An Idiots Guide to Constructing Proofs

Suppose we want to prove $A_1, \ldots, A_n \vdash C$.
We can do this systematically as follows

(1) assume each of the A_i " i (i) A_i (Ass)"
(2) remove quantifiers where necessary
(3) use tautologies to get (unquantified) form of C
(4) introduce required quantifiers.

Removing a universal quantifier.
Suppose A_i is $\forall x B_i$. Then we simply use UE:
We add the line

$$i \quad (m) \; B_i \qquad (\text{UE}(i))$$

We may then use B_i, though the assumption in force is A_i (assumption number i)

Removing an existential quantifier.
Suppose A_i is $\exists x B_i$. This is more complex.
We begin by assuming B_i. Then at the end of the proof we use EH to replace assumption B_i by A_i.

$$\begin{array}{llll} m & (m) & B_i & (\text{Ass}) \\ & \vdots & & \\ m \{j : j \neq i\} & (M) & C & (\ldots) \\ 1, \ldots, n & (M+1) & C & (\text{EH}(M)) \end{array}$$

NB We need to check that x is not free in C or in any of the A_j $(j \neq i)$.

Introducing a universal quantifier.
Suppose C has the form $\forall x D$.
We first prove D, then apply UI.

$$\begin{array}{llll} 1, \ldots, n & (m) & D & (\ldots) \\ 1, \ldots, n & (m+1) & C & (\text{UI}(m)) \end{array}$$

NB We need to check that x is not free in $A_1, \ldots, A_n$

Introducing an existential quantifier.
Suppose that C has the form $\exists x D$.
We prove D and apply EI.
There are no conditions on the status of x.

Only step 3 remains. The details will depend on the nature of C. The majority of questions involve implication or negation.

Proving an implication.
Suppose that C is $D_1 \rightarrow D_2$. As in "normal" mathematics, we assume D_1 and deduce D_2 (and use CP)

$$\begin{array}{llll} m & (m) & D_1 & (\text{Ass}) \\ & \vdots & & \\ m, 1, \ldots, n & (M) & D_2 & (\ldots) \\ 1, \ldots, n & (M+1) & D_1 \rightarrow D_2 & (\text{CP}(M)) \end{array}$$

Proving a negation.
Suppose that C is $-D$. We use "proof by contradiction"
We assume D and deduce a false statement. This generally has the form $E \& -E$ for some E (any E will do).
The details need careful handling – the heart is the tautology $((D \rightarrow (E \& -E)) \rightarrow -D)$

$$\begin{array}{llll} m & (m) & D & (\text{Ass}) \\ & \vdots & & \\ m, 1, \ldots, n & (M) & (E \& -E) & (\ldots) \\ 1, \ldots, n & (M+1) & (D \rightarrow (E \& -E)) & (\text{CP}(M)) \\ 1, \ldots, n & (M+2) & -D & (\text{Taut}(M+1)) \end{array}$$

Other forms of C.
If C is $D_1 \& D_2$, then we simply prove D_1 and D_2 separately, then use the trivial tautology

$$((D_1 \& D_2) \rightarrow (D_1 \& D_2))$$

to get C.

If C is $D_1 \vee D_2$ it is more difficult.
I have never seen it asked, but it could be tackled as follows:
Assume $-D_1$ and deduce D_2. Then apply the tautology

$$((-D_1 \rightarrow D_2) \rightarrow (D_1 \vee D_2)).$$