Compound Matrix Method and Evans Function — a quick introduction

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Abstract

In solving two-point boundary-value problems with variable coefficients, we often shoot from the two boundaries and match the solutions at an interior matching point. The Evans function $D(\lambda)$, as a function of the bifurcation parameter λ, is defined such that the two solutions match if $D(\lambda) = 0$. Thus, the problem of finding the bifurcation values is reduced to finding the roots of $D(\lambda) = 0$. The Evans function is often used together with the compound matrix method so that even stiff problems can be solved.

We consider the following eigenvalue problem:

$$\frac{d\boldsymbol{y}}{dx} = A(x, \lambda)\boldsymbol{y}, \quad a \leq x \leq b, \tag{1}$$

$$B(x, \lambda)\mathbf{y} = \mathbf{0}, \quad x = a, \tag{2}$$

$$C(x, \lambda)\mathbf{y} = \mathbf{0}, \quad x = b, \tag{3}$$

where A is a $2n \times 2n$ matrix, B and C are $n \times 2n$ matrices, all being known functions of the independent variable x and a parameter λ, and $\boldsymbol{y}$ is a $2n$-dimensional vector function. The aim is to determine values of the parameter (eigenvalues) λ so that non-trivial solutions exist. Such eigenvalue problems feature in a variety of disciplines. For instance, it often results from a linear stability/bifurcation analysis of fluid flows (e.g. Afendikov and Bridges 2001), solitary waves (e.g. Pego and Weinstein 1992), and pre-stressed elastic bodies (e.g. Fu and Pour 2002). One or both of the boundaries $x = a, b$ may be infinite.

1 Determinantal method

This would be the first method one could think of without reading any books. The idea is to shoot from one end to the other end and to iterate on λ so that the boundary condition on the other end is satisfied. We may also shoot from both ends towards a middle point and to iterate on λ so that the two solutions coincide at the middle point.

If we choose to shoot from $x = a$, then the procedure is as follows:

Assuming that matrix B has rank n, we may then always find n linearly independent vectors $\boldsymbol{y}_0^{(1)}, \boldsymbol{y}_0^{(2)}, ..., \boldsymbol{y}_0^{(n)}$ such that

$$B(a, \lambda)\boldsymbol{y}_0^{(i)} = 0, \quad i = 1, 2, ..., n. \tag{4}$$

Using each of these vectors as the initial value at $x = a$, we may integrate (1) from $x = a$ to obtain n independent solutions, say $\boldsymbol{y}^{(i)}(x), i = 1, 2, ..., n$. A general solution that satisfies (1) and the boundary condition (2) is then given by

$$\boldsymbol{y} = \sum_{i=1}^{n} k_i \boldsymbol{y}^{(i)}(x), \tag{5}$$

where $k_1, k_2, ..., k_n$ are arbitrary constants. We define $M(x, \lambda)$ to be the $2n \times n$ matrix whose i$^{\text{th}}$ column is $\mathbf{y}^{(i)}$, that is

$$M(x, \lambda) = [\mathbf{y}^{(1)}, \mathbf{y}^{(2)}, ..., \mathbf{y}^{(n)}]. \tag{6}$$

Equation (5) can then be written as

$$\boldsymbol{y} = M(x, \lambda)\boldsymbol{k}, \tag{7}$$

where

$$\boldsymbol{k} = [k_1, k_2, ..., k_n]^T.$$

On substituting (7) into the other boundary condition (3), we obtain

$$C(b, \lambda)M(b, \lambda)\boldsymbol{k} = \mathbf{0}. \tag{8}$$

Since $\boldsymbol{k} \neq \mathbf{0}$, we deduce that

$$|C(b, \lambda)M(b, \lambda)| = 0, \tag{9}$$

where a pair of vertical bars denotes the determinant of the matrix enclosed. We iterate on λ so that the determinantal equation (9) is satisfied.

When one of the boundaries or both are infinite, it is usual to shoot from $x = a$ and from $x = b$, respectively, so that the two solutions match at a middle point, say $x = d$. Denote by $\boldsymbol{y}^{(1)}(x), \boldsymbol{y}^{(2)}(x), ..., \boldsymbol{y}^{(n)}(x)$ the n solutions obtained by shooting from $x = a$ as explained in the previous paragraph. Then again, a general solution satisfying the left boundary condition is given by

$$\boldsymbol{y} = \sum_{i=1}^{n} k_i \boldsymbol{y}^{(i)}(x). \tag{10}$$

Likewise, we denote by $\boldsymbol{y}^{(n+1)}(x), \boldsymbol{y}^{(n+2)}(x), ..., \boldsymbol{y}^{(2n)}(x)$ the n solutions obtained by shooting from $x = b$. Then a general solution satisfying the right boundary condition is given by

$$\boldsymbol{y} = \sum_{i=n+1}^{2n} k_i \boldsymbol{y}^{(i)}(x), \tag{11}$$

where $k_{n+1}, k_{n+2}, ..., k_{2n}$ are another set of n constants. The two solutions (10) and (11) must match at $x = d$. Thus,

$$\sum_{i=1}^{n} k_i \boldsymbol{y}^{(i)}(x) = \sum_{i=n+1}^{2n} k_i \boldsymbol{y}^{(i)}(x), \quad \text{when } x = d, \tag{12}$$

or equivalently,

$$N(d,\lambda)\boldsymbol{c} = \mathbf{0}, \tag{13}$$

where

$$N(d,\lambda) = [\mathbf{y}^{(1)}, \mathbf{y}^{(2)}, ..., \mathbf{y}^{(n)}, \mathbf{y}^{(n+1)}, \mathbf{y}^{(n+2)}, ..., \mathbf{y}^{(2n)}], \tag{14}$$

$$\boldsymbol{c} = [k_1, k_2, ..., k_n, -k_{n+1}, -k_{n+2}, ..., -k_{2n}]^T.$$

We then iterate on λ so that the determinantal equation

$$|N(d,\lambda)| = 0 \tag{15}$$

is satisfied.

We note that whereas the $|N(d,\lambda)|$ defined above is dependent on d, the matching point, the following quantity is independent of d:

$$D(\lambda) = \mathrm{e}^{-\int_a^d \mathrm{tr}\, A(s,\lambda)\, ds} |N(d,\lambda)|. \tag{16}$$

This may easily be proved with the aid of the properties

$$\frac{d\boldsymbol{y}^{(i)}}{dx} = A(x,\lambda)\boldsymbol{y}^{(i)}, \tag{17}$$

and

$$\begin{aligned} &|A\mathbf{y}^{(1)}, \mathbf{y}^{(2)}, ..., \mathbf{y}^{(2n)}| + |\mathbf{y}^{(1)}, A\mathbf{y}^{(2)}, ..., \mathbf{y}^{(2n)}| + \cdots \\ &+ |\mathbf{y}^{(1)}, \mathbf{y}^{(2)}, ..., A\mathbf{y}^{(2n)}| = \mathrm{tr}\, A\, |\mathbf{y}^{(1)}, \mathbf{y}^{(2)}, ..., \mathbf{y}^{(2n)}|, \end{aligned} \tag{18}$$

see Chadwick (1976, p.18).

The determinantal method is conceptually easy, but for large or small values of λ the eigenvalue problem (1) usually becomes stiff and the solutions $\boldsymbol{y}^{(1)}, \boldsymbol{y}^{(2)}, ..., \boldsymbol{y}^{(n)}$, although linearly independent initially at $x = a$, quickly become linearly dependent due to the dominance of exponentially growing solutions. To address this problem, the compound matrix method was proposed by Ng and Reid (1979a, b, 1985); see also Lindsay and Rooney (1992). Bridges (1999) gave a very good differential-geometric interpretation of this method and explained why this new method works.

2 Compound matrix method

Let

$$\left\{\boldsymbol{y}^{(1)}(x), \boldsymbol{y}^{(2)}(x), ..., \boldsymbol{y}^{(n)}(x)\right\} \quad \text{and} \quad \left\{\boldsymbol{y}^{(n+1)}(x), \boldsymbol{y}^{(n+2)}(x), ..., \boldsymbol{y}^{(2n)}(x)\right\}$$

be two sets of n linearly independent solutions of (1) as defined in the previous section. The determinant of M defined by (6) has $\binom{2n}{n}$ minors and we denote them by $\phi_1, \phi_2, ...$ For instance, if $n = 2$, we have

$$\phi_1 = \begin{vmatrix} y_1^{(1)} & y_1^{(2)} \\ y_2^{(1)} & y_2^{(2)} \end{vmatrix} \stackrel{\text{def.}}{=} (1,2), \quad \phi_2 = \begin{vmatrix} y_1^{(1)} & y_1^{(2)} \\ y_3^{(1)} & y_3^{(2)} \end{vmatrix} \stackrel{\text{def.}}{=} (1,3), \tag{19}$$

and likewise

$$\phi_3 = (1,4), \quad \phi_4 = (2,3), \quad \phi_5 = (2,4), \quad \phi_6 = (3,4).$$

With the aid of (17), we may easily find the first-order differential equations satisfied by these ϕ's. For instance, when $n = 2$, we have

$$\phi_1' = \begin{vmatrix} y_1^{(1)'} & y_1^{(2)'} \\ y_2^{(1)} & y_2^{(2)} \end{vmatrix} + \begin{vmatrix} y_1^{(1)} & y_1^{(2)} \\ y_2^{(1)'} & y_2^{(2)'} \end{vmatrix}$$

$$= \begin{vmatrix} \sum_{i=1}^4 A_{1i} y_i^{(1)} & \sum_{i=1}^4 A_{1i} y_i^{(2)} \\ y_2^{(1)} & y_2^{(2)} \end{vmatrix} + \begin{vmatrix} y_1^{(1)} & y_1^{(2)} \\ \sum_{i=1}^4 A_{2i} y_i^{(1)} & \sum_{i=1}^4 A_{2i} y_i^{(2)} \end{vmatrix}$$

$$= A_{11}\phi_1 - A_{13}\phi_4 - A_{14}\phi_5 + A_{22}\phi_1 + A_{23}\phi_2 + A_{24}\phi_3.$$

Writing these $\binom{2n}{n}$ differential equations as a matrix equation, we have

$$\boldsymbol{\phi}' = \mathcal{A}(x,\lambda)\boldsymbol{\phi}, \quad a \le x \le b, \tag{20}$$

where $\boldsymbol{\phi}$ is the column vector with elements $\phi_1, \phi_2, \ldots$. When $n = 2$, the matrix $\mathcal{A}(x,\lambda)$ is given by

$$\mathcal{A}(x,\lambda) = \begin{pmatrix} A_{11}+A_{22} & A_{23} & A_{24} & -A_{13} & -A_{14} & 0 \\ A_{32} & A_{11}+A_{33} & A_{34} & A_{12} & 0 & -A_{14} \\ A_{42} & A_{43} & A_{11}+A_{44} & 0 & A_{12} & A_{13} \\ -A_{31} & A_{21} & 0 & A_{22}+A_{33} & A_{34} & -A_{24} \\ -A_{41} & 0 & A_{21} & A_{43} & A_{22}+A_{44} & A_{23} \\ 0 & -A_{41} & A_{31} & -A_{42} & A_{32} & A_{33}+A_{44} \end{pmatrix}. \tag{21}$$

The boundary condition for $\boldsymbol{\phi}$ at $x = a$ is obtained from $\boldsymbol{y}_0^{(i)}, i = 1, 2, \ldots, n$. For instance,

$$\phi_1(a) = \begin{vmatrix} y_{01}^{(1)} & y_{01}^{(2)} \\ y_{02}^{(1)} & y_{02}^{(2)} \end{vmatrix},$$

where $y_{01}^{(1)}$ is the first element in $\boldsymbol{y}_0^{(1)}$ etc. We may now integrate (20) from $x = a$ to $x = b$. The boundary condition (9) at $x = b$ can be expressed in terms of $\boldsymbol{\phi}$ by Laplace expansion. We iterate on λ so that this boundary condition is satisfied.

Alternatively, we may shoot from $x = a$ and from $x = b$, respectively, and match the two solutions at a middle point $x = d$ as in the previous section. In this case, we denote by $\boldsymbol{\phi}^{(-)}$ the $\boldsymbol{\phi}$ formed from the minors of

$$|\mathbf{y}^{(1)}, \mathbf{y}^{(2)}, \ldots, \mathbf{y}^{(n)}|,$$

and by $\boldsymbol{\phi}^{(+)}$ the $\boldsymbol{\phi}$ formed from the minors of

$$|\mathbf{y}^{(n+1)}, \mathbf{y}^{(n+2)}, \ldots, \mathbf{y}^{(2n)}|.$$

Obviously, $\boldsymbol{\phi}^{(-)}$ and $\boldsymbol{\phi}^{(+)}$ are both governed by the differential equation (20). We have explained just now how $\boldsymbol{\phi}^{(-)}$ is obtained. The $\boldsymbol{\phi}^{(+)}$ is obtained in a similar manner. The matching condition (15) can be expressed in terms of $\boldsymbol{\phi}^{(-)}$ and $\boldsymbol{\phi}^{(+)}$ using Laplace expansion. Laplace expansion is a straightforward generalization of the usual expansion of a determinant by a single row or column: it expands a determinant by any number of rows or columns. For instance, when $n = 2$, we have

$$|N(x,\lambda)| = |\,[\boldsymbol{y}^{(1)}(x), \boldsymbol{y}^{(2)}(x), \boldsymbol{y}^{(3)}(x), \boldsymbol{y}^{(4)}(x)]\,| = \begin{vmatrix} y_1^{(1)} & y_1^{(2)} & y_1^{(3)} & y_1^{(4)} \\ y_2^{(1)} & y_2^{(2)} & y_2^{(3)} & y_2^{(4)} \\ y_3^{(1)} & y_3^{(2)} & y_3^{(3)} & y_3^{(4)} \\ y_4^{(1)} & y_4^{(2)} & y_4^{(3)} & y_4^{(4)} \end{vmatrix}$$

$$= \begin{vmatrix} y_1^{(1)} & y_1^{(2)} \\ y_2^{(1)} & y_2^{(2)} \end{vmatrix} \cdot (-1)^{1+2+1+2} \begin{vmatrix} y_3^{(3)} & y_3^{(4)} \\ y_4^{(3)} & y_4^{(4)} \end{vmatrix} + \begin{vmatrix} y_1^{(1)} & y_1^{(2)} \\ y_3^{(1)} & y_3^{(2)} \end{vmatrix} \cdot (-1)^{1+2+1+3} \begin{vmatrix} y_2^{(3)} & y_2^{(4)} \\ y_4^{(3)} & y_4^{(4)} \end{vmatrix}$$

$$+ \begin{vmatrix} y_1^{(1)} & y_1^{(2)} \\ y_4^{(1)} & y_4^{(2)} \end{vmatrix} \cdot (-1)^{1+2+1+4} \begin{vmatrix} y_2^{(3)} & y_2^{(4)} \\ y_3^{(3)} & y_3^{(4)} \end{vmatrix} + \begin{vmatrix} y_2^{(1)} & y_2^{(2)} \\ y_3^{(1)} & y_3^{(2)} \end{vmatrix} \cdot (-1)^{1+2+2+3} \begin{vmatrix} y_1^{(3)} & y_1^{(4)} \\ y_4^{(3)} & y_4^{(4)} \end{vmatrix}$$

$$+ \begin{vmatrix} y_2^{(1)} & y_2^{(2)} \\ y_4^{(1)} & y_4^{(2)} \end{vmatrix} \cdot (-1)^{1+2+2+4} \begin{vmatrix} y_1^{(3)} & y_1^{(4)} \\ y_3^{(3)} & y_3^{(4)} \end{vmatrix} + \begin{vmatrix} y_3^{(1)} & y_3^{(2)} \\ y_4^{(1)} & y_4^{(2)} \end{vmatrix} \cdot (-1)^{1+2+3+4} \begin{vmatrix} y_1^{(3)} & y_1^{(4)} \\ y_2^{(3)} & y_2^{(4)} \end{vmatrix}.$$

Thus,

$$|N(x,\lambda)| = \phi_1^-\phi_6^+ - \phi_2^-\phi_5^+ + \phi_3^-\phi_4^+ + \phi_4^-\phi_3^+ - \phi_5^-\phi_2^+ + \phi_6^-\phi_1^+, \tag{22}$$

$$D(\lambda) = \mathrm{e}^{-\int_a^d \operatorname{tr} A(s,\lambda)\,ds}|N(d,\lambda)|.$$

We iterate on λ so that the matching condition $D(\lambda) = 0$ is satisfied.

The function $D(\lambda)$ is known as Evans function; it is an invariant of the differential equation (1). This Evans function was first introduced by Evans (1972, 1975), and further developed by Alexander *et al* (1990).

3 Interpretation of $\boldsymbol{\phi}$ in terms of wedge/exterior products

Write

$$u = \boldsymbol{y}^{(1)} \wedge \boldsymbol{y}^{(2)} ... \wedge \boldsymbol{y}^{(n)}, \tag{23}$$

where the wedge product obeys the following three rules:

$$(\boldsymbol{a}\wedge\boldsymbol{b})\wedge\boldsymbol{c} = \boldsymbol{a}\wedge(\boldsymbol{b}\wedge\boldsymbol{c}), \quad (k_1\boldsymbol{a}+k_2\boldsymbol{b})\wedge k_3\boldsymbol{c} = k_1k_3\boldsymbol{a}\wedge\boldsymbol{c} + k_2k_3\boldsymbol{b}\wedge\boldsymbol{c}, \quad \boldsymbol{a}\wedge\boldsymbol{b} = -\boldsymbol{b}\wedge\boldsymbol{a},$$

for all vectors $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}$ and scalars k_1, k_2, k_3.

Take $n = 2$ as an example. We denote the unit vectors in R^4 by $\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3, \boldsymbol{e}_4$. We then have

$$\boldsymbol{y}^{(1)} = \sum_{i=1}^{4} y_i^{(1)}\boldsymbol{e}_i, \quad \boldsymbol{y}^{(2)} = \sum_{j=1}^{4} y_j^{(2)}\boldsymbol{e}_j. \tag{24}$$

It then follows that

$$\boldsymbol{y}^{(1)} \wedge \boldsymbol{y}^{(2)} = \sum_{i=1}^{4}\sum_{j=1}^{4} y_i^{(1)} y_j^{(2)} \boldsymbol{e}_i \wedge \boldsymbol{e}_j = \sum_{i=1, i<j}^{4} \sum_{j=1}^{4} (y_i^{(1)} y_j^{(2)} - y_j^{(1)} y_i^{(2)}) \boldsymbol{e}_i \wedge \boldsymbol{e}_j,$$

$$= \phi_1\, \boldsymbol{e}_1 \wedge \boldsymbol{e}_2 + \phi_2 \boldsymbol{e}_1 \wedge \boldsymbol{e}_3 + \phi_3\, \boldsymbol{e}_1 \wedge \boldsymbol{e}_4 + \phi_4\, \boldsymbol{e}_2 \wedge \boldsymbol{e}_3 + \phi_5\, \boldsymbol{e}_2 \wedge \boldsymbol{e}_4 + \phi_6\, \boldsymbol{e}_3 \wedge \boldsymbol{e}_4. \tag{25}$$

Thus, we see that the variables $\phi_1, \phi_2, ..., \phi_6$ used in the compound matrix method are simply the components of the wedge product $\boldsymbol{y}^{(1)} \wedge \boldsymbol{y}^{(2)}$ relative to the basis

$$\boldsymbol{e}_1 \wedge \boldsymbol{e}_2, \quad \boldsymbol{e}_1 \wedge \boldsymbol{e}_3, \quad \boldsymbol{e}_1 \wedge \boldsymbol{e}_4, \quad \boldsymbol{e}_2 \wedge \boldsymbol{e}_3, \quad \boldsymbol{e}_2 \wedge \boldsymbol{e}_4, \quad \boldsymbol{e}_3 \wedge \boldsymbol{e}_4.$$

We may therefore conclude that if $\boldsymbol{y}^{(1)}, \boldsymbol{y}^{(2)}$ are solutions of (1), then $\boldsymbol{y}^{(1)} \wedge \boldsymbol{y}^{(2)}$ is a solution of (20).

We now write

$$\boldsymbol{y}^{(1)} \wedge \boldsymbol{y}^{(2)} = \phi_1^-(\boldsymbol{e}_1 \wedge \boldsymbol{e}_2) + \phi_2^-(\boldsymbol{e}_1 \wedge \boldsymbol{e}_3) + \phi_3^-(\boldsymbol{e}_1 \wedge \boldsymbol{e}_4) + \phi_4^-(\boldsymbol{e}_2 \wedge \boldsymbol{e}_3) + \phi_5^-(\boldsymbol{e}_2 \wedge \boldsymbol{e}_4) + \phi_6^-(\boldsymbol{e}_3 \wedge \boldsymbol{e}_4),$$

and

$$\boldsymbol{y}^{(3)} \wedge \boldsymbol{y}^{(4)} = \phi_1^+(\boldsymbol{e}_1 \wedge \boldsymbol{e}_2) + \phi_2^+(\boldsymbol{e}_1 \wedge \boldsymbol{e}_3) + \phi_3^+(\boldsymbol{e}_1 \wedge \boldsymbol{e}_4) + \phi_4^+(\boldsymbol{e}_2 \wedge \boldsymbol{e}_3) + \phi_5^+(\boldsymbol{e}_2 \wedge \boldsymbol{e}_4) + \phi_6^+(\boldsymbol{e}_3 \wedge \boldsymbol{e}_4).$$

We then have

$$\begin{aligned} \boldsymbol{y}^{(1)} \wedge \boldsymbol{y}^{(2)} \wedge \boldsymbol{y}^{(3)} \wedge \boldsymbol{y}^{(4)} \quad = \quad & \phi_1^- \phi_6^+(\boldsymbol{e}_1 \wedge \boldsymbol{e}_2 \wedge \boldsymbol{e}_3 \wedge \boldsymbol{e}_4) \\ & + \phi_2^- \phi_5^+(\boldsymbol{e}_1 \wedge \boldsymbol{e}_3 \wedge \boldsymbol{e}_2 \wedge \boldsymbol{e}_4) \\ & + \phi_3^- \phi_4^+(\boldsymbol{e}_1 \wedge \boldsymbol{e}_4 \wedge \boldsymbol{e}_2 \wedge \boldsymbol{e}_3) \\ & + \phi_4^- \phi_3^+(\boldsymbol{e}_2 \wedge \boldsymbol{e}_3 \wedge \boldsymbol{e}_1 \wedge \boldsymbol{e}_4) \\ & + \phi_5^- \phi_2^+(\boldsymbol{e}_2 \wedge \boldsymbol{e}_4 \wedge \boldsymbol{e}_1 \wedge \boldsymbol{e}_3) \\ & + \phi_6^- \phi_1^+(\boldsymbol{e}_3 \wedge \boldsymbol{e}_4 \wedge \boldsymbol{e}_1 \wedge \boldsymbol{e}_2) \end{aligned}$$

$$= (\phi_1^- \phi_6^+ - \phi_2^- \phi_5^+ + \phi_3^- \phi_4^+ + \phi_4^- \phi_3^+ - \phi_5^- \phi_2^+ + \phi_6^- \phi_1^+)\boldsymbol{e}_1 \wedge \boldsymbol{e}_2 \wedge \boldsymbol{e}_3 \wedge \boldsymbol{e}_4.$$

Thus, by comparing with (22) and generalizing to arbitrary n, we may also write the expression (16) as

$$D(\lambda) = \mathrm{e}^{-\int_a^d \mathrm{tr}\, A(s,\lambda)\, ds} \boldsymbol{y}^{(1)} \wedge \boldsymbol{y}^{(2)} \cdots \wedge \boldsymbol{y}^{(n)} \wedge \boldsymbol{y}^{(n+1)} \cdots \wedge \boldsymbol{y}^{(2n)}. \tag{26}$$

More information on exterior multiplication can be found in Arnold (1989, p.170).

4 Examples

Example 1: We first solve the eigenvalue problem

$$\varepsilon^4 \frac{d^4 w}{dx^4} + 2\varepsilon^2 \lambda \frac{d}{dx}\left[\sin(x) \frac{dw}{dx}\right] + w = 0, \quad x \in [0, \pi], \tag{27}$$

$$w = \frac{d^2w}{dx^2} = 0 \text{ for } x = 0, \pi, \tag{28}$$

where ε is known positive parameter (the problem becomes stiff if $\varepsilon \ll 1$ which can be solved using the WKB method), and the problem is to find the minimum eigenvalue of λ for which the above boundary value problem has a non-trivial solution.

Define $\boldsymbol{y} = (w, w', w'', w''')^T$. Then the above governing equation (27) can be written in the form (1) with A given by

$$A = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & A_{42} & A_{43} & 0 \end{pmatrix},$$

where

$$A_{42} = -\frac{2\lambda}{\varepsilon^2}\cos(x), \quad A_{43} = -\frac{2\lambda}{\varepsilon^2}\sin(x).$$

Since both the governing equation and boundary conditions are symmetric about $x = \pi/2$, we expect that the eigen modes are either symmetric or anti-symmetric about $x = \pi/2$. We focus on the symmetric modes which must satisfy the conditions

$$w'(\frac{\pi}{2}) = w'''(\frac{\pi}{2}) = 0,$$

and the eigenvalue problem is solved for $x \in [0, \frac{\pi}{2}]$.

Two independent vectors that satisfy the boundary conditions at $x = 0$ are

$$\boldsymbol{y}_0^{(1)} = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \quad \boldsymbol{y}_0^{(2)} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix},$$

whereas two independent vectors that satisfy the boundary conditions at $x = \pi/2$ are

$$\boldsymbol{y}_0^{(3)} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad \boldsymbol{y}_0^{(4)} = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}.$$

It then follows that

$$\phi^+(0) = (0,0,0,0,1,0)^T, \quad \phi^-(\frac{\pi}{2}) = (0,1,0,0,0,0)^T. \tag{29}$$

The compound matrix equation (20) can now be integrated subjected to $(29)_{1,2}$ to obtain the solutions ϕ^+ and ϕ^-, respectively. Plotting $D(\lambda)$ shows that $D(\lambda) = 0$ has an infinite number of positive roots, the first three being given by

$$1.10517, \quad 1.34269, \quad 1.62184$$

when $\varepsilon = 0.1$. For small ε, a WKB analysis (Coman 2004) gives the asymptotic expression $\lambda = 1 + (2m+1)\varepsilon + O(\varepsilon^2)$, $\quad m = 0, 1, 2, \ldots$. For the same $\varepsilon = 0.1$, this expression gives the first three eigenvalues $1.1, 1.3, 1.5$.

Example 2: We next consider the eigenvalue problem

$$\frac{d^4 w}{dx^4} - \frac{d^2 w}{dx^2} + 2\frac{d^2}{dx^2}(w_0 w) + \lambda w = 0, \qquad x \in (-\infty, \infty), \tag{30}$$

$$w(\pm\infty) \to 0, \tag{31}$$

where

$$w_0 = \frac{3}{2}\mathrm{sech}^2(\frac{x}{2}).$$

The problem is to find the minimum eigenvalue of λ for which the above boundary value problem has a non-trivial solution.

We first note that in the limit $x \to \pm\infty$, the governing equation (30) tends to

$$\frac{d^4 w}{dx^4} - \frac{d^2 w}{dx^2} + \lambda w = 0, \qquad x \in (-\infty, \infty). \tag{32}$$

On substituting the trial solution $w = \mathrm{e}^{kx}$ into (32), we obtain

$$k^2 = \frac{1}{2}(1 \pm \sqrt{1 - 4\lambda}).$$

Thus, since the behaviour of w in the limit $x \to \pm\infty$ must necessarily be a linear combination of all solutions of the form e^{kx}, with k determined above, decaying solutions are only possible for $\lambda < 1/4$. We make this assumption from now on.

Again define $\boldsymbol{y} = (w, w', w'', w''')^T$. Then the above governing equation (30) can be written in the form (1) with A given by

$$A = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -(2w_0'' + \lambda) & -4w_0' & 1 - 2w_0 & 0 \end{pmatrix}.$$

Since both the governing equation and boundary conditions are symmetric about $x = 0$, we expect that the eigen modes are either symmetric or anti-symmetric about $x = 0$. We focus on the symmetric modes which must satisfy the conditions

$$w'(0) = w'''(0) = 0,$$

and the eigenvalue problem is solved for $x \in [0, \infty)$.

Two independent vectors that satisfy the boundary conditions at $x = 0$ are

$$\boldsymbol{y}_0^{(1)} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad \boldsymbol{y}_0^{(2)} = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}.$$

In the limit $x \to \infty$, equation (1) reduces to

$$\boldsymbol{y}' = A_\infty \boldsymbol{y}, \quad \text{where } \; A_\infty = A(\infty, \lambda).$$

Numerically, we will have to replace ∞ by a sufficiently large positive number, say L. By considering a solution of the form $\boldsymbol{y} = \mathrm{Y}\mathrm{e}^{kx}$, we find that a decaying solution at $x = L$ must necessarily be a linear combination of

$$\boldsymbol{y}_0^{(3)} \mathrm{e}^{-k_1 L} \quad \text{and} \;\; \boldsymbol{y}_0^{(4)} \mathrm{e}^{-k_2 L},$$

or equivalently, a linear combination of

$$\boldsymbol{y}_0^{(3)} \quad \text{and} \;\; \boldsymbol{y}_0^{(4)},$$

where

$$\boldsymbol{y}_0^{(3)} = \begin{pmatrix} 1 \\ -k_1 \\ k_1^2 \\ -k_1^3 \end{pmatrix}, \quad \boldsymbol{y}_0^{(4)} = \begin{pmatrix} 1 \\ -k_2 \\ k_2^2 \\ -k_2^3 \end{pmatrix}, \quad k_1 = \sqrt{\frac{1}{2}(1 + \sqrt{1 - 4\sigma})}, \quad k_2 = \sqrt{\frac{1}{2}(1 - \sqrt{1 - 4\sigma})}.$$

The appropriate boundary conditions are then given by

$$\boldsymbol{\phi}^+(0) = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \quad \boldsymbol{\phi}^-(L) = \begin{pmatrix} -k_2 + k_1 \\ k_2^2 - k_1^2 \\ -k_2^3 + k_1^3 \\ -k_1 k_2^2 + k_1^2 k_2 \\ k_1 k_2^3 - k_2 k_1^3 \\ -k_1^2 k_2^3 + k_2^2 k_1^3 \end{pmatrix}. \tag{33}$$

The compound matrix equation (20) can now be integrated subjected to $(33)_{1,2}$ to obtain the solutions ϕ^+ and ϕ^-, respectively. Plotting $D(\lambda)$ shows that $D(\lambda) = 0$ has a single root $3/16$ in $(0, 1/4)$. It turns out that at this eigenvalue, equation (30) can be solved exactly to give

$$w = \mathrm{sech}(\frac{x}{2}) - 2\mathrm{sech}^3(\frac{x}{2}).$$

See, e.g., Pearce and Fu (2010).

References

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