Statistical Hyperbolicity

We now sketch the argument given by Dowdall-Duchin-Masur [4, Theorem 7.1] of how the thickness and separation properties imply statistical hyperbolicity. The idea is to mimic a proof of the fact that $E(\mathbb{H}^n) = 2$ for the natural measures on spheres which makes use of the δ-hyperbolicity of $\mathbb{H}^n$, and of the speed of separation of geodesics.

However, as discussed above, Teichmüller space is neither δ-hyperbolic, nor negatively-curved in the sense of Busemann. The motivation for the separation property 2.7 is to show instead that most pairs of geodesics after some threshold time become separated by a definite amount. This replaces the use of the negative curvature of $\mathbb{H}^n$. The combination of the following theorem of Dowdall-Duchin-Masur [4, Theorem A] and the thickness property 2.6 then replaces the use of the δ-hyperbolicity of $\mathbb{H}^n$.

Theorem 4.1 ([4], Theorem A)**.** *For any $\epsilon > 0$ and any $0 < \theta' \leqslant 1$, there exist constants C and L such that for any geodesic sub-interval $I \subset [x, x'] \subset \mathcal{T}(S)$ of length at least L and spending at least θ proportion of its time in $\mathcal{T}_\epsilon(S)$, we have*

$$I \cap \mathrm{Nbhd}_C([x, x''] \cup [x', x'']) \neq \varnothing,$$

for all $x'' \in \mathcal{T}(S)$.

We now sketch the proof of statistical hyperbolicity using Theorem 4.1 with the thickness and separation properties, namely 2.6 and 2.7.

Let $x \in \mathcal{T}(S)$. We choose θ large enough, say $\theta = 3/4$, and then for any $0 < p < \eta < 1/3$ we let $\epsilon = \epsilon(\theta, \eta) > 0$ be that guaranteed by the thickness property 2.6. Now choose $\theta' < \theta$, say $\theta' = 1/2$, and let C and L be the constants given by Theorem 4.1 for our choice of θ' and ϵ. The thickness and separation properties then imply that for all r large enough, there exists a subset $P_r \subset S_r(x) \times S_r(x)$ whose complement has $\nu_r \times \nu_r$-measure at most p and is such that for all $(x', x'') \in P_r$ we have $d_{\mathrm{Teich}}(x'_t, x''_t) \geqslant 3C$, and

$$\mathrm{Thick}^{\%}_{\epsilon}([x, x'_t]), \mathrm{Thick}^{\%}_{\epsilon}([x, x''_t]) \geqslant \theta,$$

for all $\eta r \leqslant t \leqslant r$. It can be checked that $B(x'_t, C) \cap [x, x''] = \varnothing$ for all such points x'_t.

Choosing r large enough, we can arrange that the interval $I_r = [x'_{\eta r}, x'_{2\eta r}]$ spends at least θ' proportion of its time in $\mathcal{T}_\epsilon(S)$, has length at least L and, since $\eta < 1/3$, is contained in $[x, x']$. By applying Theorem 4.1, we must have that $I_r \cap \mathrm{Nbhd}_C([x, x''] \cup [x', x'']) \neq \varnothing$ and, since we have already noted that $I_r \cap \mathrm{Nbhd}_C([x, x'']) = \varnothing$, it then follows that there exists a point in $[x', x'']$ at distance at most $2\eta r + C$ from x. Hence we have that

$$d_{\mathrm{Teich}}(x', x'') \geqslant (2 - 4\eta)r - 2C,$$

for all $(x', x'') \in P_r$. From which it follows that

$$\begin{aligned} E(X) &\geqslant \liminf_{r\to\infty} \frac{1}{r} \int_{S_r(x)\times S_r(x)} d_{\mathrm{Teich}}(x', x'')\; d\nu_r(x')\, d\nu_r(x'') \\ &\geqslant \liminf_{r\to\infty} \frac{1}{r}(1-p)((2-4\eta)r - 2C) \\ &= (1-p)(2-4\eta). \end{aligned}$$

Hence, the result follows as p and η can be taken to be arbitrarily small.

4.2. **Proof of Theorem 1.2.** We now give a quick proof of Theorem 1.2. The ambient geometry in $\mathbb{H}^n$ is already hyperbolic. So we can simply bypass the thickness discussion. All we need is the separation property for the harmonic measure. This follows from the exponential decay of the harmonic measure for half-spaces in $\mathbb{H}^n$.

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