THE STRATUM OF RANDOM MAPPING CLASSES.

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ABSTRACT. We consider random walks on the mapping class group whose support generates a non-elementary subgroup and contains a pseudo-Anosov map whose invariant Teichmüller geodesic is in the principal stratum. For such random walks, we show that mapping classes along almost every infinite sample path are eventually pseudo-Anosov, with invariant Teichmüller geodesics in the principal stratum. This provides an answer to a question of Kapovich and Pfaff [KP15].

1. INTRODUCTION

Let S be an orientable surface of finite type. Let $\text{Mod}(S)$ denote the mapping class group consisting of orientation preserving diffeomorphisms on S modulo isotopy. The Teichmüller space $\mathcal{T}(S)$ is the space of marked conformal structures on S and the mapping class group $\text{Mod}(S)$ acts on $\mathcal{T}(S)$ by changing the marking. This action is properly discontinuous, and the quotient $\mathcal{M}(S)$ is the moduli space of Riemann surfaces. The unit tangent space of $\mathcal{T}(S)$ may be identified with the space of unit area quadratic differentials $Q(S)$, with simple poles at the punctures of S. The space $Q(S)$ is stratified by sets consisting of quadratic differentials with a given list of multiplicities for their zeroes. The *principal stratum* consists of those quadratic differentials all of whose zeros are simple, i.e. have multiplicity one; this is the top dimensional stratum in $Q(S)$. Maher [Mah11] and Rivin [Riv08] showed that a random walk on $\text{Mod}(S)$ gives a pseudo-Anosov mapping class with a probability that tends to 1 as the length of the sample path tends to infinity. A pseudo-Anosov element preserves an invariant geodesic in $\mathcal{T}(S)$, which is contained in a single stratum. As a refinement, Kapovich and Pfaff raise the following question: what is the stratum of quadratic differentials for the invariant Teichmüller geodesic of a random pseudo-Anosov element? See [KP15, Question 1.5] and [DHM15, Question 6.1].

As a step towards answering the question, we prove the following result. We shall write d_{Mod} for the word metric on $\text{Mod}(S)$ with respect to a choice of finite generating set.

Theorem 1.1. *Let μ be a probability distribution on $Mod(S)$ such that*

(1) *μ has finite first moment with respect to d_{Mod},*
(2) *$Supp(\mu)$ generates a non-elementary subgroup H of $Mod(S)$, and*
(3) *The semigroup generated by $Supp(\mu)$ contains a pseudo-Anosov g such that the invariant Teichmüller geodesic γ_g for g lies in the principal stratum of quadratic differentials.*

Then, for almost every infinite sample path $\omega = (w_n)$, there is positive integer N such that w_n is a pseudo-Anosov map in the principal stratum for all $n \geqslant N$. Furthermore, almost every bi-infinite sample path determines a unique Teichmüller geodesic γ_ω with the same limit points, and this geodesic also lies in the principal stratum.

We will refer to condition (3) above as the *principal stratum assumption*.

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The proof follows the following strategy. Let g be a pseudo-Anosov element whose invariant Teichmüller geodesic γ_g lies in the principal stratum. We show that any Teichmüller geodesic that fellow travels γ_g for a sufficiently large distance D, depending on g, also lies in the principal stratum. Next, we show that if g lies in the semigroup generated by the support of μ, there is a positive probability that the geodesic γ_ω tracked by a sample path ω, fellow travels the invariant geodesic γ_g for distance at least D. Ergodicity of the shift map on $\text{Mod}(S)^{\mathbb{Z}}$ then implies that a positive proportion of subsegments of γ_ω of length D fellow travel some translate of γ_g. We then use work of Dahmani and Horbez [DH15] which shows that for almost all sample paths ω, for sufficiently large n, all elements w_n are pseudo-Anosov, with invariant geodesics γ_{w_n} which fellow travel γ_ω for a distance which grows linearly in n. In particular, this implies that γ_{w_n} fellow travels a sufficiently long subsegment of a translate of γ_g, and so lies in the principal stratum.

Teichmüller preliminaries

Let S be an orientable surface of finite type. For the sporadic examples in which the Euler characteristic of S is zero, namely the torus and the 4-punctured sphere, there is a single stratum of quadratic differentials in each case, so we will assume that the Euler characteristic of S is negative.

The Teichmüller metric is given by

$$d_{\mathcal{T}}(X, Y) = \tfrac{1}{2} \inf_f \log K(f),$$

where the infimum is taken over all quasiconformal maps $f\colon X \to Y$ in the given homotopy class, and $K(f)$ is the quasiconformal constant of f. As there is a unique Teichmüller geodesic connecting any pair of points in Teichmüller space, we may sometimes write $[X, Y]$ to denote the Teichmüller geodesic segment from X to Y. For detailed background about the Teichmüller metric and the geometry of quadratic differentials, see for example [Wri15].

The complex of curves $\mathcal{C}(S)$ is an infinite graph with vertices isotopy classes of simple closed curves on S. Two vertices $[\alpha], [\beta]$ are separated by an edge if the curves α and β can be isotoped to be disjoint. The graph $\mathcal{C}(S)$ is locally infinite and has infinite diameter, and Masur and Minsky showed that $\mathcal{C}(S)$ is δ-hyperbolic in the sense of Gromov [MM99].

By the uniformization theorem, a conformal class X determines a a unique hyperbolic metric on S, which we shall also denote by X. For a hyperbolic surface X, a systole of X is a simple closed curve that has the shortest length in the hyperbolic metric. The set of systoles of X is a finite set whose diameter in $\mathcal{C}(S)$ is bounded above by a constant that depends only on the topology of S. Thus, the systole defines a coarse projection map $\pi : \mathcal{T}(S) \to \mathcal{C}(S)$. For notational simplicity, we will use upper case letters for points X in $\mathcal{T}(S)$, and the corresponding lower case letters $x = \pi(X)$ for their projections to the curve complex. Masur and Minsky [MM99, 6.1] showed that π is coarsely Lipschitz, i.e. there are constants $M_1 > 0, A_1 > 0$ that depend only on S, such that for any pair of points $X, Y \in \mathcal{T}(S)$

$$d_{\mathcal{C}}(x, y) < M_1 d_{\mathcal{T}}(X, Y) + A_1. \tag{1.2}$$

Moreover, Masur and Minsky also show that Teichmüller geodesics γ project to uniformly unparameterised quasigeodesics in $\mathcal{C}(S)$. Let (M_2, A_2)-be the quasigeodesicity constants for the projection of a Teichmüller geodesic, and these constants depend only on S.

The set of hyperbolic surfaces $X \in \mathcal{T}(S)$ for which the length of the systole is less than ϵ form the ϵ-thin part $\mathcal{T}(S)_\epsilon$ of Teichmüller space. The complement $K_\epsilon = \mathcal{T}(S) \setminus \mathcal{T}(S)_\epsilon$ is called the thick part. By Mumford compactness, $\text{Mod}(S) \backslash K_\epsilon$ is compact, and furthermore a metric regular neighbourhood of the thick part is contained in a larger thick part. More precisely, for any $\epsilon > 0$, and any $D \geqslant 0$, there is a constant ϵ', depending on ϵ, D and the surface S, such that a metric D-neighbourhood of K_ϵ, in the Teichmüller metric, is contained in $K_{\epsilon'}$.

Let γ and γ' be two geodesics in a metric space (M,d). If there are are choices of (not necessarily unit speed) parameterizations $\gamma(t)$ and $\gamma'(t)$ such that there is a constant E with $d(\gamma(t),\gamma'(t)) \leqslant E$ for all t, then we say that γ and γ' are fellow travellers with fellow travelling constant E, or E-fellow travel. If $d(\gamma(t),\gamma'(t)) \leqslant E$, for all t, for the unit speed parameterizations of γ and γ', then we say that γ and γ' are parameterized E-fellow travellers.

Let γ and γ' be two Teichmüller geodesics whose projections to the curve complex $\pi(\gamma)$ and $\pi(\gamma')$ fellow travel. In general, this does not imply that the original Teichmüller geodesics fellow travel in Teichmüller space. However, we now show in the following lemma that if γ is contained in a thick part K_ϵ, and $\pi(\gamma')$ fellow travels $\pi(\gamma)$ for a sufficiently long distance in $\mathcal{C}(S)$, then γ' contains a point that is close to γ in Teichmüller space.

Lemma 1.3. *For any constants $\epsilon > 0$ and $E \geqslant 0$, there are constants $L > 0$ and $F > 0$, depending on ϵ, E and the surface S, such that if $\gamma = [X,Y]$ is a Teichmüller geodesic segment contained in the thick part K_ϵ, of length at least L, and $\gamma' = [X',Y']$ is a Teichmüller geodesic segment, whose endpoints x', y' in $\mathcal{C}(S)$ are distance at most E from the endpoints x, y of $\pi(\gamma)$, i.e. $d_{\mathcal{C}(S)}(x,x') \leqslant E$ and $d_{\mathcal{C}(S)}(y,y') \leqslant E$, then there is a point Z on γ' such that $d_{\mathcal{T}}(Z,\gamma) \leqslant F$.*

This result may also be deduced from work of Horbez [Hor15, Proposition 3.10] and Dowdall, Duchin and Masur [DDM14, Theorem A], extending Rafi [Raf14], but for the convenience of the reader, we provide a direct proof of this result in Section 3, relying only on Rafi [Raf14]. In particular, we will make extensive use of the following fellow travelling result for Teichmüller geodesics whose endpoints are close together in the thick part.

Theorem 1.4. [Raf14, Theorem 7.1] *For any constants $\epsilon > 0$ and $A \geqslant 0$, there is a constant B, depending only on ϵ, A and the surface S, such that if $[X,Y]$ and $[X',Y']$ are two Teichmüller geodesics, with X and Y in the ϵ-thick part, and*

$$d_{\mathcal{T}}(X,X') \leqslant A \text{ and } d_{\mathcal{T}}(Y,Y') \leqslant A,$$

then $[X,Y]$ and $[X',Y']$ are parameterized $B(\epsilon,A)$-fellow travellers.

We now continue with the proof of Theorem 1.1 assuming Lemma 1.3. Recall that the Gromov product based at a point $u \in \mathcal{C}(S)$ is defined to be

$$(x,y)_u = \frac{1}{2}\left(d_{\mathcal{C}}(u,x) + d_{\mathcal{C}}(u,y) - d_{\mathcal{C}}(x,y)\right).$$

Given points $x, y \in \mathcal{C}(S)$ and a constant $R > 0$, the R-shadow of y is defined to be

$$S_x(y,R) = \{z \in \mathcal{C}(S) \mid (y,z)_x \geqslant d_{\mathcal{C}}(x,y) - R\}.$$

The definition we use here for shadows follows [MT14], and may differ slightly from other sources. The following lemma follows from the thin triangles property of Gromov hyperbolic spaces, and we give a proof for the convenience of the reader.

Lemma 1.5. *There is a constant D, which only depends on δ, and a constant E, which only depends on R and δ, such that if $d_{\mathcal{C}}(x,y) \geqslant 2R + D$, then for any $x' \in S_y(x,R)$ and any $y' \in S_x(y,R)$, any geodesic segment $[x',y']$ contains a subsegment which E-fellow travels $[x,y]$.*

Proof. We shall write $O(\delta)$ to denote a constant which only depends (not necessarily linearly) on δ.

Let p be the nearest point projection of x' to $[x,y]$, and let q be the nearest point projection of y' to $[x,y]$. The nearest point projection of the shadow $S_x(y,R)$ is contained in an $(R+O(\delta))$-neighbourhood of y, see for example [MT14, Proposition 2.4], so $d_{\mathcal{C}}(x,p) \leqslant R + O(\delta)$ and $d_{\mathcal{C}}(y,q) \leqslant R + O(\delta)$. Recall that if $d_{\mathcal{C}}(p,q) \geqslant O(\delta)$ then any geodesic from x' to y' passes within an $O(\delta)$-neighborhood of both p and q, see for example [MT14, Proposition 2.3]. Therefore, if $d(x,y) \geqslant 2R + O(\delta)$, then this implies that if p' is the closest point on $[x',y']$ to p, and q' is the closest point on $[x', y']$ to q, then $[p', q']$ E-fellow travels $[x, y]$, where E is a constant which only depends on R and δ, as required. ☐

Remark 1.6. *One can replace the geodesic segments $[x, y]$ and $[x', y']$ by (M_2, A_2)-quasigeodesic segments. The constants D and E now change, and in addition to R and δ, they now depend on the quasigeodesicity constants.*

We shall write PMF for the set of projective measured foliations on the surface S, which is Thurston's boundary for Teichmüller space. A projective measured foliation is uniquely ergodic if the foliation supports a unique projective measure class. Let UE be the subset of PMF consisting of uniquely ergodic foliations. We shall give UE the corresponding subspace topology. A uniquely ergodic foliation determines a class of mutually asymptotic geodesic rays in $\mathcal{T}(S)$, as shown by Masur [Mas80]. These rays project to a class of mutually asymptotic quasigeodesic rays in $\mathcal{C}(S)$, and so determines a point in the Gromov boundary of the curve complex. This boundary map is injective on uniquely ergodic foliations, see for example Hubbard and Masur [HM79]. Thus, UE is also a subset of $\partial\mathcal{C}(S)$. Klarriech [Kla] showed that $\partial\mathcal{C}(S)$ is homeomorphic to the quotient of the set of minimal foliations in PMF by the equivalence relation which forgets the measure. In particular, this implies that the two subspace topologies on UE, induced from inclusions in PMF and $\partial\mathcal{C}(S)$, are the same.

Let γ be a Teichmüller geodesic in a thick part K_ϵ. Let λ^+ and λ^- be the projective classes of vertical and horizontal measured foliations of γ. By the work of Kerckhoff, Masur and Smillie [KMS86, Theorem 3], vertical foliations of Teichmüller rays that are recurrent to a thick part are uniquely ergodic, so the foliations λ^+ and λ^- are uniquely ergodic, and by Hubbard and Masur [HM79] such a pair (λ^-, λ^+) determines a unique bi-infinite Teichmüller geodesic. Given two points X and Y in Teichmüller space, and a constant $r \geqslant 0$, define $\Gamma_r(X, Y)$ to be the set of all oriented geodesics with uniquely ergodic vertical and horizontal foliations, which intersect both $B_r(X)$ and $B_r(Y)$, and furthermore, whose first point of intersection with either $B_r(X)$ or $B_r(Y)$ lies in $B_r(X)$. A Teichmüller geodesic with uniquely ergodic vertical foliation λ^+ and uniquely ergodic horizontal foliation λ^- determines a point (λ^-, λ^+) in UE $\times$ UE. Therefore $\Gamma_r(X, Y)$ determines a subset of UE $\times$ UE, which, by abuse of notation, we shall also denote by $\Gamma_r(X, Y)$.

Proposition 1.7. *For any Teichmüller geodesic γ contained in a thick part K_ϵ, with vertical foliation λ^+ and horizontal foliation λ^-, there is a constant $r > 0$, depending on ϵ, such that for any pair of points X and Y on γ, the set $\Gamma_r(X, Y)$ contains an open neighbourhood of (λ^-, λ^+) in UE $\times$ UE.*

Proof. As $\mathrm{Mod}(S)$ acts coarsely transitively on the curve complex $\mathcal{C}(S)$, there is a constant $R > 0$, depending only on S, such that for all x and y in $\mathcal{C}(S)$, the limit set of the shadow $\overline{S_x(y, R)}$ contains a non-empty open set in $\partial\mathcal{C}(S)$, see for example [MT14, Propositions 3.18–19]. Given such an R, let D and E be the constants in Lemma 1.5, such that if $d(x, y) \geqslant D$ then for any $x' \in S_y(x, R)$ and $y' \in S_y(x, R)$, a geodesic $[x', y']$ has a subsegment which E-fellow travels with $[x, y]$. Given ϵ and E, let L and F be the constants in Lemma 1.3, i.e. if γ and γ'' are two Teichmüller geodesics of length at least L, whose endpoints in $\mathcal{C}(S)$ are distance at most E apart, then the distance from γ to γ' is at most F.

As γ lies in the thick part K_ϵ, there is a constant D', depending only on ϵ, such that if $d_\mathcal{T}(X, Y) \geqslant D'$, then $d_\mathcal{C}(x, y) \geqslant D$. Let Z_1 and Z_2 be points along γ such that $[X, Y] \subset [Z_1, Z_2]$, the orientations of the segments agree, $d_\mathcal{T}(X, Y) \geqslant D'$, $d_\mathcal{T}(Z_1, X) > L$ and $d_\mathcal{T}(Y, Z_2) > L$. Consider the limits sets $\overline{S_{z_1}(z_2, R)}$ and $\overline{S_{z_2}(z_1, R)}$ in $\partial\mathcal{C}(S)$, and let ξ^+ and ξ^- be uniquely ergodic foliations in $\overline{S_{z_1}(z_2, R)}$ and $\overline{S_{z_2}(z_1, R)}$, respectively. Let γ' be the Teichmüller geodesic with vertical foliation ξ^+ and the horizontal foliation ξ^-. By Lemma 1.5, the projection $\pi(\gamma')$ fellow travels $\pi(\gamma)$ with constant E between z_1 and z_2. For clarity, denote by Z_1', X', Y' and Z_2' the points of γ' whose projections z_1', x', y' and z_2' are coarsely the closest points to z_1, x, y and z_2 respectively, i.e. the distances

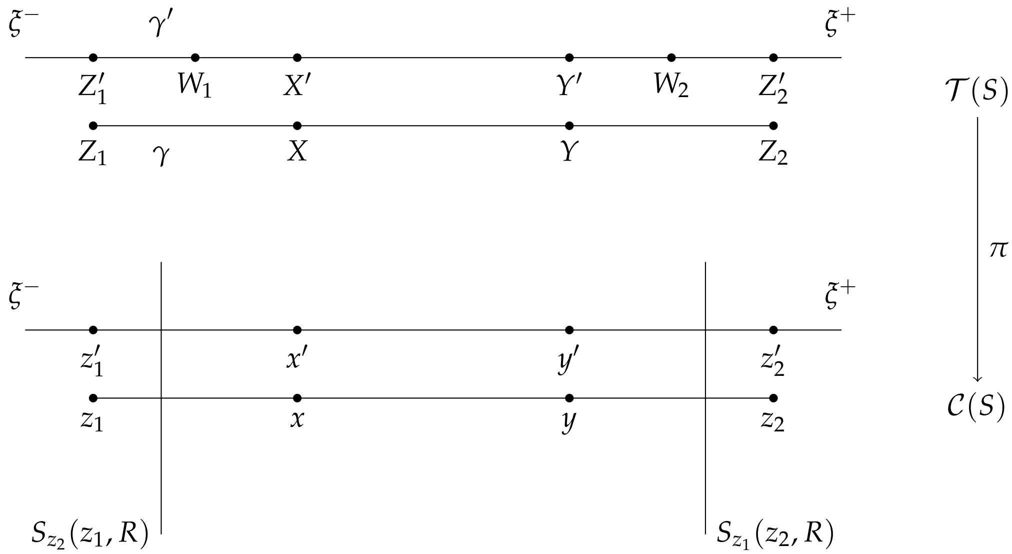

FIGURE 1.8. Shadows in $\mathcal{C}(S)$.

$d_{\mathcal{C}}(z_1', z_1), d_{\mathcal{C}}(x', x), d_{\mathcal{C}}(y', y)$ and $d_{\mathcal{C}}(z_2', z_2)$ are all at most E. By Lemma 1.3 applied to the segments $[Z_1', X']$ and $[Z_1, X]$ there is a point $W_1 \in [Z_1', X']$ such that $d_{\mathcal{T}}(W_1, [Z_1, X]) \leqslant F$. Similarly, there is a point $W_2 \in [Y', Z_2']$ such that $d_{\mathcal{T}}(W_2, [Y, Z_2]) \leqslant F$.

By the fellow travelling result, Theorem 1.4, the Teichmüller geodesic segment $[W_1, W_2] \subset \gamma'$ fellow travels γ with the constant $r = B(\epsilon, F)$. In particular, γ' passes through $B_r(X)$ and $B_r(Y)$, and hence lies in $\Gamma_r(X, Y)$, and so this set contains an open neighbourhood of (λ^-, λ^+). We have shown this as long as $d_{\mathcal{T}}(X, Y) \geqslant D'$, but for $r' = 2r + D'$, every pair of balls $B_{r'}(X')$ and $B_{r'}(Y')$ contain smaller balls $B_r(X)$ and $B_r(Y)$ with $d_{\mathcal{T}}(X, Y) \geqslant D'$, so the stated result follows. □

2. FELLOW TRAVELLING OF INVARIANT AND TRACKED GEODESICS

In this section, we establish that along almost every sample path ω, for sufficiently large n, the invariant Teichmüller geodesic for the pseudo-Anosov element w_n, has a subsegment, whose length grows linearly in n, which fellow travels the Teichmüller geodesic sublinearly tracked by ω. This uses a result of Dahmani and Horbez [DH15] and the fellow travelling result, Theorem 1.4. We fix a basepoint $X \in \mathcal{T}(S)$.

We require a slight rephrasing of a result of Dahmani and Horbez. Let ℓ be the drift of the random walk in the Teichmüller metric. Kaimanovich and Masur [KM96] showed that almost every bi-infinite sample path ω converges to distinct uniquely ergodic measured foliations λ_ω^+ and λ_ω^-, with $w_n X$ converging to λ_ω^+, and $w_{-n} X$ converging to λ_ω^- as $n \to \infty$. Let γ_ω be the unique bi-infinite Teichmüller geodesic determined by these foliations, and we shall give γ_ω a unit speed parameterization, such that $\gamma_\omega(0)$ is a closest point on γ_ω to X, and as $t \to \infty$ the geodesic $\gamma_\omega(t)$ converges to λ^+. If w_n is pseudo-Anosov, then we shall write γ_{ω_n} for its invariant Teichmüller geodesic.

Steps 1 and 3 in the proof of [DH15, Theorem 2.6], stated in the context of Teichmüller space, can be rephrased as follows:

Proposition 2.1. *Given $\epsilon > 0$, there are constants $F > 0$ and $0 < e < \frac{1}{2}$, such that for almost every ω, there exists N, such that for all $n \geqslant N$, there are points Y_0 and Y_1 of γ_{w_n} and points $\gamma_\omega(T_0)$ and $\gamma_\omega(T_1)$ of γ_ω, such that*

(1) $d_{\mathcal{T}}(\gamma_\omega(T_0), Y_0) \leqslant F$,
(2) $d_{\mathcal{T}}(\gamma_\omega(T_1), Y_1) \leqslant F$,
(3) $0 \leqslant T_0 \leqslant e\ell n \leqslant (1-e)\ell n \leqslant T_1 \leqslant \ell n$, *and*
(4) $\gamma_\omega(T_0)$ *and* $\gamma_\omega(T_1)$ *are in the thick part* K_ϵ.

Dahmani and Horbez state condition (4) in terms of a "contraction" property that they define: $\gamma_\omega(T_0)$ and $\gamma_\omega(T_1)$ are "contraction" points on γ_ω for the projection map to the curve complex. In effect, the property being used by them is that under the projection to the curve complex γ_ω makes definite progress at $\gamma_\omega(T_0)$ and $\gamma_\omega(T_1)$. See the discussion related to [DH15, Propositions 3.6 and 3.7]. We recall their precise definition [DH15, Definition 3.5] for definite progress here:

Definition 2.2. *Given constants $B, C > 0$, a Teichmüller geodesic γ makes (B, C)-progress at a point $Y = \gamma(T)$ if the image under π of the subsegment of γ of length B starting at Y has diameter at least C in the curve complex.*

For completeness, we prove that definite progress implies thickness.

Lemma 2.3. *If γ makes (B, C)-progress at Y, then there is a constant $\epsilon > 0$, which depends on B and C, such that Y lies in the thick part K_ϵ.*

Proof. Let α be the systole for the hyperbolic surface Y. For any point Y' on the subsegment, Wolpert's lemma implies

$$\ell_{Y'}(\alpha) \leqslant e^B \ell_Y(\alpha).$$

We will use the following version of the Collar Lemma, due to Matelski [Mat76], which states that a simple closed geodesic of length ℓ is contained in an embedded annular collar neighbourhood of width at least w_ℓ, where a lower bound for w_ℓ is given by

$$\sinh^{-1}\left(\frac{1}{\sinh(\ell/2)}\right),$$

and furthermore, this lower bounds holds for all $\ell > 0$. Thus the width of the collar neighbourhood for α in the hyperbolic metric corresponding to Y' is bounded below by

$$\sinh^{-1}\left(\frac{1}{\sinh(e^B \ell_Y(\alpha)/2)}\right),$$

and the bound tends to infinity monotonically as $\ell_Y(\alpha)$ tends to zero. Suppose β is the systole at Y', and $d_{\mathcal{C}}(\alpha, \beta) \geqslant C$. This implies that the intersection number satisfies

$$i(\alpha, \beta) \geqslant \frac{C-1}{2}.$$

From the lower bound on the width of the collar, the length of β has to satisfy

$$\ell_{Y'}(\beta) \geqslant \frac{C-1}{2} \sinh^{-1}\left(\frac{1}{\sinh(e^B \ell_Y(\alpha)/2)}\right).$$

Since β is the systole at Y', the length of β at Y' is at most the length of α at Y', so one obtains

$$e^B \ell_Y(\alpha) \geqslant \frac{C-1}{2} \sinh^{-1}\left(\frac{1}{\sinh(e^B \ell_Y(\alpha)/2)}\right).$$

Note that sinh is monotonically increasing, zero at zero, and unbounded, so this gives a lower bound ϵ on how small $\ell_Y(\alpha)$ can be, which depends on B and C. ☐

Remark 2.4. *Lemma 2.3 implies that the points $\gamma_\omega(T_0)$ and $\gamma_\omega(T_1)$ in Proposition 2.1 are in a thick part K_ϵ. By the fellow travelling result, Theorem 1.4 the geodesics γ_ω and γ_{w_n} fellow travel between $\gamma_\omega(T_0)$ and $\gamma_\omega(T_1)$. Let $s = B(\epsilon, F)$ be the constant for fellow traveling of γ_ω and γ_{w_n}.*

We now show that for a pseudo-Anosov element g in the support of μ, there is a positive probability that the geodesic γ_ω fellow travels the invariant geodesic γ_g. We shall write ν for the harmonic measure on UE, and $\check{\nu}$ for the reflected harmonic measure, i.e the harmonic measure arising from the random walk generated by the probability distribution $\check{\mu}(g) = \mu(g^{-1})$.

Lemma 2.5. *Let g be a pseudo-Anosov element contained in the support of μ with invariant Teichmüller geodesic γ_g. Then there is a constant $r > 0$ such that $\check{\nu} \times \nu(\Gamma_r(X, Y)) > 0$ for all X and Y on γ_g.*

Furthermore, there is a constant $\rho > 0$, depending on g, such that for all constants $D \geqslant 0$, there is a positive probability (that depends on D) for the subsegment of γ_ω of length D, centered at a closest point on γ_ω to the basepoint, to ρ-fellow travel with γ_g.

Proof. Let λ^+ and $\lambda^- \in$ PMF be the vertical and horizontal foliations of γ_g. Fix an $\epsilon > 0$ such that the thick part K_ϵ contains the geodesic γ_g. Let r be the constant in Proposition 1.7, i.e. for any points X and Y on γ_g, the set $\Gamma_r(X, Y)$ contains an open neighbourhood of (λ^-, λ^+). We recall:

Proposition 2.6. [MT14, Proposition 5.4] *Let G be a non-elementary, countable group acting by isometries on a separable Gromov hyperbolic space X, and let μ be a non-elementary probability distribution on G. Then there is a number R_0 such that for any group element g in the semigroup generated by the support of μ, the closure of the shadow $S_{x_0}(gx_0, R_0)$ has positive hitting measure for the random walk determined by μ.*

Let $x_0 = \pi(X_0)$ be the projection of the basepoint X_0 into the curve complex. We may assume that $\Gamma_r(X, Y)$ contains an open neighbourhood of (λ^-, λ^+) of the form $U^- \times U^+$, where U^- is an open neighbourhood of λ^- in UE, and U^+ is an open neighbourhood of λ^+ in UE. As

$$\bigcap_{i \in \mathbb{N}} \overline{S_{x_0}(g^{-i}x_0, R_0)} = \lambda^- \text{ and } \bigcap_{i \in \mathbb{N}} \overline{S_{x_0}(g^{i}x_0, R_0)} = \lambda^+,$$

there is an integer i, such that the limit sets of the shadows are contained in the open neighbourhoods of λ^+ and λ^-, i.e.

$$\overline{S_{x_0}(g^{-i}x_0, R_0)} \cap \mathrm{UE} \subset U^- \text{ and } \overline{S_{x_0}(g^{i}x_0, R_0)} \cap \mathrm{UE} \subset U^+.$$

The element g^{-1} is in the semigroup generated by the inverses of $\mathrm{Supp}(\mu)$, i.e. $g^{-1} \in \mathrm{Supp}(\check{\mu})$. Hence, by Proposition 2.6,

$$\check{\nu} \times \nu \left(\overline{S_{x_0}(g^{-i}x_0, R_0)} \times \overline{S_{x_0}(g^{i}x_0, R_0)} \right) > 0,$$

and so $\check{\nu} \times \nu(\Gamma_r(X, Y)) > 0$, as required.

The final statement then follows from Theorem 1.4, which implies that there is a $\rho > 0$ such that any geodesic in $\Gamma_r(X, Y)$ must ρ-fellow travel $[X, Y]$, as required. Here we may choose X and Y on γ_g such that the geodesic $[X, Y]$ contains a subsegment of length D centered at any closest point on γ_g to the basepoint X_0; as γ_g is contained in a thick part K_ϵ, the set of closest points on γ_g to X_0 has bounded diameter, depending only on ϵ and the surface S. □

We now make use of the principal stratum assumption, i.e. that the semigroup generated by $\mathrm{Supp}(\mu)$ contains a pseudo-Anosov g whose invariant Teichmüller geodesic γ_g lies in the principal stratum. We first prove the following proposition:

Proposition 2.7. *Let g be a pseudo-Anosov element of $\mathrm{Mod}(S)$, whose invariant Teichmüller geodesic is contained in the principal stratum. For any $\rho > 0$, there is a constant $D > 0$, depending on ρ and g, such that for any pair of points X, Y on γ_g with $d_{\mathcal{T}}(X, Y) \geqslant D$, any Teichmüller geodesic in $\Gamma_\rho(X, Y)$ lies in the principal stratum.*

Proof. The invariant geodesic γ_g projects to a closed geodesic in moduli space, and so lies in the thick part K_ϵ, for some ϵ depending on g. If a geodesic γ passes through $B_\rho(X)$ and $B_\rho(Y)$ for $X, Y \in \gamma_g$ then by the fellow travelling result, Theorem 1.4 it $B(\epsilon, \rho)$-fellow travels $[X, Y]$.

To derive a contradiction, suppose that there is a sequence ϕ_n of geodesic segments in non-principal strata such that the ϕ_n fellow travel γ_g for distances d_n with $d_n \to \infty$ as $n \to \infty$. As the cyclic group generated by g acts coarsely transitively on γ_g, we may assume that the midpoints of the ϕ_n are all a bounded distance from the basepoint X in Teichmüller space. By convergence on compact sets we can pass to a limiting geodesic ϕ which lies in a non-principal strata, as the principal stratum is open. The geodesics ϕ and γ_g fellow travel in the forward direction for all times. By [Mas80, Theorem 2], this implies that ϕ and γ_g have the same vertical foliation. This is a contradiction since ϕ is in a non-principal stratum. □

We now complete the proof of Theorem 1.1.

Proof of Theorem 1.1. We fix a pseudo-Anosov element g in the support of μ for which the invariant Teichmüller geodesic γ_g is contained in the principal stratum. Without loss of generality, we fix the basepoint X to be on γ_g.

Let $\epsilon > 0$ be sufficiently small such that γ_g is contained in the thick part K_ϵ. Given this ϵ, let $F_0 > 0$ and $0 < e_0 < \frac{1}{2}$ be the constants from Proposition 2.1. Let $\rho > 0$ be the constant in Lemma 2.5 that ensures ρ-fellow travelling for any length $D > 0$ between γ_w and γ_g with a positive probability, depending on D. By Proposition 2.7, there is a D_0 such that any Teichmüller geodesic which $(\rho + F_0)$-fellow travels with γ_g distance at least D_0 is contained in the principal stratum. We shall set $D = D_0 + 2F_0$.

Let $k > 0$ be the smallest positive integer such that $d_{\mathcal{T}}(g^{-k}X, g^k X) \geqslant D$. By Theorem 1.4, any geodesic in $\Gamma_r(g^{-k}X, g^k X)$ ρ-fellow travels the subsegment $[g^{-k}X, g^k X]$ of γ_g. Let $\Omega \subset \text{Mod}(S)^{\mathbb{Z}}$ consist of those sample paths ω such that the sequences $w_{-n}X$ and $w_n X$ converge to distinct uniquely ergodic foliations $(\lambda^-, \lambda^+) \in \Gamma_r(g^{-k}X, g^k X)$. Lemma 2.5 implies that the subset Ω has positive probability $p > 0$.

Let $\sigma : \text{Mod}(S)^{\mathbb{Z}} \to \text{Mod}(S)^{\mathbb{Z}}$ be the shift map. Ergodicity of σ implies that for almost every ω, there is some $n \geqslant 0$ such that $\sigma^n(\omega) \in \Omega$. For such n, the subsegment of γ_ω of length D, centered at the closest point on γ_ω to the point $w_n X$, ρ-fellow travels with a translate of $w_n \gamma_g$. In particular, this implies that γ_ω lies in the principal stratum, giving the final claim in Theorem 1.1.

For almost every ω, the proportion of times $1 \leqslant n \leqslant N$ such that $\sigma^n(\omega) \in \Omega$ tends to p as $N \to \infty$. Choose numbers e_1 and e_2 such that $e_0 < e_1 < e_2 < \frac{1}{2}$, then this also holds for N replaced with either $e_1 N$ or $(1 - e_1)N$. So this implies that the proportion of times $e_1 N \leqslant n \leqslant (1 - e_1)N$ with this property also tends to p as $N \to \infty$. This implies that given ω, there is an N_0 such that for all $N \geqslant N_0$, there is an n with $e_1 N \leqslant n \leqslant (1 - e_1)N$ and $\sigma^n(\omega) \in \Omega$.

Recall that by sublinear tracking in Teichmüller space, due to Tiozzo [Tio15], there is a constant $\ell > 0$ such that for almost all ω,

$$\lim_{n\to\infty} \tfrac{1}{n} d_{\mathcal{T}}(w_n X, \gamma_\omega(\ell n)) = 0,$$

where γ_ω is parameterized such that $\gamma_\omega(0)$ is a closest point on γ_ω to the basepoint. Therefore, possibly replacing N_0 with a larger number, we may also assume that $d_{\mathcal{T}}(x_N X, \gamma_\omega(\ell N)) \leqslant (e_2 - e_1)N$ for all $N \geqslant N_0$.

Choose numbers ℓ_1 and ℓ_2, with $\ell_1 < \ell < \ell_2$, and choose them sufficiently close to ℓ so that $e_0 \ell < e_1 \ell_1$ and $(1 - e_1)\ell_2 < (1 - e_0)\ell$. Therefore the geodesic $[\gamma_\omega(e_2 \ell_1 N - \rho), \gamma_\omega((1 - e_2)\ell_2 N + \rho)]$ contains a subsegment of length at least D which ρ-fellow travels with a translate of γ_g. By our choice of ℓ_1 and ℓ_2, the geodesic $[\gamma_\omega(e_0 \ell_1 N - \rho), \gamma_\omega((1 - e_0)\ell_2 N + \rho)]$ is contained in $[\gamma_\omega(e_2 \ell N), \gamma_\omega((1 - e_2)\ell N)]$ for N sufficiently large. Now using Proposition 2.1, this implies that the invariant geodesic γ_{w_n} $(\rho + F_0)$-fellow travels with γ_g for a distance at least $D - 2F_0 \geqslant D_0$. Then by Proposition 2.7, γ_{w_n} is contained in the principal stratum, as required. □

3. Fellow travelling in Teichmüller space

We now provide a direct proof of Lemma 1.3, relying only on results from Rafi [Raf14]. The first result we shall use is the fellow travelling result for Teichmüller geodesics with endpoints in the thick part, Theorem 1.4. The second result is a thin triangles theorem for triangles in Teichmüller space, where one side has a large segment contained in the thick part.

Theorem 3.1. [Raf14, Theorem 8.1] *For every $\epsilon > 0$, there are constants C and L, depending only on ϵ and S, such that the following holds. Let X, Y and Z be three points in $\mathcal{T}(S)$, and let $[X', Y']$ be a segment of $[X, Y]$ with $d_{\mathcal{T}}(X', Y') > L$, such that $[X', Y']$ is contained in the ϵ-thick part of $\mathcal{T}(S)$. Then, there is a point $W \in [X', Y']$, such that*

$$\min\{d_{\mathcal{T}}(W, [X, Z]), d_{\mathcal{T}}(W, [Y, Z])\} \leqslant C.$$

We now prove Lemma 1.3.

Proof. The projection of an ϵ_i-thick Teichmüller geodesic makes definite progress in the curve complex, i.e. there exist constants P_i and Q_i, depending on ϵ_i and the surface S, such that for any points X, Y on γ we have the estimate

$$d_{\mathcal{C}}(x, y) \geqslant P_i d_{\mathcal{T}}(X, Y) - Q_i. \tag{3.2}$$

Set $\epsilon_1 = \epsilon$. Let L_1 and C_1 be the corresponding constants from the thin triangle result, Theorem 3.1. Let $B_1 = B(\epsilon_1, C_1 + L_1/2)$ be the constant in the fellow travelling theorem, Theorem 1.4. Set $\epsilon_2 = \epsilon(\epsilon_1, B_1)$, i.e. the B_1-neighbourhood of K_{ϵ_1} is contained in K_{ϵ_2}. Given this ϵ_2, let L_2 and C_2 be the corresponding constants from the thin triangle result, Theorem 3.1. Now that all the constants we need are defined, we shall choose L to be the maximum of the following three terms

$$\begin{gathered} \tfrac{3}{P_1}\left(M_1 C_1 + Q_1 + M_2 E + A_2 + A_1\right) + \tfrac{3}{2} L_1, \\ 3L_2 + 3L_1 + 6C_1, \\ \tfrac{3}{P_2}\left(M_1 C_2 + Q_2 + M_2 E + A_2 + A_1\right) + \tfrac{3}{2} L_1 + 3B_1. \end{gathered} \tag{3.3}$$

Let Z_1 be the point that is $1/3$ of the way along $[X, Y]$. Let γ_1 be the geodesic segment of γ centered at Z_1 with length L_1. Similarly, let Z_2 be the point that is $2/3$ of the way along $[X, Y]$. Let γ_2 be the geodesic segment of γ centered at Z_2 with length L_1. The second term of (3.3) implies that $L > 3L_1$. Figure 3.4 illustrates this setup.

Applying the thin triangles result, Theorem 3.1, to X, Y and Y', there is a point W_1 on $[X, Y'] \cup [Y, Y']$ within distance C_1 of γ_1. Similarly, there is a point W_2 on $[X, Y'] \cup [Y, Y']$ within distance C_1 of γ_2.

We now show that there is a lower bound on the distance of γ_2 from $[Y, Y']$. In particular, the same is true for the distance of γ_1, from $[Y, Y']$.

Claim 3.5. *The Teichmüller distance from γ_2 to $[Y, Y']$ is at least C_1.*

Proof. The Teichmüller distance of Y from γ_2 is at least $\frac{1}{3}L - \frac{1}{2}L_1$, i.e.

$$d_{\mathcal{T}}(\gamma_2, Y) \geqslant \tfrac{1}{3}L - \tfrac{1}{2}L_1.$$

As ϵ_1-thick geodesics make definite progress in $\mathcal{C}(S)$, (3.2), this implies

$$d_{\mathcal{C}}(\pi(\gamma_2), y) \geqslant P_1(\tfrac{1}{3}L - \tfrac{1}{2}L_1) - Q_1.$$

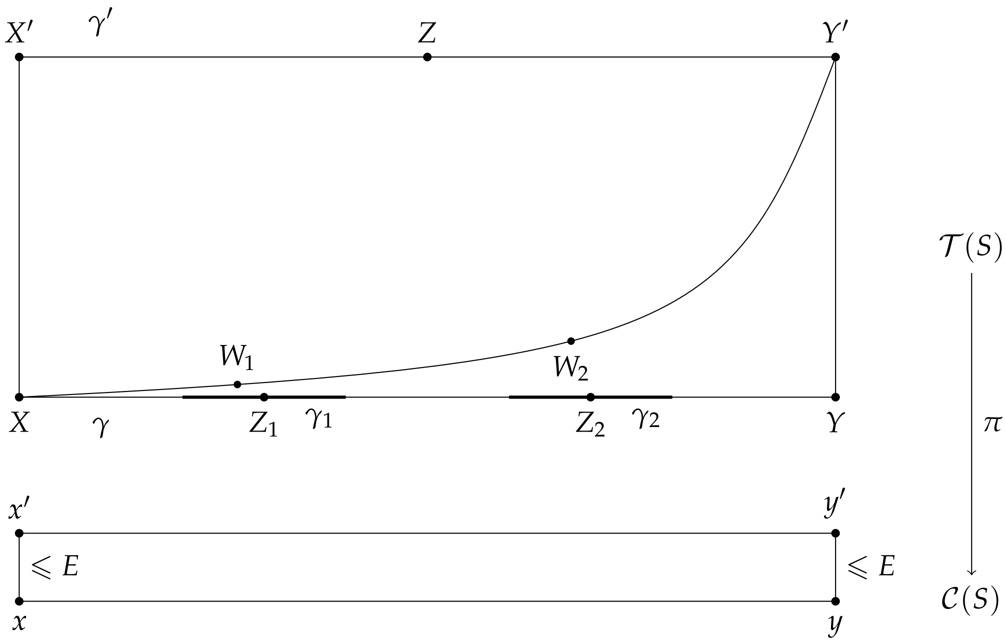

FIGURE 3.4. Fellow travelling geodesics in $\mathcal{T}(S)$.

Teichmüller geodesic segments project to (M_2, A_2)-quasigeodesics in $\mathcal{C}(S)$. Since the endpoints of γ and γ' are distance at most E apart in $\mathcal{C}(S)$, this implies,

$$d_{\mathcal{C}}(\pi(\gamma_2), \pi([Y, Y'])) \geqslant P_1(\tfrac{1}{3}L - \tfrac{1}{2}L_1) - Q_1 - M_2E - A_2.$$

As the curve complex distance is a coarse lower bound on the Teichmüller distance, (1.2), this implies

$$d_{\mathcal{T}}(\gamma_2, [Y, Y']) \geqslant \frac{1}{M_1}(P_1(\tfrac{1}{3}L - \tfrac{1}{2}L_1) - Q_1 - M_2E - A_2 - A_1).$$

Finally, a comparison with the first term of (3.3) shows that

$$d_{\mathcal{T}}(\gamma_2, [Y, Y']) > C_1,$$

as required. □

This implies that W_2 lies on $[X, Y']$ and not on $[Y, Y']$. As γ_1 is further away from $[Y, Y']$ along γ than γ_2, the same argument implies that W_1 lies on $[X, Y']$. Furthermore, $d_{\mathcal{T}}(W_1, Z_1) \leqslant C_1 + L_1/2$. Similarly $d_{\mathcal{T}}(W_2, Z_2) \leqslant C_1 + L_1/2$.

The segment $[X, Z_2]$ is in the ϵ_1-thick part. The endpoints of $[X, W_2]$ are within distance $C_1 + L_1/2$ of the endpoints of $[X, Z_2]$. So by the fellow travelling result, i.e. Theorem 1.4, $[X, W_2]$ and $[X, Z_2]$ are B_1-fellow travellers, where $B_1 = B(\epsilon_1, C_1 + L_1/2)$. Recall that B_1 depends on $\epsilon_1, C_1 + L_1/2$, and the surface S.

Recall that $\epsilon_2 = \epsilon'(\epsilon_1, B_1)$, i.e. the B_1-neighbourhood of K_{ϵ_1} is contained in K_{ϵ_2}. Note that ϵ_2 depends only on the constants $\epsilon = \epsilon_1, B_1$ and the surface S. In particular, the geodesic $[X, W_2]$ is contained in the ϵ_2-thick part. Given ϵ_2, recall that L_2 and C_2 are the corresponding constants from the thin triangle result, Theorem 3.1.

By the triangle inequality,

$$d_{\mathcal{T}}(Z_1, W_1) + d_{\mathcal{T}}(W_1, W_2) + d_{\mathcal{T}}(W_2, Z_2) \geqslant d_{\mathcal{T}}(Z_1, Z_2).$$

Thus, the Teichmüller distance between W_1 and W_2 is at least

$$d_{\mathcal{T}}(W_1, W_2) \geqslant \frac{1}{3}L - 2C_1 - L_1,$$

The second term of (3.3) implies that the right hand side above is at least L_2. So we may apply the thin triangles result, Theorem 3.1, to X, X' and Y' to conclude that there is a point Z on $[X, X'] \cup [X', Y']$ within distance C_2 of $[W_1, W_2]$.

We now show a lower bound for the distance between $[W_1, W_2]$ and $[X, X']$.

Claim 3.6. *The distance between $[W_1, W_2]$ and $[X, X']$ is at least C_2.*

Proof. Let W be a point of $[W_1, W_2]$ that is closest to X. Let V be the point of γ that is closest to W. Then

$$B_1 \geqslant d_{\mathcal{T}}(W, V) \quad \text{and} \quad d_{\mathcal{T}}(X, V) \geqslant \tfrac{1}{3}L - \tfrac{1}{2}L_1$$

Thus, by the triangle inequality

$$d_{\mathcal{T}}(X, W) \geqslant d_{\mathcal{T}}(X, V) - d_{\mathcal{T}}(W, V) > \tfrac{1}{3}L - \tfrac{1}{2}L_1 - B_1,$$

or equivalently

$$d_{\mathcal{T}}([W_1, W_2], X) \geqslant \tfrac{1}{3}L - \tfrac{1}{2}L_1 - B_1.$$

As ϵ_2-thick geodesics make definite progress in $\mathcal{C}(S)$, (3.2) implies

$$d_{\mathcal{C}}(\pi([W_1, W_2]), x) \geqslant P_2(\tfrac{1}{3}L - \tfrac{1}{2}L_1 - B_1) - Q_2.$$

As the distance between x and x' in $\mathcal{C}(S)$ is at most E, this implies,

$$d_{\mathcal{C}}(\pi([W_1, W_2]), \pi([X, X'])) \geqslant P_2(\tfrac{1}{3}L - \tfrac{3}{2}L_1 - C_1) - Q_2 - M_2E - A_2.$$

As the curve complex distance is a coarse lower bound on the Teichmüller metric (1.2), this implies

$$d_{\mathcal{T}}([W_1, W_2], [X, X']) \geqslant \frac{1}{M_1}(P_2(\tfrac{1}{3}L - \tfrac{1}{2}L_1 - B_1) - Q_2 - M_2E - A_2 - A_1).$$

A comparison with the third term in (3.3) then shows that

$$d_{\mathcal{T}}([W_1, W_2], [Y, Y']) > C_2,$$

as required. □

Claim 3.6 implies that Z lies on $[X', Y']$ and not on $[X, X']$. The segments $[W_1, W_2]$ and $[Z_1, Z_2]$ are B_1-fellow travellers. As Z lies within distance C_2 of $[W_1, W_2]$, the distance of Z from γ is at most $C_2 + B_1$. To conclude the proof of Lemma 1.3, we may set $F = C_2 + B_1$, which depends only on ϵ, A and the surface S, as required. □

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