2N Class Test 2007.

1. If $a = 3k+r$, with $k \in \mathbf{Z}$, then $a(a^2-1) = a(a-1)(a+1) = (3k+r)(3k+r-1)(3k+r+1)$.

When $r = 0$, $a(a^2-1) = 3.\{k(3k-1)(3k+1)\}$.
When $r = 1$, $a(a^2-1) = 3.\{ (3k+1)k(3k+2)\}$.
When $r = 2$, $a(a^2-1) = 3.\{(3k+2)(3k+1)(k+1)\}$.

We are given that we can choose $k \in Z$, so that $r = 0$, 1 or 2.
Thus, in any case, $3 \mid a(a^2-1)$.

2. Suppose that $d \mid (9a+5)$ and $d \mid (7a+2)$.
Then $d \mid \{7.(9a+5)-9.(7a+2)\}$, i.e. $d \mid 17$.

Now d_n is a *positive* common divisor,
As 17 is prime, we must have $d_n = 1$ or 17.

3.

$$\begin{aligned} 121 &= 1.77 + 44 \\ 77 &= 1.44 + 33 \\ 44 &= 1.33 + 11 \\ 33 &= 3.11 + 0 \qquad \text{so that } \gcd(121,77) = 11 \end{aligned}$$

As 11 does not divide 1000, there are no integer solutions.

4.

$$\begin{aligned} 105 &= 1.91 + 14 \\ 91 &= 6.14 + 7 \\ 14 &= 2.7 + 0 \qquad \text{so that } \gcd(91,105) = 7 \end{aligned}$$

Backtracking, $7 = 91 – 6.14 = 91 – 6.(105 – 91) = 7.91 – 6.105$.

Thus, $x = 7$, $y = -6$ is a solution of $91x + 105y = 7$.

Now, $700 = 100.7$, so $x = 700$, $y = -600$ is a solution of $91x + 105y = 700$.

The general solution is

$x = 700 – (105/7)t = 700 – 15t$, $y =- -600 +(91/7)t = -600 + 13t$, $t \in \mathbf{Z}$.