SEQUENCES

An unending list of real numbers $a_1, a_2, a_3, \dots$ with n^{th} term a_n and denoted by $\{a_n\}$.

Examples

(i) $3, 3, 3, 3, 3, \dots$ $\quad a_n = 3 \to 3$ (constant)

(ii) $2, 3, 4, 5, 6, \dots$ $\quad a_n = n+1 \to \infty$ (strictly increasing)

(iii) $-\frac{1}{2}, \frac{1}{3}, -\frac{1}{4}, \frac{1}{5}, -\frac{1}{6}, \dots$ $\quad a_n = \frac{(-1)^n}{n+1} \to 0$ (alternating signs)

(iv) $1, 2, 3, -2, -19, \dots$ $\quad a_n = n - (n-1)(n-2)(n-3) \to -\infty$

(v) $1, 1, 1, \frac{29}{32}, \frac{101}{128}, \dots$ $\quad a_n = 1 - \frac{(n-1)(n-2)(n-3)}{n^3} \to 0$ (decreasing)

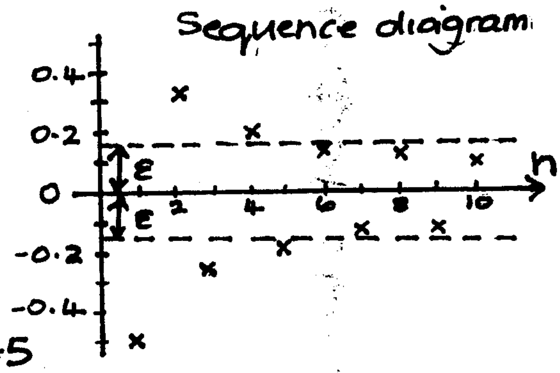
(vi) $1, 2, 2\frac{1}{3}, 2\frac{2}{5}, 2\frac{2}{3}, \dots$ $\quad a_n = \frac{3n-2}{n} \to 3$ (strictly increasing)

Null sequences

The sequence $\{a_n\}$ is null if for each positive number ε, there is an integer N such that $|a_n| < \varepsilon$, for all $n > N$.

Example (iii) $\quad a_n = \frac{(-1)^n}{n+1}$

Sequence diagram



In particular,

$$|a_n| < 0.15 \Leftrightarrow \left|\frac{(-1)^n}{n+1}\right| < 0.15$$
$$\Leftrightarrow \frac{1}{n+1} < 0.15$$
$$\Leftrightarrow n+1 > \frac{1}{0.15} = \frac{20}{3}$$
$$\Leftrightarrow n > \frac{20}{3} - 1 = \frac{17}{3}$$

Hence $|a_n| < 0.15$ for all $n > N = [17/3] = 5$

In general, (proof)

$$|a_n| < \varepsilon \Leftrightarrow \left|\frac{(-1)}{n+1}\right| < \varepsilon \Leftrightarrow \frac{1}{n+1} < \varepsilon \Leftrightarrow n+1 > \frac{1}{\varepsilon} \Leftrightarrow n > \frac{1}{\varepsilon} - 1.$$

Hence $|a_n| < \varepsilon$ for all $n > N = [1/\varepsilon - 1]$. (Note: If $\varepsilon > \frac{1}{2}$, $N = 1$).

Basic null sequences

(a) $\{\frac{1}{n^p}\}$, $p > 0$ $\quad$ e.g. $\{\frac{1}{n^3}\}$

(b) $\{c^n\}$, $|c| < 1$ $\quad$ e.g. $\{\frac{3^n}{4^n}\} = \{(\frac{3}{4})^n\}$

(c) $\{n^p c^n\}$, $p > 0$, $|c| < 1$ $\quad$ e.g. $\{n^3 (\frac{3}{4})^n\}$

(d) $\{\frac{c^n}{n!}\}$, $c \in \mathbb{R}$ $\quad$ e.g. $\{\frac{100^n}{n!}\}$

(e) $\{\frac{n^p}{n!}\}$, $p > 0$ $\quad$ e.g. $\{\frac{n^3}{n!}\}$

Note: $\{\frac{100^n}{n!}\}$ has first few terms
$100, 5000, 166\,666\frac{2}{3}, 4\,166\,666\frac{2}{3}, 83\,333\,333\frac{1}{3}$
but is a null sequence!

Convergent sequences

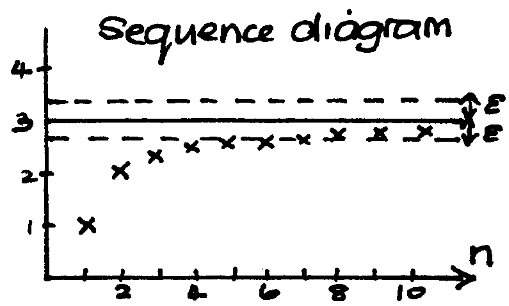
The sequence $\{a_n\}$ is convergent with limit ℓ if $\{a_n-\ell\}$ is a null sequence, that is, if for each positive number ε, there is an integer N such that

$$|a_n-\ell| < \varepsilon, \quad \text{for all } n>N.$$

Then we write

$$a_n \to \ell \text{ as } n\to\infty \quad \text{or} \quad \lim_{n\to\infty} a_n = \ell$$

Example (vi) $\quad a_n = \dfrac{3n-2}{n}$



$$|a_n-3|<\varepsilon \Leftrightarrow |3-\tfrac{2}{n}-3|<\varepsilon$$
$$\Leftrightarrow \tfrac{2}{n}<\varepsilon$$
$$\Leftrightarrow n > 2/\varepsilon$$

so $|a_n|<\varepsilon$ for all $n>N=[2/\varepsilon]$

(eg if $\varepsilon = 1/4$, $N=[2/\tfrac{1}{4}]=8$

Note: the ε-band is now centred on $\ell=3$

Combination rules for sequences

If $a_n\to\ell$ and $b_n\to m$ as $n\to\infty$, then as $n\to\infty$

$a_n+b_n \to \ell+m$	Sum Rule
$\lambda a_n \to \lambda\ell$	Multiple Rule
$a_n b_n \to \ell m$	Product Rule
$a_n/b_n \to \ell/m$, if $m\neq 0$	Quotient Rule

Example

Find the limit of $a_n = \dfrac{4n^3-2^n+3(5^n)}{5^n+3n^2+2n}$

Solution

Dividing by the dominant term 5^n,

$$a_n = \frac{4n^3/5^n-2^n/5^n+3}{1+3n^2/5^n+2n/5^n} = \frac{4(n^3/5^n)-(2/5)^n+3}{1+3(n^2/5^n)+2(n/5^n)}.$$

Since $\{n^3/5^n\}$, $\{(2/5)^n\}$, $\{n^2/5^n\}$ and $\{n/5^n\}$ are basic null sequences, by the combination Rules

$$\lim_{n\to\infty} a_n = \frac{0-0+3}{1+0+0} = \frac{3}{1} = 3.$$

Monotone Convergence Theorem

If the sequence $\{a_n\}$ is increasing and bounded above or decreasing and bounded below, then $\{a_n\}$ converges.

Example (vi) $\quad \{a_n\} = \{\frac{3n-2}{n}\}$ converges as:

- $a_{n+1}-a_n = (3-\frac{2}{n+1})-(3-\frac{2}{n}) = 2(\frac{1}{n}-\frac{1}{n+1}) = \frac{2}{n(n+1)} > 0$

  so $\{a_n\}$ is strictly increasing
- $0<\frac{2}{n}\leq 2 \Leftrightarrow -2\leq -\frac{2}{n}<0 \Leftrightarrow 1\leq 3-\frac{2}{n}<3$

  so $\{a_n\}$ is bounded above by 3.

CONTINUITY

A sequence $\{x_n\}$ allows us to step along the x-axis so that if $x_n \to a$ as $n \to \infty$, we can let x approach a and investigate the behaviour of function $f(x)$ by considering the sequence $\{f(x_n)\}$ as $x_n \to a$.

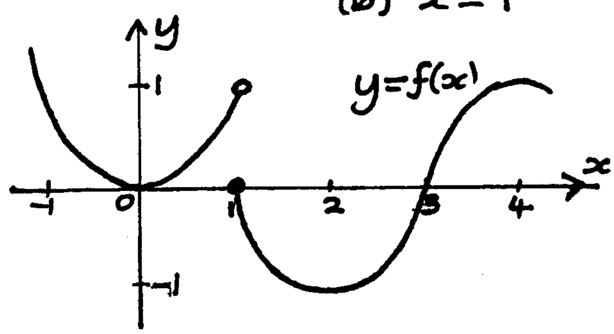
Definition

A function $f: A \to \mathbb{R}$ is continuous at $a \in A$, if, for <u>each</u> sequence $\{x_n\}$ in A such that $x_n \to a$, $f(x_n) \to f(a)$.

Example

Is $f(x) = \begin{cases} x^2, & x < 1 \\ \cos(\pi x/2), & x \geq 1 \end{cases}$ continuous at (a) $x = -1$? (b) $x = 1$?

Solution



$f(-1) = 1$ and $f(1) = 0$

(a) Let $x_n \to -1$ (a general seq) then, as $n \to \infty$,

$f(x_n) = x_n^2 \to (-1)^2 = 1$ (Product Rule)

so $x_n \to -1 \Rightarrow f(x_n) \to f(1)$

and <u>f is continuous at $x = -1$</u>.

(b) Let $x_n = 1 - 1/n$ (a particular seq) then, as $n \to \infty$, $x_n \to 1$ but $f(x_n) = (1 - 1/n)^2 \to 1 \neq 0 = f(1)$, ie $x_n \to 1$ but $f(x_n) \not\to f(1)$

so <u>f is not continuous at $x = 1$</u>.

Strategy: To show f is not continuous at $a \in A$
Find one sequence $\{x_n\}$ in A such that $x_n \to a$ but $f(x_n) \not\to f(a)$

<u>To show f is continuous at a</u> choose from:

1. Definition for a general sequence $x_n \to a$
2. Basic continuous functions
3. Combination Rules
4. Composition Rule
5. Squeeze Rule
6. Local Rule
7. Glue Rule

<u>Limit of a sequence using continuity</u>

Example: Find the limit of $a_n = \sin(\pi/2 - 1 + e^{1/n^2})$

Solution

Let ① $x_n = 1/n^2$, ② $f(x) = \sin(\pi/2 - 1 + e^x)$.

Then ① $x_n \to 0$ as $n \to \infty$

② $f(x)$ is continuous by the Composition Rule

Hence f is continuous at 0 so as $n \to \infty$,

$x_n \to 0 \Rightarrow f(x_n) \to f(0) = \sin(\pi/2 - 1 + 1) = \sin\pi/2 = 1$

and $\lim_{n\to\infty} f(x_n) = \lim_{n\to\infty} \sin(\pi/2 - 1 + e^{1/n^2}) = 1$.

$$f(x) = \begin{cases} x, & \text{if } x \text{ is rational} \\ -x, & \text{if } x \text{ is irrational} \end{cases}$$

<u>$x = 1$</u>: $f(1) = 1$

Let $x_n = 1 + \frac{\sqrt{2}}{n}$, then x_n is an irrational sequence and, as $n \to \infty$,

$$x_n \to 1$$

but $\quad f(x_n) = -x_n = -1 - \frac{\sqrt{2}}{n} \to -1 \neq 1$

so $\quad f(x_n) \not\to f(1)$

Hence f is not continuous at 1

<u>$x = \sqrt{2}$</u>: $f(\sqrt{2}) = -\sqrt{2}$

As $\sqrt{2} = 1.41421356\ldots$, let x_n be the sequence of rational numbers

$$\frac{1}{1}, \frac{14}{10}, \frac{141}{100}, \frac{1{,}414}{1{,}000}, \frac{14{,}142}{10{,}000}, \frac{141{,}421}{100{,}000}, \ldots$$

then as $n \to \infty$,

$$x_n \to \sqrt{2}$$

but $\quad f(x_n) = x_n \to \sqrt{2} \neq -\sqrt{2}$

so $\quad f(x_n) \not\to f(\sqrt{2})$

Hence f is not continuous at $\sqrt{2}$

<u>$x = 0$</u>: $f(0) = 0$

Let $g(x) = -|x|$, $h(x) = |x|$, then f and g are basic continuous functions and

$$-|x| \leq f(x) \leq |x|, \quad x \in \mathbb{R}$$

so:
1. $g(x) \leq f(x) \leq h(x), \quad x \in \mathbb{R}$
2. $g(0) = f(0) = h(0) = 0$
3. g and h are continuous at 0.

Hence, by the Squeeze Rule, f is continuous at 0