UNIVERSITY OF GLASGOW

DEPARTMENT OF MATHEMATICS

MATHEMATICS-2N

Number Theory Examples

1. Determine the integral parts of $1/\pi$, 6, $-\sqrt{2}$, $17/3$, -15, $e-5$.

2. Prove that $[x]+[y] \leqslant [x+y]$ for all real numbers x, y. Find real numbers x, y such that $[x]+[y] < [x+y]$.

3. Is it true that $[x][y] \leqslant [xy]$ for all real numbers x, y? If it is true then construct a proof of the result, otherwise exhibit a counter-example.

4. Let x be a real number. Prove that
$$[x]+[-x] = \begin{cases} 0 & \text{if } x \in \mathbb{Z}, \\ -1 & \text{if } x \notin \mathbb{Z}. \end{cases}$$

5. Find integers q and r such that
$$b = qa + r \quad \text{and} \quad 0 \leqslant r < a$$
in the cases: (i) $a = 8$, $b = 34$; (ii) $a = 9$, $b = 45$; (iii) $a = 12$, $b = 7$; (iv) $a = 8$, $b = -34$.

6. Find integers q and r such that
$$b = qa + r \quad \text{and} \quad 0 \leqslant r < |a|$$
in the cases: (i) $a = -5$, $b = 23$; (ii) $a = -5$, $b = -23$.

7. Represent the number $(342501)_7$ in the base 5 and the base 16.

8. Let a, b, c be integers such that $a \mid b$ and $b \mid c$. Prove that $a \mid c$.

9. Find integers a, b, c such that $a \mid bc$ but $a \nmid b$ and $a \nmid c$.

10. Let a, b be integers such that $a \mid b$ and $b \mid a$. Prove that $a = \pm b$.

11. Let a, b, n be integers with $n \neq 0$. Prove that
$$a \mid b \text{ if and only if } na \mid nb.$$

12. Let a, b, c be integers such that $ab \mid c$. Prove that $a \mid c$ and $b \mid c$. Find integers a, b, c such that $a \mid c$ and $b \mid c$ but $ab \nmid c$.

13. Let a, b, c, d be integers such that $a \mid b$ and $c \mid d$. Prove that $ac \mid bd$.

14. Prove that the product of two consecutive integers is even and deduce that $8 \mid (a^2-1)$ for all odd integers a.

15. Show that $\gcd(15a+1, 13a-2) = 1$ or 43 for all integers a.

16. Prove that $\gcd(39a+5, 26a-1) = 1$ for all integers a.

17. Find the greatest common divisor of 901 and 1219 and list all their common divisors. Find integers x, y such that
$$901x + 1219y = \gcd(901, 1219).$$

18. Find integers x, y such that

$$425x + 1649y = \gcd(425, 1649)$$

and determine the general solution of this Diophantine equation.

19. Find the general solution of the Diophantine equation

$$52x + 161y = 3.$$

State the general solution of the Diophantine equation

$$161x - 52y = 3$$

and find all the integral solutions satisfying $0 < y < 3x + 12$.

20. Find all positive integral solutions of

$$14x + 17y = 79.$$

21. Show that the equation

$$51x + 85y = 100$$

has no integral solutions.

22. Find the general solution of the Diophantine equation

$$46x + 32y = 1000.$$

How many positive integral solutions are there?

23. A concert ticket costs £3.60 with a special rate of £1.50 for students. At a particular concert when both categories of patrons were present and the students were outnumbered by more than 2 to 1, the takings amounted to £180. How many people were in the audience?

24. Let a, b be integers, not both zero. Prove that

$$\gcd(na, nb) = n \gcd(a, b)$$

for all positive integers n.

25. Let a, b, c be integers, with a, b not both zero, such that $a \mid c$, $b \mid c$ and $\gcd(a, b) = 1$. Prove that $ab \mid c$.

26. Let a, b, c be integers such that a and c are coprime, and b and c are coprime. Prove that ab and c are coprime. Given pairwise coprime integers m_1, m_2, $\ldots$, m_n $(n \geqslant 3)$, deduce that $m_1 m_2 \ldots m_{n-1}$ and m_n are coprime.

27. Let m_1, m_2, $\ldots$, m_n be pairwise coprime integers $(n \geqslant 2)$ and c any integer. Deduce from Examples 25 and 26 that

$$m_i \mid c \ (i = 1, 2, \ldots, n) \iff m_1 m_2 \ldots m_n \mid c.$$

28. Find integers x, y, z such that

$$24x + 52y + 78z = 98.$$

Find the general solution of this Diophantine equation.
[Hint: Consider the equations $24x + 26u = 98$, $2y + 3z = u$.]

29. Which of the following integers are prime: (i) 101, (ii) 111, (iii) 1111, (iv) 1001, (v) 11111?

30. Let p_1, p_2, p_3, … be the primes in increasing order. Find the least integer n such that

$$p_1 p_2 \dots p_n + 1$$

is composite.

31. Express 290472 as a product of primes.

32. Let a, b, c be positive integers and p_1, p_2, p_3, … the primes (in increasing order). Suppose that

$$a = \prod_{i=1}^{\infty} p_i^{\alpha_i}, \qquad b = \prod_{i=1}^{\infty} p_i^{\beta_i}, \qquad c = \prod_{i=1}^{\infty} p_i^{\gamma_i}$$

for some non-negative integers α_i, β_i, γ_i ($i = 1, 2, 3, \dots$). Show that:

(i) $ab = c \iff \alpha_i + \beta_i = \gamma_i \ (i = 1, 2, 3, \dots)$;

(ii) $a \,|\, b \iff \alpha_i \leqslant \beta_i \ (i = 1, 2, 3, \dots)$;

(iii) $\gcd(a, b) = \prod_{i=1}^{\infty} p_i^{\min(\alpha_i, \beta_i)}$;

(iv) $\operatorname{lcm}(a, b) = \prod_{i=1}^{\infty} p_i^{\max(\alpha_i, \beta_i)}$.

33. Use the formulae for gcd and lcm in terms of products of primes to show that

$$\gcd(a, b)\operatorname{lcm}(a, b) = ab$$

for all positive integers a, b. Use this formula to find lcm(4655,12075).

34. Let E be the set of even integers. An *E-prime* is a positive integer in E which cannot be expressed as the product of two or more positive members of E (not necessarily distinct). Find the first six E-primes. Show that every positive member of E can be expressed as a product of E-primes, but that this factorisation is not unique. Find the least two positive members of E with more than one factorisation into E-primes.

35. A *perfect number* is a positive integer which is the sum of all its positive divisors other than itself. For example, $6 = 1 + 2 + 3$ and so 6 is perfect. Verify that 28 and 496 are perfect. Show also that if $2^n - 1$ is prime for some positive integer n then $2^{n-1}(2^n - 1)$ is perfect. Hence find another perfect number.

36. Find the least positive integer k such that $6k - 1$ is not prime and the least positive integer k such that $6k + 1$ is not prime.

37. Prove that every prime $p > 3$ satisfies

$$p^2 \equiv 1 \pmod{24}.$$

38. Let a, b, m be integers with $m > 0$. Which of the following are true and which false:

(i) $a \equiv b \pmod{m} \implies 2a \equiv 2b \pmod{m}$;

(ii) $2a \equiv 2b \pmod{m} \implies a \equiv b \pmod{m}$;

(iii) $a \equiv b \pmod{m} \implies a \equiv b \pmod{2m}$;

(iv) $a \equiv b \pmod{2m} \implies a \equiv b \pmod{m}$?

Justify your answers.

39. Let a, b, m be integers, with $m > 0$, such that

$$a \equiv b \pmod{m}.$$

Prove that

$$\gcd(a, m) = \gcd(b, m).$$

40. Let a, b, k, m be integers, with $m > 0$, such that

$$ak \equiv bk \pmod{m}.$$

Prove that

$$a \equiv b \pmod{m/d},$$

where $d = \gcd(k, m)$.

41. Construct the addition and multiplication tables (mod 10). Comment on (i) primes (mod 10), (ii) unique factorisation (mod 10).

42. Verify that $4^2 \equiv 3^3 \equiv 5 \pmod{11}$. Deduce that $2.4^{2n+1} + 3^{3n+1}$ is divisible by 11 for all positive integers n.

43. Show that

$$\text{(i) } 3^{2n} - 2^n \equiv 0 \pmod{7}, \qquad \text{(ii) } 2^{2n-1} + 3^{2n-1} \equiv 0 \pmod{5}$$

for all positive integers n.

44. Show that 11 divides $3^{5n} + 4^{5n+2} + 5^{5n+1}$ for all positive integers n.

45. Without performing any division, determine whether the integers

$$17654212521, \quad 15023567847$$

are divisible by 3, 9 or 11.

46. Solve the following linear congruences:
   (i) $3x \equiv 5 \pmod{17}$;
   (ii) $8x \equiv 6 \pmod{18}$;
   (iii) $8x \equiv 3 \pmod{19}$.

47. Solve the following systems of linear congruences:
   (i) $x \equiv 1 \pmod{2}$, $x \equiv 4 \pmod{5}$, $x \equiv 3 \pmod{11}$;
   (ii) $x \equiv 1 \pmod{3}$, $2x \equiv 5 \pmod{7}$, $x \equiv 4 \pmod{8}$;
   (iii) $x \equiv 1 \pmod{2}$, $x \equiv 2 \pmod{3}$, $x \equiv 3 \pmod{5}$, $x \equiv 5 \pmod{7}$.

48. Solve the linear congruence:

$$17x \equiv 11 \pmod{468}.$$

49. Use two different methods to solve the linear congruence:

$$13x \equiv 3 \pmod{110}.$$

50. Which of the following sets are CSR (mod 6):
   (i) $\{7, -2, -3, 2, 11\}$,
   (ii) $\{12, 15, -2, -10, 13, 3\}$,
   (iii) $\{4, -9, 7, 0, -1, 8\}$,
   (iv) $\{3, 14, 17, -6, 10, -5, 11\}$,
   (v) $\{3, 5, 6, 4, 2, 7, 5\}$?

51. Find the values of (i) $\phi(102)$, (ii) $\phi(108)$, (iii) $\phi(504)$.

52. Find (i) $\phi(20)$, (ii) $\phi(1001)$, (iii) $\phi(2002)$.

53. Show that $13 \,|\, (1981^{123} + 1891^{321})$.

54. Show that $333^{333} + 777^{777}$ is divisible by 11.

55. Which of the following sets are RSR (mod 12):
    (i) $\{17, 11, -23, 29\}$,
    (ii) $\{5, 15, 13, -5\}$,
    (iii) $\{13, -1, 19, 5\}$,
    (iv) $\{1, 11, 19\}$?

56. Determine the units digit in the decimal representation of 2^{400}.

57. Simplify (i) $3^{17} \pmod 7$, (ii) $3^{17} \pmod 8$, (iii) $40^{60} \pmod{61}$, (iv) $7^{1014} \pmod{31}$, (v) $5^{1166} \pmod{41}$.

58. Show that:
    (i) $a^{13} \equiv a \pmod{2730}$ for every integer a;
    (ii) $a^{17} \equiv a \pmod{8160}$ for every odd integer a.

59. Do there exist infinitely many positive integers n such that $\phi(n) = \phi(2n)$?

60. Show that any set of m consecutive integers, where m is a positive integer, contains exactly one multiple of m. Deduce that:
    (i) $2^n - 1$ and $2^n + 1$ cannot both be prime for an integer $n > 2$;
    (ii) $a^5 - 5a^3 + 4a \equiv 0 \pmod{60}$ for all integers a.

61. Find the solution of each of the following equations, where n is a positive integer:

$$\text{(i) } \phi(n) = \frac{1}{2}n, \quad \text{(ii) } \phi(3n) = 3\phi(n), \quad \text{(iii) } \phi(n) = 12.$$

62. Find $\operatorname{ord}_{13} a$ for $a = 1, 2, \ldots, 12$.