AB3 Power Series

Motivation $\frac{1}{1-x} = 1 + x + x^2 + \cdots$ (for $|x|<1$).

A power series for $f(x)$ at a $\quad f(x) = \sum_{n=0}^{\infty} a_n (x-a)^n$

What are the a_n?
When does the series converge?
How close an approximation is the n^{th} partial sum?

Suppose that $f(x) = a_0 + a_1(x-a) + a_2(x-a)^2 + a_3(x-a)^3 + \cdots$

$$
\begin{array}{lll}
x=a: & f(a) = a_0 & \\
 & f'(x) = a_1 & + 2a_2(x-a) + 3a_3(x-a)^2 + \cdots \\
x=a: & f'(a) = a_1 & \\
 & f''(x) = & 2a_2(x-a) + 3.2a_3(x-a) + \cdots \\
x=a: & f''(a) = 2a_2 \quad \text{so} & a_2 = f''(a)/2
\end{array}
$$

Generalizing $\quad a_n = f^{(n)}(a)/n!$

The Taylor polynomial of degree n for f at a

$$T_n(x) = f(a) + \frac{f'(a)}{1!}(x-a) + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n$$

Taylor's Theorem $\quad f(x) = T_n(x) + R_n(x)$, where

$$R_n(x) = \frac{f^{(n+1)}(c)}{(n+1)!}(x-a)^{n+1} \quad (c \text{ between } a \text{ and } x)$$

If we can estimate $R_n(x)$, we have the error $|f(x) - T_n(x)|$
If $|R_n(x)| \to 0$, the series converges to $f(x)$.

For example, $e^x = 1 + x + x^2/2! + \cdots = \sum_{n=0}^{\infty} x^n/n!$

Example Calculate $T_2(x)$ for $f(x) = \frac{2x}{3+x}$ at 1.
Show that $T_2(x)$ approximates $f(x)$ with error less than 0.003 on $[1, 1.5]$.

Solution $\quad f(x) = \frac{2x}{3+x}$ $\qquad f(1) = 1/2$

$$f'(x) = \frac{(3+x)2 - 1.2x}{(3+x)^2} = \frac{6}{(3+x)^2} \qquad f'(1) = 3/8$$

$$f''(x) = \frac{(-2).6}{(3+x)^3} = \frac{-12}{(3+x)^3} \qquad f''(1) = -3/16$$

Hence, $T_2(x) = \frac{1}{2} + \frac{3}{8}\cdot\frac{1}{1!}(x-1) + \left(-\frac{3}{16}\right)\frac{1}{2!}(x-1)^2$

$$= \frac{1}{2} + \frac{3}{8}(x-1) - \frac{3}{32}(x-1)^2$$

$f'''(x) = \frac{-36}{(3+x)^4}$ For $x \geq 1$, $|f'''(x)| \leq \left|\frac{-36}{(3+x)^4}\right| = \frac{9}{64}$

(Taylor) $|R_2(x)| \leq \frac{9}{64}\cdot\frac{1}{3!}|x-1|^3$ $x \in [1, 1.5]$

$$\leq \frac{9}{64}\cdot\frac{1}{6}(1.5-1)^3 = \frac{3}{1024} < 0.003$$

Suppose we apply the Ratio Test (for series) to $\sum_{n=0}^{\infty} a_n(x-a)^n$

$$\left|\frac{a_{n+1}(x-a)^{n+1}}{a_n(x-a)^n}\right| = \left|\frac{a_{n+1}}{a_n}\right||x-a|.$$

The series converges if this is less than 1, and diverges if it is greater than 1

(when the ratio is 1, we need to use different tests)

In general, series converges for $|x-a|<R$: R radius of convergence

Ratio Test for Radius of Convergence.

Suppose that $\sum a_n(x-a)^n$ has $\left|\frac{a_{n+1}}{a_n}\right| \longrightarrow L$

- If $L=\infty$, series converges only for $x=a$ $(R=0)$
- if $L=0$, series converges for all x $(R=\infty)$
- if $L>0$, series converges for $|x-a|<1/L$ $(R=1/L)$

In the third case, series converges for $|x-a|<R$, ie $]a-R, a+R[$.
The end-points must be considered separately (giving the interval of convergence)

In $\sum_{n=0}^{\infty} a_n(x-a)^n$, each term can be differentiated (or integrated).

FACT $\sum n a_n (x-a)^{n-1}$ (derivatives)

and $\sum \frac{a_n}{n+1}(x-a)^{n+1}$ (integrals)

have the same radius of convergence as $\sum a_n(x-a)^n$

Example Determine the interval of convergence of the series
$$\sum_{n=0}^{\infty} \frac{(-1)^n}{2^n(n+1)}(x-1)^n$$
State the radius of convergence of the series
$$\sum_{n=0}^{\infty} \frac{(-1)^n}{2^n(n+1)^2}(x-1)^{n+1},$$
explaining your answer.

Solution The series has $a_n = \dfrac{(-1)^n}{2^n(n+1)}$

$$\left|\frac{a_{n+1}}{a_n}\right| = \frac{1}{2^{n+1}(n+2)} \cdot \frac{2^n(n+1)}{1} = \frac{1}{2}\frac{(n+1)}{(n+2)} = \frac{1}{2}\left(\frac{1+1/n}{1+2/n}\right) \to \frac{1}{2}$$

By Ratio Test, radius of convergence is $1/(1/2) = 2$

i.e. series converges for $|x-1| < 2$

ie $-1 < x < 3$

For $x = -1$, series is $\sum \frac{(-1)^n}{2^n(n+1)}(-2)^n = \sum \frac{1}{n+1}$

this is divergent (comare $\sum 1/n$)

For $x = 3$, series is $\sum \frac{(-1)^n}{2^n(n+1)}(2)^n = \sum \frac{(-1)^n}{n+1}$

this is convergent (Alternating Test)

Thus, interval of convergence is $]-1, 3]$

The second series is obtained by integrating term by term
This does not affect the radius of convergence
Thus the radius is 2.

[By looking at $x = -1, 3$, you can check that interval of convergence is $[-1, 3]$, but this was not asked]

General Binomial Theorem

$$(1+x)^\alpha = \sum_{n=0}^{\infty} \binom{\alpha}{n} x^n \qquad \text{for } |x|<1$$

where $\binom{\alpha}{n} = \underbrace{\alpha(\alpha-1)\dots(\alpha-n+1)}_{n \text{ terms}}/n!$

Example Use the General Binomial Theorem to determine the first three non-zero terms of the Taylor series at 0 for

$$f(x) = (1-2x^3)^{3/4}$$

Solution

$$\begin{aligned}(1+X)^{3/4} &= 1 + \binom{3/4}{1} X + \binom{3/4}{2} X^2 + \dots \\ &= 1 + \frac{3/4}{1!} X + \frac{(3/4)(3/4-1)}{2!} X^2 + \dots \\ &= 1 + \frac{3}{4} X - \frac{3}{32} X^2 + \dots \qquad (|X|<1\end{aligned}$$

Now put $(-2x^3)$ for X:

$$\begin{aligned}(1-2x^3)^{3/4} &= 1 + \frac{3}{4}(-2x^3) - \frac{3}{32}(-2x^3)^2 + \dots \\ &= 1 - \frac{3}{2}x^3 - \frac{3}{8}x^6 + \dots\end{aligned}$$

[It converges for $|-2x^3| < 1$ ie $|x| < 1/2^{1/3}$]

Optional Example Find the Taylor polynomial for e^x at 0
Find the error in using $T_n(x)$ for e^x in $[-1,1]$
Determine the interval of convergence of the Taylor series.

Solution If $f(x) = e^x$, then $f'(x) = e^x$, $f''(x) = e^x, \dots, f^{(n)}(x) = e^x$
Thus $f^{(n)}(0) = 1$, so $a_n = 1/n!$ $(n = 1, 2, 3, \dots)$

$$T_n(x) = 1 + \frac{1}{1!}x + \frac{1}{2!}x^2 + \dots + \frac{1}{n!}x^n$$

$$R_n(x) = \frac{f^{(n+1)}(c)}{(n+1)!} x^{n+1} \qquad c \in [-1,1] \text{ so } e^c \le e$$

error $\le e/(n+1)!$ (small!)

Here $a_n = 1/n!$, so $\left|\frac{a_{n+1}}{a_n}\right| = \frac{1}{(n+1)!} / \frac{1}{n!} = \frac{1}{n+1} \to 0 \ (n \to \infty)$

Hence series $\sum_{n=0}^{\infty} x^n/n!$ converges on $]-\infty, \infty[$