

PART I

- (i) You should attempt as many questions as you can in this part.
- (ii) Write your answers in the answer book provided, beginning each question on a new page.
- (iii) Questions in this part do not necessarily carry equal marks. The mark allocation is indicated for each question.

Question 1

Draw a sketch of the graph of the function f defined by

$$f(x) = \frac{3 - 2x}{x + 1}.$$

Your sketch should include:

- (a) any asymptotes for the graph;
- (b) any points where the graph crosses the axes. [4]

Question 2

Express the complex number $z = \left(\frac{7 - i}{3 + i}\right)^2$ in Cartesian form.

Write down $|z|$ and $\sin \theta$ where θ is the argument of z . [5]

Question 3

Find the matrix of the linear transformation

$$\begin{aligned} f: \mathbb{R}^2 &\longrightarrow \mathbb{R}^2 \\ (x, y) &\longmapsto (2x + y, 3x - y) \end{aligned}$$

with respect to

- (a) the standard basis in both domain and codomain;
- (b) the basis $\{(1, 1), (2, -3)\}$ in the domain and the standard basis in the codomain;
- (c) the basis $\{(1, 1), (2, -3)\}$ in both domain and codomain. [5]

Question 4

The set S is defined by $S = \{(a, b, 3b - a) : a, b \in \mathbb{R}\}$.

- (a) Show that S is a subspace of \mathbb{R}^3 .
- (b) Show that the vectors $(1, 0, -1)$ and $(1, 1, 2)$ belong to S . Express the general vector $(a, b, 3b - a)$ as a linear combination of these two vectors.

Explain why the set $\{(1, 0, -1), (1, 1, 2)\}$ is a basis for S . [5]

Question 5

Find the solution set of the inequality

$$\frac{3x}{x^2 - 4} < 1. \quad [5]$$

Question 6

Determine whether or not each of the following sequences $\{a_n\}$ converges, naming any result or test that you use. If it does converge, find its limit.

(a) $a_n = \frac{n^3 - 11n + 4(n!)}{2n^2 + n! - 6}$

(b) $a_n = \frac{2n^4 + 5^n + 8}{n^3 + 11n + 3^n}$

[6]

Question 7

Show that the function

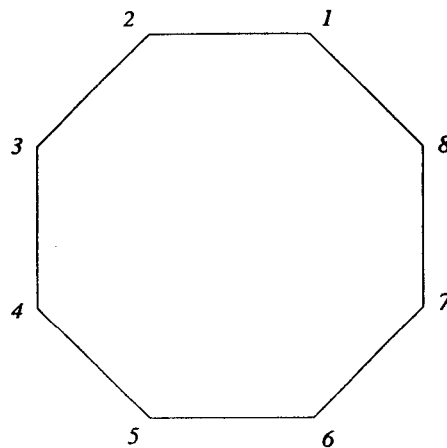
$$f : x \mapsto \begin{cases} \sin 2\pi x, & x \leq 1, \\ x^2 - 1, & x > 1, \end{cases}$$

is continuous on \mathbb{R} . You should name any test or result that you use to justify your answer.

[5]

Question 8

This question concerns the symmetry group G of the regular octagon, shown below.



Let $g \in G$ be the anticlockwise rotation of the octagon through an angle of $\frac{3\pi}{4}$ about its centre, and let $h \in G$ be the reflection of the octagon in the line through the vertices at locations 3 and 7.

- Write g , g^2 and h in cycle form, using the numbering of the locations of the vertices as shown above. State the order of g .
- Express the conjugate ghg^{-1} , of h by g , in cycle form, and describe ghg^{-1} geometrically.
- Are the symmetries $k = (15)(26)(37)(48)$ and $l = (12)(38)(47)(56)$ conjugate? Give a brief reason for your answer.

[5]

Question 9

In this question

$$G = \{1, 2, 4, 5, 8, 10, 11, 13, 16, 17, 19, 20\}$$

which is a group under the operation of multiplication modulo 21. (You are NOT asked to prove this result.)

- (a) Find a subgroup of G of order 6.
(b) Write down the homomorphism property as it would apply to the specific case of a function

$$f: G \rightarrow \mathbb{Z}_4.$$

- (c) Specify (by giving the image of each element of G) a homomorphism

$$f: G \rightarrow \mathbb{Z}_4$$

that has as kernel the subgroup that you found in part (a).

[5]

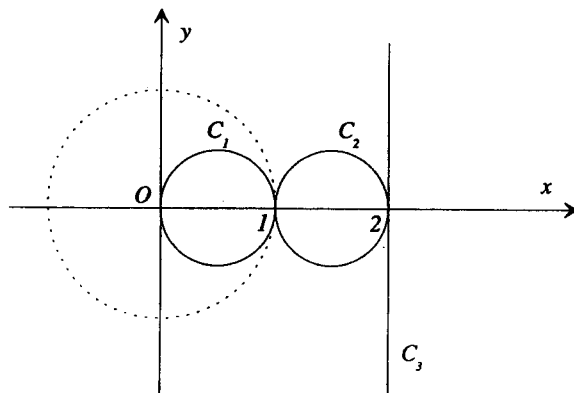
Question 10

- (a) Find an affine transformation $t: \mathbb{R}^2 \rightarrow \mathbb{R}^2$ which sends the points $(0, 0)$, $(1, 0)$, $(0, 1)$ to the points $(2, 3)$, $(5, 7)$, $(3, 5)$, respectively.
(b) Find the inverse of t , and hence or otherwise find the image of the line $y = 2x$ under t .

[6]

Question 11

Let C_1 and C_2 be circles of radius $\frac{1}{2}$ centred at $(\frac{1}{2}, 0)$ and $(\frac{3}{2}, 0)$, respectively, and let C_3 be the extended line with equation $x = 2$.



Draw a sketch showing the images of C_1 , C_2 and C_3 under inversion in the unit circle \mathcal{C} . Mark clearly which image is which.

[4]

Question 12

- (a) Find the equation of the Line which passes through the Points $A = [1, 2, 3]$ and $B = [1, 1, 1]$. Check that the Points $C = [2, -1, -4]$ and $D = [3, 1, -1]$ lie on this Line.
(b) Calculate the cross-ratio $(ABCD)$.

[5]

Question 13

Prove that

$$\lim_{x \rightarrow 1} \frac{x - (3 - 2x)^{1/2}}{x^2 + x - 2}$$

exists and determine its value.

[5]

Question 14

This question concerns $I_n = \int_1^e x (\log_e x)^n dx$, $n \geq 0$.

(a) Evaluate I_0 .

(b) Show that $I_n = \frac{e^2}{2} - \frac{n}{2} I_{n-1}$, where $n \geq 1$, and hence evaluate I_2 .

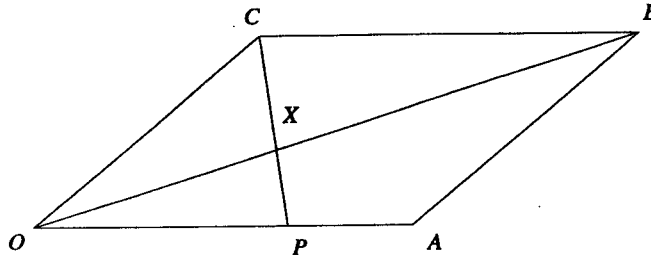
[5]

PART II

- (i) You should attempt no more than **THREE** questions from this part.
- (ii) Each question carries 10 marks. The mark allocation for each section of a question is given in square brackets beside the section.
- (iii) Start each question on a new page of your answer book.

Question 15

The parallelogram $OABC$ has vertices at O (the origin) and at points A, B, C with position vectors $\mathbf{a}, \mathbf{b}, \mathbf{c}$ respectively. The point P lies two-thirds of the way along OA from O . The lines OB, CP intersect at X .



- (a) Write down the position vectors \mathbf{b} and \mathbf{p} of the points B and P , respectively, in terms of \mathbf{a} and \mathbf{c} . [2]
- (b) Find the vector form of the equations of the lines OB and CP , in terms of \mathbf{a} and \mathbf{c} and hence find the position vector of the point X . In what ratio does X divide OB ? [4]
- (c) If $\mathbf{a} = (3, 1)$ and $\mathbf{c} = (1, 2)$ show that CP is perpendicular to OB , and find the cosine of the angle between OC and OB . [4]

Question 16

The matrix $\mathbf{A} = \begin{pmatrix} 2 & 0 & 1 \\ 0 & 2 & -1 \\ 1 & -1 & 1 \end{pmatrix}$ represents a linear transformation f with respect to the standard basis in both the domain and the codomain.

- (a) Determine the characteristic equation of \mathbf{A} and solve this equation to find the eigenvalues of f . [3]
- (b) Find a basis for \mathbb{R}^3 consisting of eigenvectors of f . [4]
- (c) The matrix \mathbf{C} of f with respect to the eigenvector basis that you chose in part (b) in both domain and codomain can be expressed as $\mathbf{C} = \mathbf{P}^T \mathbf{A} \mathbf{P}$. Find a suitable matrix \mathbf{P} , and write down the matrix \mathbf{C} . [3]

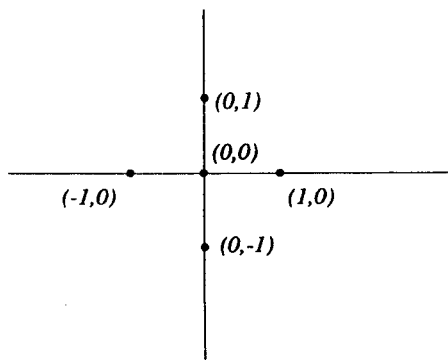
Question 17

Determine whether or not each of the following series is convergent, naming any result or test that you use.

- (a) $\sum_{n=1}^{\infty} \frac{n+1}{n^2+n-1}$ [3]
- (b) $\sum_{n=1}^{\infty} \frac{n^3 2^n}{n!}$ [3]
- (c) $\sum_{n=1}^{\infty} \frac{1+2\cos n}{2n^2+1}$ [4]

Question 18

The diagram below shows the five points $(0,0)$, $(1,0)$, $(0,1)$, $(-1,0)$ and $(0,-1)$ in \mathbb{R}^2 .



- (a) List the set G of rotations, r_θ , and reflections, q_ϕ , of the plane that fix the set $\{(0,0), (1,0), (0,1), (-1,0), (0,-1)\}$ (as a set).

[3]

You may assume now that the group G acts on the plane \mathbb{R}^2 in the natural way; that is, for $g \in G$ and $x \in \mathbb{R}^2$,

$$g \wedge x = g(x).$$

(You are NOT asked to prove that this is a group action.)

- (b) Find the orbit and the stabiliser of

- (i) $(0,0)$,
- (ii) $(0,1)$,
- (iii) $(1,1)$,
- (iv) $(2,1)$.

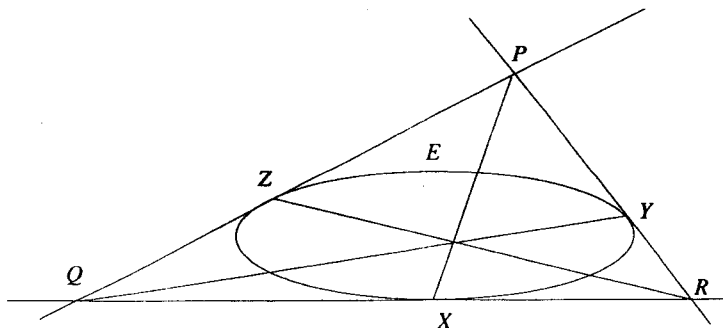
[7]

Question 19

- (a) Let E be the non-degenerate projective conic with equation

$$xy + yz + zx = 0.$$

- (i) Show that E passes through the Points $X = [1,0,0]$, $Y = [0,1,0]$ and $Z = [0,0,1]$.
- (ii) Find an equation for the tangent to E at each of the Points X , Y and Z .



- (iii) Determine the Point P where the tangents at Y and at Z meet. Hence or otherwise find the Point Q where the tangents at X and at Z meet and the Point R where the tangents at X and at Y meet.
- (iv) Show that the Lines XP , YQ and ZR meet at a single Point.

[8]

- (b) Let ABC be an arbitrary triangle which touches a non-degenerate projective conic F at Points U , V , W . Use your answer to part (a) to explain why the Lines UA , VB , WC are concurrent.

[2]

Question 20

This question concerns the linear flow for which the velocity function is

$$V(x, y) = (3x + 2y, x + 4y).$$

- (a) Write down
- (i) the matrix A of the flow;
 - (ii) the first order differential equations satisfied by the co-ordinate functions f and g of any flow function $\alpha = (f, g)$ for this flow;
 - (iii) a second order differential equation satisfied by both f and g . [3]
- (b) Find the general solution of the differential equation in part (a)(iii). [3]
- (c) Determine the flow function α for V which satisfies $\alpha(0) = (1, 4)$. [4]

[END OF QUESTION PAPER]