Three-dimensional nonlinear analysis of axisymmetric deformation of elastic cylindrical tubes

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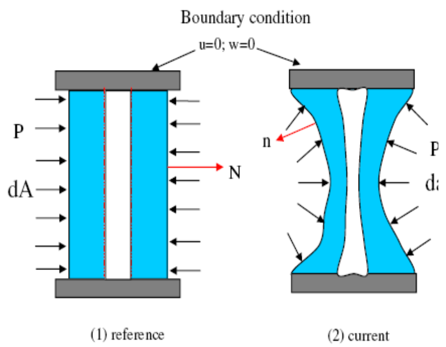
Abstract: Cylindrical tube deforms in a strongly nonlinear fashion when subject to large external pressure, a problem that frequently appear in many biological applications. Engineering approaches to this problem often used linear deformation, which can give very inaccurate predictions. In this project, a totally nonlinear analysis is conducted for axisymmetricly deformed thick-walled cylindrical tube subject to external pressure, and the results are compared to the linear predictions.



Cylindrical tube with ends constrained and subject to external pressure P. *u*, *w* are the displacements in radial and axial directions

Equilibrium Equation

$$\begin{cases} S_{Rr,R} + S_{Zr,Z} + \frac{1}{R}(S_{Rr} - S_{\Theta\theta}) = 0 \\ S_{Rz,R} + S_{Zz,Z} + \frac{1}{R}S_{Rz} = 0 \end{cases}$$

Boundary Condition

$$-P\mathbf{n}da = -PJ\mathbf{F}^{-\mathbf{T}}\mathbf{N}dA = \mathbf{S}^{\mathbf{T}}\mathbf{N}dA$$

$$\begin{cases} u = 0 \quad \text{at Z=0,L} \\ w = 0 \quad \text{at Z=0,L} \end{cases}$$

where *S* is the nominal stress, and *R* is the radius in the reference configuration.

Numerical Methods

The system of the nonlinear equations is solved using the C++ package LIBMESH, where the library SNES is used with Newton methods incorporated with line search and trust region techniques.

$$\mathrm{U}_{r+1} = \mathrm{U}_r - \mathrm{J}^{-1}(\mathrm{U}_r)\Re(\mathrm{U}_r)$$

$$\mathrm{J}(\mathrm{U}_r) = \frac{\partial\Re(\mathrm{U}_r)}{\partial\mathrm{U}} = \mathrm{K}(\mathrm{U}_r) + \frac{\partial\mathrm{K}(\mathrm{U}_r)}{\partial\mathrm{U}}\mathrm{U}_r - \frac{\partial\mathrm{F}(\mathrm{U}_r)}{\partial\mathrm{U}}$$

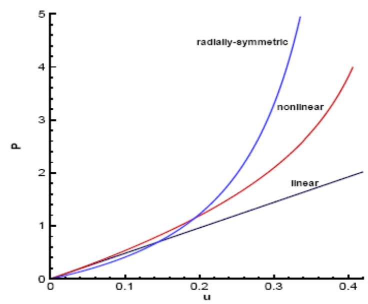
The neo-Hookean nonlinear material is used where the strain energy is determined by the three principle stretches:

$$W = \frac{1}{2}(\lambda_1^2 + \lambda_2^2 + \lambda_3^2 - 3)$$

- Isotropic
- Incompressible
- Hyperelastic

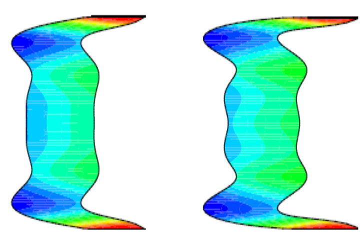
Deformation Gradient

$$\mathbf{F} = \begin{pmatrix} 1+u_R & 0 & u_Z \\ 0 & 1+u/R & 0 \\ w_R & 0 & 1+w_Z \end{pmatrix}$$

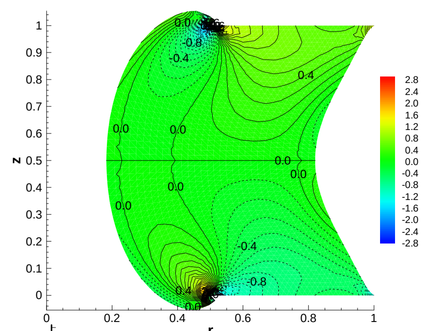


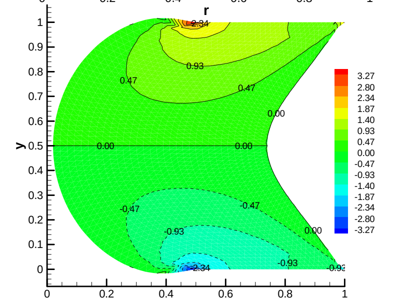
Comparison of the displacement *u* at (R=0.5, Z=0.5) for different P predicted by the linear, nonlinear, and radially symmetric models. Agreement is only good at small pressure loading.

modes



Higher mode (where the deformation has more than one humps) is also one of the main features of the nonlinear prediction, while the linear one typically only presents lower modes.





The nonlinear (upper) shear stress distribution over a cross section is significantly different from the linear prediction (lower), with the two inner corners bulges out.

Conclusions

For small deformation, both linear and nonlinear models give very similar results. However, cylindrical tubes behaves very differently under large external pressure, and the dominate features are the corner bulging and higher modes. This is the first time that a totally nonlinear analysis is carried out for thick walled tubes, and the results may have significant implications to many physiological applications involving soft vessels undergoing large deformation.

References

1.Kirk, B., Peterson, J.W., Stogner, R. H., and Carey, G. F., 2006. libMesh: A C++ Library for Parallel Adaptive Mesh Refinement/Coarsening Simulations. Engineering with Computers, vol.22, no. 3--4, pp. 237--254.
2.Ogden, R.W., 1997. Non-linear Elastic Deformations. Dover Publications, New York.
3. Y. Zhu, X.Y. Luo, R.W. Ogden, 2008. Asymmetric bifurcations of thick-walled circular cylindrical elastic tubes under axial loading and external pressure. International Journal of Solids and Structures 45,3410-3429.
4. Y. Zhu, X.Y. Luo, R.W. Ogden, 2009,Three dimensional nonlinear analysis of axisymmetric deformation of elastic cylindrical tubes (to be submitted).