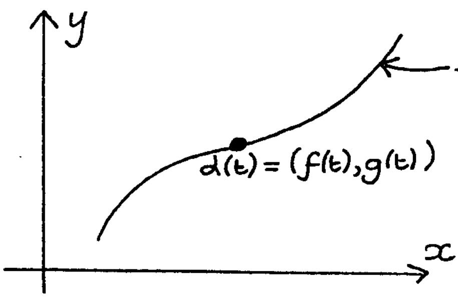
FLOWS ...

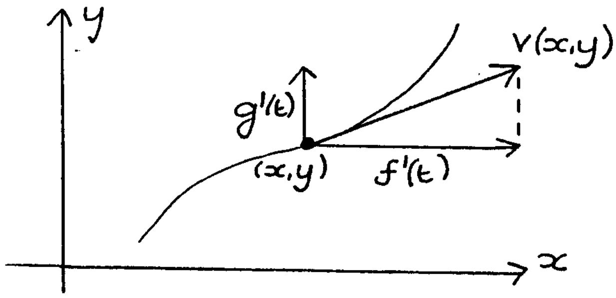
... the flow of a fluid, e.g a shallow river

Velocity vector $V(x,y)$, based at the point (x,y), gives the speed and direction of the fluid flow at that point.

Flow function $\alpha(t) = (x,y) = (f(t), g(t))$ gives the position of a fluid particle at time t.

Velocity function $V(x,y) = V(f(t), g(t)) = (f'(t), g'(t))$





Handbook p. 84

Linear flow $\quad V(x,y) = (ax+by, cx+dy)$

Matrix of flow $\quad \underline{A} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$;

$\text{tr}\,\underline{A} = a+d$ and $\det \underline{A} = ad-bc$

$$V\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} ax+by \\ cx+dy \end{pmatrix}$$

$$\Rightarrow V\begin{pmatrix} f(t) \\ g(t) \end{pmatrix} = \begin{pmatrix} af(t)+bg(t) \\ cf(t)+dg(t) \end{pmatrix} = \begin{pmatrix} f'(t) \\ g'(t) \end{pmatrix}$$

so the first order differential equations are

$$f'(t) = af(t)+bg(t) \ ①, \quad g'(t) = cf(t)+dg(t) \ ②$$

Hence
$$\begin{aligned} f''(t) &= af'(t)+bg'(t) && (\text{differentiating } ①) \\ &= af'(t)+bcf(t)+bdg(t) && (\text{from } ②) \\ &= af'(t)+bcf(t)+df'(t)-adf(t) && (\text{from } ①) \end{aligned}$$

$$\Rightarrow \quad f''(t) - (a+d)f'(t) + (ad-bc)f(t) = 0,$$

so the second-order differential equation (satisfied by both $f(t)$ and $g(t)$) is

$$f''(t) - (\text{tr}\,\underline{A})f'(t) + (\det \underline{A})f(t) = 0$$

(For solutions, see Handbook p. 90. § 4)

Example

Velocity function $V(x,y) = (2x+6y,\ 3x-y)$

Flow function α satisfies $\alpha(0) = (1,1)$.

Matrix of flow: $\underline{A} = \begin{pmatrix} 2 & 6 \\ 3 & -1 \end{pmatrix}$, so $\operatorname{tr} \underline{A} = 1$, $\det \underline{A} = -20$.

Second order differential equation: $f''(t) - f'(t) - 20f(t) = 0$

Auxiliary / Characteristic equation: $\lambda^2 - \lambda - 20 = 0$.

(i) SKETCH: The characteristic equation is

$$\begin{vmatrix} 2-\lambda & 6 \\ 3 & -1-\lambda \end{vmatrix} = 0 \quad \text{or} \quad \lambda^2 - \lambda - 20 = 0$$

so $(\lambda+4)(\lambda-5) = 0$, giving eigenvalues $\lambda = -4, 5$.

When $\lambda = -4$, the equations are

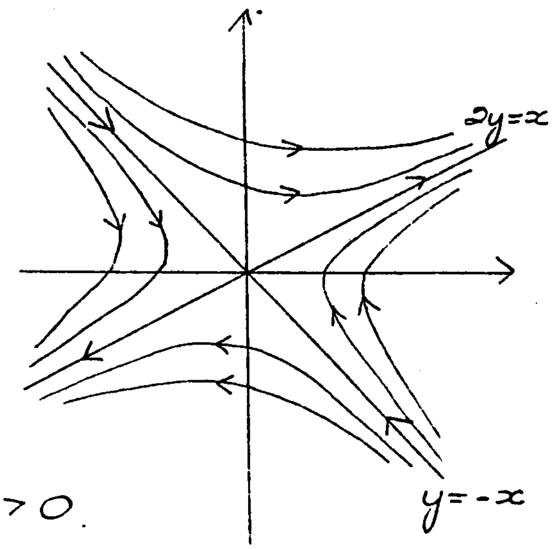
$$\left.\begin{aligned} 6x + 6y &= 0 \\ 3x + 3y &= 0 \end{aligned}\right\} \Rightarrow y = -x,$$

which is a barrier line with flow towards the origin, as $\lambda < 0$.

When $\lambda = 5$, the equations are

$$\left.\begin{aligned} -3x + 6y &= 0 \\ 3x - 6y &= 0 \end{aligned}\right\} \Rightarrow 2y = x$$

which is a barrier line with flow outwards from the origin, as $\lambda > 0$.



(ii) SOLUTION: The general solution for the differential equation is

$$f(t) = c_1 e^{-4t} + d_1 e^{5t}, \qquad \text{since } \lambda = -4, 5.$$

From the velocity function,

$$f'(t) = 2f(t) + 6g(t) \Rightarrow 6g(t) = f'(t) - 2f(t)$$

so

$$g(t) = \tfrac{1}{6}\left[-4c_1 e^{-4t} + 5d_1 e^{5t} - 2c_1 e^{-4t} - 2d_1 e^{5t}\right]$$
$$= -c_1 e^{-4t} + \tfrac{1}{2} d_1 e^{5t}$$

Hence the flow function is

$$\alpha(t) = \left(c_1 e^{-4t} + d_1 e^{5t},\ -c_1 e^{-4t} + \tfrac{1}{2} d_1 e^{5t}\right).$$

If $\alpha(0) = (1,1)$, then

$$\left.\begin{aligned} c_1 + d_1 &= 1 \\ -c_1 + \tfrac{1}{2} d_1 &= 1 \end{aligned}\right\} \Rightarrow d_1 = 4/3,\ c_1 = -1/3$$

so the flow function satisfying $\alpha(0) = (1,1)$ is

$$\alpha(t) = \left(-\tfrac{1}{3} e^{-4t} + \tfrac{4}{3} e^{5t},\ \tfrac{1}{3} e^{-4t} + \tfrac{2}{3} e^{5t}\right)$$

Further Examples

(a) Velocity function $V(x,y) = (3y-4x, 2y-6x)$
Flow function α satisfies $\alpha(0) = (1,2)$

Matrix of flow: $\underline{A} = \begin{pmatrix} -4 & 3 \\ -6 & 2 \end{pmatrix}$, $\operatorname{tr}(\underline{A}) = -2$, $\det \underline{A} = 10$

First order diff. eqns: $f'(t) = -4f(t) + 3g(t)$
$g'(t) = -6f(t) + 2g(t)$

Second order diff. eqn: $f''(t) + 2f'(t) + 10f(t) = 0$

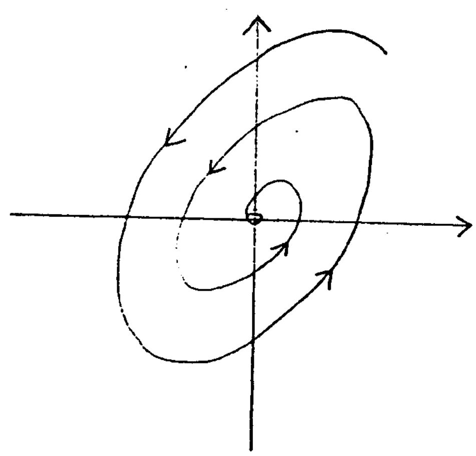
The characteristic eqn. is
$$\begin{vmatrix} -4-\lambda & 3 \\ -6 & 2-\lambda \end{vmatrix} = 0 \quad \text{or} \quad \lambda^2 + 2\lambda + 10 = 0$$
so $\lambda = \frac{1}{2}(-2 \pm \sqrt{4-40}) = -1 \pm 3i$.

As the roots are complex there are no barrier lines and as $\operatorname{Re}(\lambda) < 0$ flow is towards the origin.

The auxiliary eqn is $\lambda^2 + 2\lambda + 10 = 0$ so the general solution of the second order diff. eqn is
$$f(t) = e^{-t}(c\cos 3t + d\sin 3t).$$
From the velocity function, $f'(t) = -4f(t) + 3g(t)$
so
$$\begin{aligned} 3g(t) &= f'(t) + 4f(t) \\ &= e^{-t}(-c\cos 3t - d\sin 3t - 3c\sin 3t + 3d\cos 3t \\ &\qquad + 4c\cos t + 4d\sin 3t) \\ &= e^{-t}((3c+3d)\cos 3t + (3d-3c)\sin 3t) \\ \Rightarrow g(t) &= e^{-t}((c+d)\cos 3t + (d-c)\sin 3t). \end{aligned}$$
The flow function is
$$\alpha(t) = (e^{-t}(c\cos 3t + d\sin 3t), e^{-t}((c+d)\cos 3t + (d-c)\sin 3t)).$$

If $\alpha(0) = (1,2)$, then
$$\left.\begin{aligned} c &= 1 \\ c+d &= 2 \end{aligned}\right\} \Rightarrow \begin{aligned} c &= 1 \\ d &= 1 \end{aligned}.$$
so the flow function satisfying $\alpha(0) = (1,2)$ is

$$\alpha(t) = (e^{-t}(\cos 3t + \sin 3t), 2e^{-t}\cos 3t)$$



Further Examples

(b) Velocity function $V(x,y) = (2x-y, x+4y)$

Flow function α satisfies $\alpha(0) = (2,1)$.

Matrix of flow: $\underline{A} = \begin{pmatrix} 2 & -1 \\ 1 & 4 \end{pmatrix}$, $\operatorname{tr}(\underline{A}) = 6$, $\det(\underline{A}) = 9$

First order diff. eqns: $f'(t) = 2f(t) - g(t)$
$g'(t) = f(t) + 4g(t)$

Second order diff. eqn: $f''(t) - 6f'(t) + 9f(t) = 0$

The characteristic eqn. is

$$\begin{vmatrix} 2-\lambda & -1 \\ 1 & 4-\lambda \end{vmatrix} = 0 \quad \text{or} \quad \lambda^2 - 6\lambda + 9 = 0$$

so $(\lambda-3)^2 = 0 \Rightarrow \lambda = 3$ (twice).

As the roots are repeated and $\underline{A} \neq kI$, there is one barrier line and, as $\lambda > 0$, flow is directed away from the origin. The eigenvector eqns. are $\left.\begin{array}{l} -x-y=0 \\ x+y=0 \end{array}\right\}$ so $y = -x$ is the barrier line.

The auxiliary eqn. is $\lambda^2 - 6\lambda + 9 = 0$ so the general solution of the second order eqn. is

$$f(t) = e^{3t}(c+dt).$$

From the velocity function, $f'(t) = 2f(t) - g(t)$

so
$$\begin{aligned} g(t) &= 2f(t) - f'(t) \\ &= 2e^{3t}(c+dt) - e^{3t}(3c+3dt+d) \\ &= e^{3t}(-(c+d) - dt). \end{aligned}$$

The flow function is

$$\alpha(t) = (e^{3t}(c+dt), -e^{3t}(c+d+dt))$$

If $\alpha(0) = (2,1)$, then

$$\left.\begin{array}{r} c = 2 \\ -(c+d) = 1 \end{array}\right\} \Rightarrow \begin{array}{l} c = 2 \\ d = -3 \end{array}$$

so the flow function satisfying $\alpha(0) = (2,1)$ is

$$\alpha(t) = (e^{3t}(2-3t), e^{3t}(1+3t)).$$