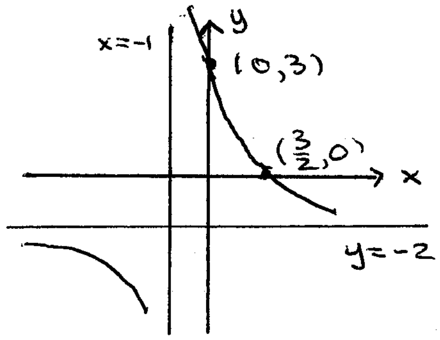
M203 Examination 1996 Solutions.

Handbook Reference

1.



(a) asymptotes are $x=-1$ and $y=-2$ — p5, §2.5

(b) crosses axes where $x=0$ $(0,3)$ and $y=0$ $(3/2,0)$.

2. $\frac{7-i}{3+i} = \frac{(7-i)(3-i)}{(3+i)(3-i)} = \frac{20-10i}{10} = 2-i$ — p13 §5.6, see "Strategy"

Hence $\underline{z = (2-i)^2 = 3-4i}$. (1)

$\underline{|z| = \sqrt{3^2+(-4)^2} = 5}$ — p13 §5.5

Then $z = 5(\cos\theta + i\sin\theta)$, so, from (1) $\underline{\sin\theta = -4/5}$ — p13 §5.7

3. (a) From the formula for f, the matrix is $\underline{A = \begin{pmatrix} 2 & 1 \\ 3 & -1 \end{pmatrix}}$. — We use p 24 §§ 1.2, 1.6

(b) The transition matrix for the change is $P = \begin{pmatrix} 1 & 2 \\ 1 & -3 \end{pmatrix}$,

so the required matrix is $B = AP = \begin{pmatrix} 2 & 1 \\ 3 & -1 \end{pmatrix}\begin{pmatrix} 1 & 2 \\ 1 & -3 \end{pmatrix} = \underline{\begin{pmatrix} 3 & 1 \\ 2 & 9 \end{pmatrix}}$ — we could also use p 20 § 3.4

c) Since we change basis in both domain and codomain, the matrix is $C = P^{-1}AP = P^{-1}B$

Now $P^{-1} = \frac{1}{-5}\begin{pmatrix} -3 & -2 \\ -1 & 1 \end{pmatrix}$, so $C = -\frac{1}{5}\begin{pmatrix} -3 & -2 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} 3 & 1 \\ 2 & 9 \end{pmatrix} = \underline{\begin{pmatrix} \frac{13}{5} & \frac{21}{5} \\ 1/5 & -8/5 \end{pmatrix}}$

4. (We show that, for $\underline{x}, \underline{y} \in S$, $\alpha \in \mathbb{R}$, $\underline{x}+\underline{y} \in S$ and $\alpha\underline{x} \in S$) — p 16 §1.5

(a) Let $\underline{x} = (a_1, b_1, 3b_1 - a_1)$, $\underline{y} = (a_2, b_2, 3b_2 - a_2)$

Then $\underline{x} + \underline{y} = (a_1+a_2, b_1+b_2, 3(b_1+b_2) - (a_1+a_2)) \in S$

and $\alpha\underline{x} = (\alpha a_1, \alpha b_1, 3(\alpha b_1) - (\alpha a_1)) \in S$, so S a subspace

(b) $a=1, b=0$ gives $(1,0,-1) \in S$, $a=1, b=1$ gives $(1,1,2) \in S$.

The general vector $(a, b, 3b-a) = \alpha(1,0,-1) + \beta(1,1,2)$

$$\Leftrightarrow \begin{cases} a = \alpha + \beta \\ b = \beta \quad \text{- so } \beta = b \\ 3b-a = -\alpha + 2\beta \end{cases} \qquad \alpha = a-\beta = a-b$$

$\alpha = a-b$, $\beta = b$ also satisfy 3rd equation. — Note the check!

Thus $\underline{(a,b,3b-a) = (a-b)(1,0,-1) + b(1,1,2)}$.

It is a basis since it spans S and is linearly independent as vectors are non-parallel — p17 §2.7, p 17 §2.6

5. $\dfrac{3x}{x^2-4} < 1 \iff 1 - \dfrac{3x}{x^2-4} > 0 \iff \dfrac{x^2-4-3x}{x^2-4} > 0$ p28 §2.1(i) Note the "$\iff$" signs.

$\iff \dfrac{(x-4)(x+1)}{(x-2)(x+2)} > 0$

<u>sign diagram:</u>

$x-4$	$-$	$-\ -$	$-\ -\ -\ -$	$-\ +$	$+$
$x-1$	$-$	$-\ -$	$-\ +\ +\ +$	$+\ +$	$+$
$x-2$	$-$	$-\ -$	$-\ -\ -\ +$	$+\ +$	$+$
$x+2$	$-$	$-\ +$	$+\ +\ +\ +$	$+\ +$	$+$

$+\ \ -2\ \ \cancel{+}\ \ 1\ +\ 2\ \ -\ \ 4\ +$

Solution set is $\cancel{\{x}$ $\underline{]-\infty,-2[\ \cup\ ]1,2[\ \cup\ ]4,\infty[}$

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6. (a) $a_n = \dfrac{(n^3/n!) - 11(n/n!) + 4}{2(n^2/n!) + 1 - 6.(1/n!)}$ [dividing by $n!$, the dominant term] p31 §3.4

$\to \dfrac{0 - 11.0 + 4}{2.0 + 1 - 6.0} = 4$ as $n\to\infty$ [by the Combination Rules as $\{n^p/n!\}$ is basic null] The reasons are <u>vital</u>.

Hence <u>$\{a_n\}$ converges to 4</u>

(b) (Here the dominant term appears only on the top, so we suspect $a_n \to \infty$. We use the Reciprocal Rule) p31 §4.3(a)

1) a_n is <u>always</u> positive (so "eventually positive") Note that <u>both</u> conditions must be checked.

2) $1/a_n = \dfrac{(n^3/5^n) + 11(n/5^n) + (3/5)^n}{2(n^4/5^n) + 1 + 8(1/5^n)}$ (5^n is dominant)

$\to 0$ as $n\to\infty$ [By Combination Rules, as $\{n^p(\frac{1}{5})^n\}, \{(\frac{3}{5})^n\}$ basic null]

Hence, by the Reciprocal Rule $a_n \to \infty$, ie <u>$\{a_n\}$ diverges</u>.

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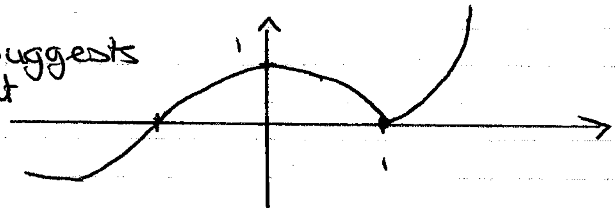
7. (A rough sketch suggests that graphs "fit" at $x = 1$) p88 gives graphs. A graph is <u>not</u> required



Let $g(x) = \sin 2\pi x$, $h(x) = x^2 - 1$, (basic continuous

For $a<1$, $f(x) = g(x)$ on $]-\infty,1[$, $a\in]-\infty,1[$, and g is continuous at a, so f is continuous at a by Local Rule p35 §3.1 Rule is <u>vital</u>

For $a>1$, $f(x) = h(x)$ on $]1,\infty[$, $a\in]1,\infty[$ and h is continuous at a, so f continuous at a by Local Rule

Let $I = \mathbb{R}$. We have 1) $\begin{cases} f(x) = g(x) & x<1 \\ f(x) = h(x) & x>1 \end{cases}$

also, 2) $g(1) = f(1) = h(1)$ $(=0)$

and 3) g, h continuous at 1, so f is continuous

[illegible] p35 §3.2

8. (a) $g = (1\,6\,3\,8\,5\,2\,7\,4)$ (3/8 of a full turn)

$g^2 = (1\,3\,5\,7)(2\,4\,6\,8)$ (3/4 turn)

$h = (1\,5)(2\,4)(6\,8)$.

(b) $ghg^{-1} = (g(1)\,g(5))(g(2)\,g(4))(g(6)\,g(8)) = (6\,2)(7\,1)(3\,5)$ — p41 §4.2

This is a reflection in the line joining 4 and 8.

(c) No – k is a rotation (through π), ℓ is a reflection in the axis bisecting 12 + 56, ie they are of different kinds

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9. (a) (If we can find an element of order 6, this will generate a suitable ~~cyclic~~ subgroup. We're lucky...) — Very hard!

$2^1 = 2,\ 2^2 = 4,\ 2^3 = 8,\ 2^4 = 16,\ 2^5 = 11$ (in G), $2^6 = 1$ (in G) — p39 §1.5

Hence a suitable subgroup is $\langle 2\rangle = \{1, 2, 4, 8, 11, 16\}$.

(b) $f(a \times_{21} b) = f(a) +_4 f(b)$, for $a, b \in G$. — p47 §1.5

(c) (By the definition of kernel), $f(x) = 0$ for $x \in \langle 2\rangle$ — p 48 §3.3

(As $\langle 2\rangle$ has index 6, it has only cosets $\langle 2\rangle$, $5\langle 2\rangle$ By Correspondence Theorem) — p 48 §4.3

$$f(x) = z \neq 0,\ x \in 5\langle 2\rangle = \{5, 10, 13, 17, 19, 20\}$$

Finally, as $5 \times_{21} 5 = 4 \in H$, $z +_4 z = 0$, ie $z = 2$ — by part (b).

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10. (a) $t: \underline{x} \mapsto A\underline{x} + \underline{b}$, where $\underline{b} = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$ (image of $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$) — p.52 §3.2

and A has columns $\begin{pmatrix} 5 \\ 7 \end{pmatrix} - \begin{pmatrix} 2 \\ 3 \end{pmatrix} = \begin{pmatrix} 3 \\ 4 \end{pmatrix}$, $\begin{pmatrix} 3 \\ 5 \end{pmatrix} - \begin{pmatrix} 2 \\ 3 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$

ie $t: \underline{x} \longmapsto \begin{pmatrix} 3 & 1 \\ 4 & 2 \end{pmatrix}\underline{x} + \begin{pmatrix} 2 \\ 3 \end{pmatrix}$

(b) Inverse is $t^{-1}: x \mapsto A^{-1}\underline{x} - A^{-1}\underline{b}$ ($A, \underline{b}$ as above) — p 52 §2.2

$A^{-1} = \frac{1}{2}\begin{pmatrix} 2 & -1 \\ -4 & 3 \end{pmatrix}$ so $A^{-1}\underline{b} = \frac{1}{2}\begin{pmatrix} 2 & -1 \\ -4 & 3 \end{pmatrix}\begin{pmatrix} 2 \\ 3 \end{pmatrix} = \begin{pmatrix} 1/2 \\ 1/2 \end{pmatrix}$

Hence $t^{-1}: \underline{x} \mapsto \frac{1}{2}\begin{pmatrix} 2 & -1 \\ -4 & 3 \end{pmatrix}\underline{x} - \begin{pmatrix} 1/2 \\ 1/2 \end{pmatrix}$

$$\begin{pmatrix} x \\ y \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 2 & -1 \\ -4 & 3 \end{pmatrix}\begin{pmatrix} x' \\ y' \end{pmatrix} - \begin{pmatrix} \frac{1}{2} \\ 1/2 \end{pmatrix} = \begin{pmatrix} x' - \frac{1}{2}y' - \frac{1}{2} \\ -2x' + \frac{3}{2}y' - \frac{1}{2} \end{pmatrix}$$ — p 52 §3.1

Long!

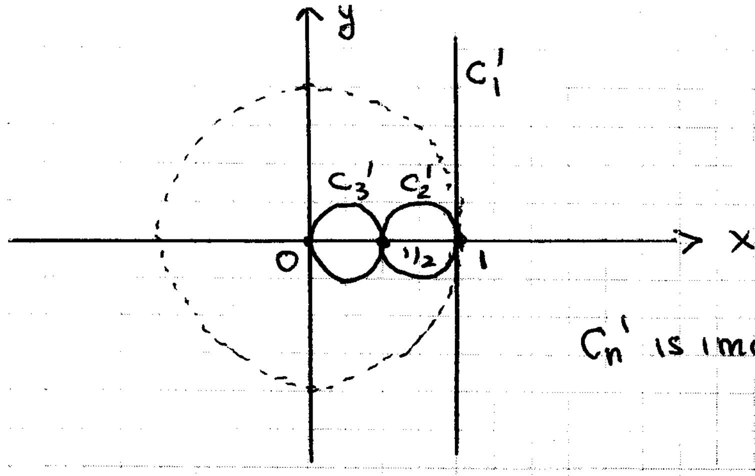
so $x = x' - \frac{1}{2}y' - \frac{1}{2}$, $y = -2x' + \frac{3}{2}y' - \frac{1}{2}$ + $y = 2x$

gives $\left(-2x' + \frac{3}{2}y' - \frac{1}{2}\right) = 2\left(x' - \frac{1}{2}y' - \frac{1}{2}\right)$

ie $\quad 4x' - \frac{5}{2}y' - \frac{1}{2} = 0$

ie. $\quad 8x - 5y - 1 = 0$

11.



C_n' is image of C_n

C_2', C_3' pass through the image of $(2,0)$ ie through $(\frac{1}{2},0)$ and are symmetrical about x-axis

p55 §1.7

12. (a) AB has equation $\begin{vmatrix} x & y & z \\ 1 & 2 & 3 \\ 1 & 1 & 1 \end{vmatrix} = 0$, ie $\underline{-x+2y-z=0}$ p65 §2.6

For C, $-2+2(-1)-(-4)=0$, so C on AB

For D, $-3+2(1)-(-1)=0$, so D on AB.

(b) $\underline{a}=(1,2,3)$, $\underline{b}=(1,1,1)$, $\underline{c}=(2,-1,-4)$, $\underline{d}=(3,1,-1)$ p67 §5.1

$(2,-1,-4)=\underline{c}=\alpha\underline{a}+\beta\underline{b}=\alpha(1,2,3)+\beta(1,1,1)$: $\left.\begin{array}{l} 2=\alpha+\beta \\ -1=2\alpha+\beta \\ -4=3\alpha+\beta \end{array}\right\}\begin{array}{l}\alpha=-3\\ \beta=5\end{array}$

$(3,1,-1)=\underline{d}=\gamma\underline{a}+\delta\underline{b}=\gamma(1,2,3)+\delta(1,1,1)$: $\left.\begin{array}{l} 3=\gamma+\delta \\ -1=2\gamma+\delta \\ -1=3\gamma+\delta \end{array}\right\}\begin{array}{l}\gamma=-2\\ \delta=5\end{array}$

$(ABCD)=\dfrac{\beta/\alpha}{\delta/\gamma}=\dfrac{5/-3}{5/-2}=\underline{\dfrac{2}{3}}.$

13. Let $f(x)=x-(3-2x)^{1/2}$, $g(x)=x^2+x-2$

Then f, g are differentiable on $]-\infty, 3/2[$, and $f(1)=g(1)=0$ p75 §5.3

Also, $\dfrac{f'(x)}{g'(x)}=\dfrac{1-\frac{1}{2}(3-2x)^{-1/2}(-2)}{2x+1}=\dfrac{1+(3-2x)^{-1/2}}{2x+1}\to\dfrac{1+1}{2+1}=\dfrac{2}{3}$ as $x\to 1$

($(3-2x)^{1/2}$ not defined for $x>3/2$)

Hence by L'Hôpital's Rule, $\lim\limits_{x\to 1}\dfrac{f(x)}{g(x)}=\dfrac{2}{3}$

14. (a) $I_0=\int_1^e x(\log x)^0\,dx=\int_1^e x\,dx=\left[\frac{1}{2}x^2\right]_1^e=\underline{\frac{1}{2}(e^2-1)}.$

(b) Let $f(x)=(\log x)^n$, $g'(x)=x$, so $g(x)=\frac{1}{2}x^2$ p77 §2.7 – we can't differentiate $(\log x)^n$ so it is $f(x)$

$$I_n=\left[\tfrac{1}{2}x^2(\log x)^n\right]_1^e-\int_1^e\tfrac{1}{2}x^2.\,n(\log x)^{n-1}\tfrac{1}{x}\,dx$$

$$=\tfrac{1}{2}e^2-0-\tfrac{n}{2}\int_1^e x(\log x)^{n-1}\,dx$$

$$=\underline{\frac{e^2}{2}-\frac{n}{2}I_{n-1}}$$

$n=1$ gives $I_1=\frac{e^2}{2}-\frac{1}{2}I_0=\frac{1}{4}(e^2+1)$

$n=2$ gives $I_2=\frac{e^2}{2}-\frac{2}{2}I_1=\frac{e^2}{2}-\frac{1}{4}(e^2+1)=\underline{\frac{1}{4}(e^2-1)}$

15. (a) $\underline{b} = \vec{OA} + \vec{AB} = \underline{a} + \underline{c}$, $\underline{p} = \frac{2}{3}\vec{OA} = \frac{2}{3}\underline{a}$.

(b) OB : $\underline{r} = \lambda\underline{0} + (1-\lambda)\underline{b} = (1-\lambda)\underline{a} + (1-\lambda)\underline{c}$ — p12 §3.16

CP : $\underline{r} = \mu\underline{c} + (1-\mu)\underline{p} = \frac{2}{3}(1-\mu)\underline{a} + \mu\underline{c}$ — Don't use λ again!

Where these meet $(1-\lambda)\underline{a} + (1-\lambda)\underline{c} = \frac{2}{3}(1-\mu)\underline{a} + \mu\underline{c}$

Comparing coefficients of $\underline{a}, \underline{c}$, $\begin{cases} 1-\lambda = \frac{2}{3}(1-\mu) \\ 1-\lambda = \mu \end{cases}$ ie $\begin{cases} 1-\lambda = \frac{2}{3}\lambda \\ \lambda = \frac{3}{5},\ \mu = 2/5 \end{cases}$

Thus $\underline{x} = \frac{2}{5}(\underline{a}+\underline{c})$ ie $OX : XB = 2 : 3$

(c) $\vec{CP} = \underline{p} - \underline{c} = \frac{2}{3}\underline{a} - \underline{c} = \frac{2}{3}(3,1) - (1,2) = (1, -4/3)$

$\vec{OB} = \underline{b} - \underline{0} = \underline{a} + \underline{c} = (3,1) + (1,2) = (4,3)$

$\vec{OB} \cdot \vec{CP} = 4.1 + 3(-4/3) = 0$ so $OB \perp CP$. — p12 §4.1,3

$\vec{OC} = \underline{c} = (1,2)$. Let θ be angle between OC and OB

Then $(1,2).(4,3) = \|(1,2)\|\|(4,3)\|\cos\theta$ — p12 §4.1

ie $10 = \sqrt{1^2+2^2}\sqrt{4^2+3^2}\cos\theta$

so $\cos\theta = \frac{10}{\sqrt{5}\cdot 5} = \frac{2}{\sqrt{5}}$.

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16. (a) $\det(A-\lambda I) = \begin{vmatrix} 2-\lambda & 0 & 1 \\ 0 & 2-\lambda & -1 \\ 1 & -1 & 1-\lambda \end{vmatrix} = (2-\lambda)((2-\lambda)(1-\lambda)-1) + 0(\ .\ ) + 1(0(-1) - 1(2-\lambda))$

$= (2-\lambda)\{(2-3\lambda+\lambda^2-1)-1\} = (2-\lambda)(\lambda^2-3\lambda)$

$= (2-\lambda)\lambda(\lambda-3)$ – roots 2, 0, 3

Hence eigenvalues are 0, 2, 3.

Strategy is on p25, §2.5 (Note common factor $2-\lambda$ in general)

(b) Eigenvector equations:

$\lambda=0$: $2x + z = 0$, $2y - z = 0$, $x - y + z = 0$	$\lambda=2$: $z = 0$, $-z = 0$, $x - y - z = 0$	$\lambda=3$: $-x + z = 0$, $-y - z = 0$, $x - y - 2z = 0$
eg $(1,-1,-2)$	eg $(1,1,0)$	eg $(1,-1,1)$

Here A is symmetric so we can use p25 §3.5 as well

Hence a suitable basis is

$\{(1,-1,-2), (1,1,0), (1,-1,1)\}$

It is a basis by p25 §3.4

(c) The vectors have length $\sqrt{1+(-1)^2+(-2)^2} = \sqrt{6}$, $\sqrt{1^2+1^2+0^2} = \sqrt{2}$ — p25 §3.5

and $\sqrt{1^2+(-1)^2+1^2} = \sqrt{3}$ respectively, so a suitable P

is $P = \begin{pmatrix} 1/\sqrt{6} & 1/\sqrt{2} & 1/\sqrt{3} \\ -1/\sqrt{6} & 1/\sqrt{2} & -1/\sqrt{3} \\ -2/\sqrt{6} & 0 & 1/\sqrt{3} \end{pmatrix}$ and $C = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix}$

unit eigenvectors are columns of P, diagonal entries of C are the eigenvalues

17. (a) $a_n = \frac{n+1}{n^2+n+1}$ has positive terms (ultimately like $\frac{1}{n}$)

Let $b_n = \frac{1}{n}$. Then $\frac{a_n}{b_n} = \frac{n+1}{n^2+n+1} \cdot \frac{n}{1} = \frac{1+1/n}{1+1/n+\frac{1}{n^2}} \to 1$ as $n \to \infty$

Now limit $\neq 0$ and $\sum b_n$ is basic divergent, so $\sum a_n$ is divergent by Limit Comparison Test.

Only powers of n so use Limit Comparison

p.33 §2.2

(b) $a_n = \frac{n^3 2^n}{n!}$ has positive terms

$$\frac{n!}{(n+1)!} = \frac{1}{n+1}$$

$$\frac{a_{n+1}}{a_n} = \frac{(n+1)^3 2^{n+1}}{(n+1)!} \cdot \frac{n!}{n^3 2^n} = \frac{(n+1)^3 \cdot 2}{(n+1)\, n^3}$$

$$= \frac{2}{n}\left(1+\frac{1}{n}\right)^2 \to 0 \text{ as } n \to \infty$$

As limit < 1, $\sum a_n$ is convergent by Ratio Test.

Factorials and exponentials — use Ratio Test p33 § 2.3

(c) (Here $a_n = \frac{1+2\cos n}{2n^2+1}$ may take either sign (and not alternating — we look at $\sum |a_n|$ & use the Absolute Convergence Test)

$$|a_n| = \left|\frac{1+2\cos n}{2n^2+1}\right| \leq \frac{1+2|\cos n|}{2n^2+1} \leq \frac{3}{2n^2+1} \text{ as } |\cos n| \leq 1$$

$$< \frac{3}{2} \cdot \frac{1}{n^2}$$

As $\sum \frac{1}{n^2}$ is basic convergent, $\sum |a_n|$ is convergent by Comparison Test. Hence $\sum a_n$ is convergent by Absolute Convergence Test

18. (a) G is really $S(\square)$ – square with vertices $(0,1), (1,0), (0,-1), (-1,0)$.

$G = \{e, r_{\pi/4}, r_{\pi/2}, r_{3\pi/4}, q_0, q_{\pi/4}, q_{\pi/2}, q_{3\pi/4}\}$.

Rotations thro' multiples of $\frac{\pi}{4}$ and reflection in axes of symmetry

(b)(i) $\mathrm{Orb}(0,0) = \{(0,0)\}$ (No element of G moves $(0,0)$)

$\mathrm{Stab}(0,0) = G$. (so all elements leave it alone!)

(ii) $\mathrm{Orb}(0,1) = \{(0,1), (1,0), (0,-1), (-1,0)\}$ ($(0,1)$ must go to another vertex of the square)

$\mathrm{Stab}(0,1) = \{e, q_{\pi/2}\}$

(Only reflection in "vertical" axis fixes $(0,1)$)

(iii) $\mathrm{Orb}(1,1) = \{(1,1), (1,-1), (-1,1), (-1,-1)\}$ (Think of the square with these vertices)

$\mathrm{Stab}(1,1) = \{e, q_{\pi/4}\}$

(iv) $\mathrm{Orb}(2,1) = \{(2,1), (1,2), (1,-2), (-1,-2), (-1,2), (-2,1), (-2,-1), (2,-1)\}$

$\mathrm{Stab}(2,1) = \{e\}$

(For (iv), just think of what each symmetry does to $(2,1)$)

Definitions of Orb, Stab are on pp 49–50 §2

Note For (i)–(iv) we can "check" using the Orbit-Stabiliser Theorem p50, §3.

19. (a)(i) $1.0+0.0+0.1=0$ so $X=[1,0,0]$ lies on $E: xy+yz+zx=0$ | Could quote p71 §3.3

Similarly, Y and Z lie on E.

(ii) By Joachimsthal, tangent at $[a,b,c]$ has equation | p70 §2.7

$$\tfrac{1}{2}(xb+ay)+\tfrac{1}{2}(yc+bz)+\tfrac{1}{2}(za+cx)=0$$

Hence tangent at X is $\tfrac{1}{2}(x.0+1.y)+\tfrac{1}{2}(y.0+0.z)+\tfrac{1}{2}(z.1+0.x)=0$

ie $\underline{y+z=0}$

By symmetry, tangent at $\underline{Y \text{ is } x+z=0}$, at $\underline{Z \text{ is } x+y=0}$

(iii) Tangents at Y, Z meet where $\begin{cases} x+z=0 \\ x+y=0 \end{cases}$, so

$\underline{P=[1,-1,-1]}$. By symmetry $\underline{Q=[-1,1,-1]}$, $\underline{R=[-1,-1,1]}$ | Put $x=1$ in equations to get P

(iv) $X=[1,0,0]$, $P=[1,-1,-1]$ so, by observation $\underline{XP \text{ is } y=z}$ | (or use p65 §2.6)

By symmetry $\underline{YQ \text{ is } x=z}$, $\underline{ZR \text{ is } x=y}$.

XP, YQ meet where $\begin{cases} y=z \\ x=z \end{cases}$ ie at $S=[1,1,1]$ | (put $z=1$ & solve)

S lies on ZR: $x=y$, so $\underline{XP, YQ, ZR \text{ meet at } S}$.

(b) By the Three Points Theorem there is a projective transformation t with $t(E)=F$, $t(X)=U$, $t(Y)=V$, $t(Z)=W$. | p70 §3.2

As t preserves tangency $t(P)=A$, $t(Q)=B$, $t(R)=C$. As XP, YQ, ZR are concurrent, so are their images UA, VB, WC.

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20. (a) (i) $\underline{A=\begin{pmatrix} 3 & 2 \\ 1 & 4 \end{pmatrix}}$ (ii) $\underline{\begin{aligned} f' &= 3f+2g \\ g' &= f+4g \end{aligned}}$ | p80 §1.3, p80 §2.1

(iii) $f''-(\operatorname{tr}A)f'+(\det A)f=0$, ie $\underline{f''-7f'+10f=0}$ | Numerical version required here

(b) The auxiliary equation is $\lambda^2-7\lambda+10=0$ ie $(\lambda-2)(\lambda-5)=0$ | p82 §4

Since roots are real & distinct, the solution is $\underline{f(t)=ce^{2t}+de^{5t}}$

(c). From (a)(ii) $f'=3f+2g$ so

$$\begin{aligned} 2g(t) &= (ce^{2t}+de^{5t})'-3(ce^{2t}+de^{5t}) \\ &= (2ce^{2t}+5de^{5t})-3(ce^{2t}+de^{5t}) \\ &= (-ce^{2t}+2de^{5t}) \end{aligned}$$

so $\underline{g(t)=-\tfrac{1}{2}ce^{2t}+de^{5t}}$

$\alpha(0)=(1,4)$ so $f(0)=1$, $g(0)=4$ so $\begin{cases} 1=c+d \\ 4=-\tfrac{1}{2}c+d \end{cases}$ | Note $e^{2.0}=e^{5.0}=1$

ie $d=3$, $c=-2$

Hence $\underline{\alpha(t)=(-2e^{2t}+3e^{5t},\ e^{2t}+3e^{5t})}$.