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NONSTANDARD FINITE DIFFERENCE METHOD FOR NONLINEAR RIESZ SPACE FRACTIONAL REACTION-DIFFUSION EQUATION

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Abstract. In this paper, a modified nonstandard finite difference method for the two-dimensional Riesz space fractional reaction-diffusion equations is developed. The space fractional derivative is discretized by the shifted Grünwald-Letnikov method and the nonlinear reaction term is approximated by Taylor formula instead of Micken's. Multigrid method is introduced to reduce the computation time of the traditional Gauss-Seidal method. The stability and convergence of the nonstandard implicit difference scheme are strictly proved. The method is extended to simulate the fractional FitzHugh-Nagumo model. Numerical results are provided to verify the theoretical analysis.

Key words. Riesz fractional derivative, nonstandard finite difference method, shifted Grünwald-Letnikov method.

1. Introduction

Reaction-diffusion models are widely used in patten formulae in biology, chemistry, physics and engineering [26]. The computation of electrical wave propagation in the heart is one of the most important applications of reaction-diffusion models in physiology [28]. The simplest two-dimensional reaction-diffusion model can be described by

$$\frac{\partial u}{\partial t} = \nabla \cdot (\mathbf{K}\nabla u) + f(u), \tag{1}$$

where $\mathbf{K}$ is the diffusion coefficient and $f(u)$ is a nonlinear function representing the reaction source, u is a normalized transmembrane potential. If $f(u) = u(1-u)(u-a)$, Eq. (1) reduces to the Nagumo reaction-diffusion equation [6, 24].

Over the last few decades, fractional calculus has become famous of its ability to model anomalous diffusion phenomena, which has attracted more and more attention from researchers in various fields of science and engineering. Compared with the traditional integer order, fractional-derivative models has the advantages of describing the memory and hereditary properties of various processes. By applying the space Riesz fractional operator [8, 22] to the Eq. (1), the fractional system is given as following

$$\frac{\partial u}{\partial t} = \mathbf{K}\mathbf{R}^{\alpha} u + f(u). \tag{2}$$

Here $\mathbf{R}^{\alpha} = (R_x^{\alpha}, R_y^{\alpha}) = (\partial^{\alpha}/\partial|x|^{\alpha}, \partial^{\alpha}/\partial|y|^{\alpha})$ is the Riesz fractional order operator

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with fractional power $1 < \alpha \leq 2$. Due to the extensive applications of fractional-derivative models, it is becoming increasingly important to find the effective methods to solve them. The methods include several analytical techniques, such as the Fourier transform method, the Laplace transform method, and the Green function method [25]. Some numerical methods are also developed. For example, Meerschaert [16] obtained the solution of the one-component fractional reaction-diffusion equation by using a finite difference method; Liu [9, 10] proposed finite difference method (FDM) and alternating direction implicit (ADI) method for the two-dimension space fractional reaction-diffusion equation, and verified the stability as well as convergence; Zeng [31] developed a Crank-Nicolson ADI spectral method for fractional diffusion equations; Cai [4] proposed a high-resolution semi-discrete Hermite central-upwind scheme for multidimensional reaction-diffusion equation.

In addition to standard finite difference methods, numerical solution can also be obtained by applying the nonstandard finite difference method (NSFD) [18], which has the following advantages. Firstly, the NSFD can be applied to the structurally unstable planar dynamical system, for example, the Lotka-Volterra equations [17]. Secondly, the NSFD preserves the physical properties of the epidemic model and the numerical results are qualitatively equivalent to the real dynamics of the epidemic model [23]. Thirdly, a scheme based on NSFD is shown to be free of numerical instabilities and contrived behaviours regardless of the step-size used in the numerical simulations [7]. Finally, the NSFD has been applied to the fractional order ODE [29] and PDE [13], and the results are in good agreement with the already existing ones.

In this paper, we consider the following 2-D Riesz space fractional reaction-diffusion equation (2D-RFRDE) on a finite domain $\Omega = [a, b] \times [c, d]$ as

$$\frac{\partial u}{\partial t} = k_x \frac{\partial^{\alpha_1} u}{\partial |x|^{\alpha_1}} + k_y \frac{\partial^{\alpha_2} u}{\partial |y|^{\alpha_2}} + f(u, x, y, t) \qquad (x, y, t) \in \ \Omega \times (0, T), \tag{3}$$

with initial condition:

$$u(x, y, 0) = \phi(x, y) \qquad (x, y) \in \Omega, \tag{4}$$

and Dirichlet boundary conditions:

$$\begin{array}{ll} u(a, y, t) = 0, & u(b, y, t) = 0, \\ u(x, c, t) = 0, & u(x, d, t) = 0. \end{array} \tag{5}$$

Here $1 < \alpha_1, \alpha_2 \leq 2$, and k_x, $k_y > 0$ are diffusion coefficients. The space Riesz fractional derivative operator $\frac{\partial^{\alpha_1} u}{\partial |x|^{\alpha_1}}$ in [15] is defined as

$$\frac{\partial^{\alpha_1} u}{\partial |x|^{\alpha_1}} = -\frac{1}{2\cos(\pi\alpha_1/2)} \left(\frac{\partial^{\alpha_1} u}{\partial_+ x^{\alpha_1}} + \frac{\partial^{\alpha_1} u}{\partial_- x^{\alpha_1}} \right). \tag{6}$$

Here the left-handed (+) and the right-handed (−) fractional derivative are defined later. Similarly, we can define the space Riesz fractional derivative $\frac{\partial^{\alpha_2} u}{\partial |y|^{\alpha_2}}$ of order α_2 with respect to y. An improved nonstandard finite difference scheme is constructed to obtain the numerical solution of Eqs. (3)-(5).

The outline of this paper is showed as follows. In Section 2, we introduce some notations and lemmas which are needed later on. In Section 3, the nonstandard finite difference (NSFD) method for the 2D-RFRDE is proposed. The stability and convergence are discussed in Section 4. Some numerical results are given in Section 5. we draw the conclusions in Section 6.