UDC 517.9

Gaussian beam behaviour in the anisotropic medium with interface

Kirpichnikova A.S.

March 15, 2006

We consider a region which consists of two parts with anisotropic Riemannian metrics. The metric has a jump on the interface. The reflected and transmitted from the interface asymptotic solutions to the wave equation - Gaussian beams ("quasiphotons") are constructed.

1 Results

1.1 Main definitions.

Consider two anisotropic media Ω_- and Ω_+ with a common part which we call the common ***interface*** γ. Assume that the dimension of $\Omega_\pm$ is n, and the interface γ is a hypersurface. Consider $\Omega_\pm$ to be $C^\infty-$ smooth up to the boundary $\partial\Omega_\pm$ Riemannian manifolds. Denote Riemannian metric tensor on $\Omega_\pm$ by $g^\pm$, i.e. $g^\pm = g(\Omega_\pm)$, assuming that

$$g^-|_\gamma \neq g^+|_\gamma. \tag{0.1}$$

1.1.1 Coordinates.

We use boundary normal coordinates (also called semi-geodesic) (see [1], [2], [4]), i.e. coordinates corresponding to the interface γ, such that $(\mathbf{q}, \sigma^\pm) = (q^1, .., q^{n-1}, \sigma^\pm) = \{q^\alpha, \sigma^\pm\}, \alpha = 1, .., n-1$, where $\mathbf{q}$ are some smooth coordinates on γ and $\sigma^\pm$ is the distance to γ in the metric $g^\pm$, i.e. $\sigma = \begin{cases} \sigma^+ > 0, \, \Omega_+, \\ \sigma = 0, \, \gamma, \\ -\sigma^- < 0, \, \Omega_-, \end{cases}$. We use Greek letters

$\alpha, \beta, \delta, ...$ to count $(n-1)$ interface coordinates $\{q^\alpha\} = \{q^1, .., q^{n-1}\}$. Sometimes we need a notation for any regular (inner) coordinates $\mathbf{x} = (x^1, ..., x^n) = \{x^i\},\ i = 1, .., n$, i.e. coordinates, smooth in some vicinity $V \subset \Omega_- \bigcup \Omega_+$ of the point on γ. Latin letters count inner coordinates of media $\{x^i\} = \{x^1, .., x^n\}$. Choose an origin such that it belongs to γ, for instance choose point M_1 with coordinates $\sigma = 0, q^\alpha = 0$ to be the origin of semi-geodesic coordinates.

As σ is orthogonal to all q^α, the metric tensor matrix in semi-geodesic coordinates takes the form

$$\{g^\pm_{ij}\} = \begin{pmatrix} g^\pm_{\alpha\beta} & 0 \\ 0 & 1 \end{pmatrix},$$

where $g^\pm_{\alpha\beta} = (\frac{d\mathbf{x}}{dq^\alpha}, \frac{d\mathbf{x}}{dq^\beta})_{g^\pm}$ is $(n-1) \times (n-1)-$ smooth matrix of tangential components of the metric.

1.1.2 Laplace-Beltrami operator, the Dirichlet Problem.

Consider Laplace-Beltrami operator $\Delta_\pm$ on both sides from γ, it has the following form in local semi-geodesic coordinates

$$\Delta_g = \frac{1}{\sqrt{g}}(\partial_\alpha g^{\alpha\beta}\sqrt{g}\partial_\beta + \partial_n\sqrt{g}\partial_n), \quad g = \det g_{ij},\ i, j = 1, .., n, \tag{0.2}$$

where $\partial_\alpha := \frac{\partial}{\partial q^\alpha},\ \partial_n := \partial_\sigma := \frac{\partial}{\partial\sigma}$ and

$$g = \begin{cases} g^+, & \sigma > 0 \\ g^-, & \sigma < 0 \end{cases}, \text{ or } g = \begin{cases} g^+, & (\mathbf{q}, \sigma^+) \in \Omega_+ \\ g^-, & (\mathbf{q}, \sigma^-) \in \Omega_-. \end{cases}$$

Recall the metric jump condition on the interface (0.1). The worth case (in the sense of smoothness class of the operator) is the case when metric tensor g has a jump on the zero measure set γ, i.e. $g \in L^\infty$. The Dirichlet form is determined for such an operator. The latter means that the operator is determined in the weak sense, in the sense of quadratic forms and it can be integrated with Sobolev class functions H^1_0. We state a spectral Dirichlet problem, supplying the operator with continuity conditions on the interface and with boundary conditions such that this problem will have a unique solution. Thus,

$$\begin{cases} -\Delta_g \varphi_k(\mathbf{x}) = \lambda_k \varphi_k(\mathbf{x}), \\ \varphi_k|_{(\Gamma_0)\backslash\gamma} = 0 \text{ Dirichlet type boundary conditions (DBC)}, \\ \text{continuity conditions of the function and its normal derivative} \\ \text{on the interface}\, \gamma, \end{cases} \tag{0.3}$$

where $\varphi_k(\cdot,\cdot) = \begin{cases} \varphi_k^+, \ (\mathbf{q}, \sigma^+) \\ \varphi_k^-, \ (\mathbf{q}, \sigma^-) \end{cases}$ are the eigenfunctions of the operator considered on the union of our media, in the whole region, and λ_k are the corresponding eigenvalues. The domain of the operator is

$$\mathcal{D}(\Delta_g) = \left\{ f^{\pm} = H_0^1(\Omega_{\pm}) : f^+|_\gamma = f^-|_\gamma, \ \sqrt{g^+}\frac{\partial f^+}{\partial \sigma^+}|_\gamma = \sqrt{g^-}\frac{\partial f^-}{\partial \sigma^-}|_\gamma \right\}.$$

We need the following useful notation:

$$\Box_g = \frac{\partial^2}{\partial t^2} - \Delta_g \ \text{ is a D'Alambert's wave operator.} \tag{0.4}$$

It is useful to consider a few variants of possible DBC, different by their domains. Thus, DBC can be determined, say on Γ_0 which is

- the whole boundary of the manifold or the part of this boundary, in our case $\Gamma_0 = (\partial\Omega_+ \bigcup \partial\Omega_-)\backslash\gamma$, or $\Gamma_0 \subset ((\partial\Omega_+ \bigcup \partial\Omega_-)\backslash\gamma)$ correspondingly.
- the part of the connected sub-domain $\mathcal{D}$ of codimension one inside $(\Omega_+ \bigcup \Omega_-)$.

The continuity conditions on the interface necessary for the solution uniqueness will be considered later. Without a loss of generality we assume that the solution during time period $(0, t_0 > 0)$ won't leave the domain of regularity of semi-geodesic coordinates and won't reach the manifold boundary $\partial\Omega_+ \bigcup \partial\Omega_-$.

1.2 Gaussian beams - "quasiphotons".

We seek the solution to the wave equation in the form of a Gaussian beam reviewing the well-known procedure from papers of Babich V., Ulin V., [1], Katchalov A., [3], Ralston J., [7] and others. *Gaussian beam* is a complex-valued asymptotic solution to the wave equation such that

✓ starting from the boundary it is concentrated at time t near the point $\mu(t)$ on some geodesic. In other words, Gaussian beam decays fast on increasing the distance from that point,

✓ it propagates with unit velocity along the geodesic $\mu(t)$.

Such a solution can be obtained as a unique solution of the initial-boundary value problem (IBVP) for the wave equation, assuming that the source $f^0(\varepsilon; t, \mathbf{z})$, $\mathbf{z} \in \Gamma_0$ is situated on the boundary Γ_0 in the vicinity of the initial point $M_0(\mathbf{z}_0, \tau_0)$ at time $-t_0 < 0$, $\mathbf{z}_0 \in \Gamma_0$, where (z, τ) are semi-geodesic coordinates corresponding to Γ_0, i.e. $\Gamma_0 : \{\tau = \tau_0 = 0,\ \mathbf{z}^\alpha$ are some smooth coordinates on$\Gamma_0\}$. Denote by

$$U(t, \mathbf{z}, \tau) = \begin{cases} U^+(t, \mathbf{z}, \tau), & \sigma > 0 \\ U^-(t, \mathbf{z}, \tau), & \sigma < 0 \end{cases}$$

the solution of the following problem

$$\begin{cases} \partial_t^2 U - \Delta_g U = 0, \\ U|_{-t_0} = \partial_t U|_{-t_0} = 0, \\ U|_{\Gamma_0} = f^0(\varepsilon; t, \mathbf{z}). \end{cases} \tag{0.5}$$

Here $f^0(\varepsilon; t, \mathbf{z}) := M_\varepsilon \chi^0(t, \mathbf{z}) \exp\{i\varepsilon^{-1}\Theta^0(t, \mathbf{z})\} V^0(\mathbf{z})$ is a functional class on $\Gamma_0 \times \mathbb{R}$, χ^0 is a smooth characteristic function in the vicinity of point $(-t_0, \mathbf{z}_0)$, where $M_\varepsilon := (\pi\varepsilon)^{-\frac{n}{4}}$, $V^0(\mathbf{z})$ is a given smooth function, ε is a small parameter, $0 < \varepsilon < 1$. The amplitude function can be presented by a sum of smooth homogeneous polynomials on the distances $(\mathbf{z} - \mathbf{z}_0)$ and $(t + t_0)$ with complex coefficients, the phase function has the following form

$$\Theta^0(t, \mathbf{z}) = -(t + t_0) + \frac{1}{2}(H^0(\mathbf{z} - \mathbf{z}_0), (\mathbf{z} - \mathbf{z}_0)) + \frac{i}{2}(t + t_0)^2, \tag{0.6}$$

where $(\cdot, \cdot)$ is a euclidian inner product, $(H^0)^t = H^0$, $\operatorname{Im} H^0 > 0$. Consider a **definition: a Finite Gaussian beam (FGb)** N *is a function* $U_N(\varepsilon; t, \boldsymbol{q}, \sigma)$ *of the following form (we follow papers of Katchalov A., [4], [3], and Katchalov A., Kurylev Ya., Lassas M., [5]):*

$$U_N(\varepsilon; t, \boldsymbol{q}, \sigma) \asymp^N M_\varepsilon \exp\{-(i\varepsilon)^{-1}\Theta_N(t, \boldsymbol{q}, \sigma)\} \sum_{l=0}^{N} u_l(t, \boldsymbol{q}, \sigma)(i\varepsilon)^l, \tag{0.7}$$

where phase function $\Theta_N(t, \boldsymbol{q}, \sigma)$ *satisfies conditions:*

$$\operatorname{Im}\Theta_N(t, \mu(t)) = 0, \quad \mu(t) = (\boldsymbol{q}(t), \sigma(t)) \text{ is a geodesic,} \tag{0.8}$$

$$\operatorname{Im}\Theta_N(t, \boldsymbol{q}, \sigma) \geq C_0(t) dist^2(\xi, \xi(t)),\ dist(\xi, \xi(t)) \neq 0, \tag{0.9}$$

here C_0 *is a continuous positive function. We have an inequality valid for the beam:*

$$|\Box_g U_N(\varepsilon; t, \boldsymbol{q}, \sigma)| \leq C(2t_0) M_\varepsilon \varepsilon^N, \tag{0.10}$$

where $C(2t_0)$ *does not depend on* ε,

$$\xi = \begin{pmatrix} \boldsymbol{q} \\ \sigma \end{pmatrix},\ \xi(t) = \begin{pmatrix} \boldsymbol{q}(t) \\ \sigma(t) \end{pmatrix},\ \boldsymbol{Y} = \xi - \xi(t), \tag{0.11}$$

where $0 < \eta < 1/6$*. The phase and amplitude function expansions have the form:*

$$\Theta_N = \theta_1 + \theta_2 + ...\theta_{K(N)},\ K(N) = \frac{2(N+2-l)}{1-2\eta}, \tag{0.12}$$

$$u_l = u_{l(0)} + ... + u_{l(L(N))},\ L(N) = \frac{2(N+1-l)}{1-2\eta}, \tag{0.13}$$

at that $dist(\xi, \xi(t)) = dist((\boldsymbol{q}, \sigma), (\boldsymbol{q}(t), \sigma(t)))$ *is the distance in* $\Omega_\pm$*.*

Remark: Lemma 2.49 from [5] allows one to restrict himself on construction of a finite number of terms in phase and amplitude expansions, see (0.12), (0.13).

Remark: The finite Gaussian beam (FGb) $U_N(\varepsilon; t, \mathbf{q}, \sigma)$ introduced above is concentrated near $\xi(t)$, i.e. $\|U_N(\varepsilon; t, \cdot)|_{B_\rho}\| = \mathcal{O}(1)$. FGb decays exponentially

$$\|U_N(\varepsilon; t, \cdot)|_{(\Omega_- \bigcup \Omega_+)\setminus B_\rho}\| = \mathcal{O}(\varepsilon^p),$$

for any $p > 0$ outside of the ball $B_\rho(\xi(t))$ of radius $\rho > 0$, $\rho \sim \varepsilon^{\frac{1}{2}-\eta}$ (i.e. on the distances more than $\mathcal{O}(\varepsilon^{1/2-\eta})$, $0 < \eta < 1/6$ from the geodesic $\mu(t)$).

Useful notation 1: We write $\asymp^k$ when there is an equality of formal series up to the order k over all powers $\varepsilon, t, q^1, .., q^{n-1}, \sigma$ and their combinations takes place (we do not care about the convergence here); we denote a formal asymptotic (algebraic) expansion by $\asymp^k$, i.e. we write $f(t, \mathbf{Y}(t)) \asymp^k 0$ when $\partial_Y^\delta f(t, \mathbf{Y}(t))|_{\mathbf{Y}(t)=0} = 0$, where $|\delta| \leq k$ is a multi-index. If index k is absent then the expansion is true for any k.

Useful notation 2: The first terms of the phase expansion Θ_N of the Gaussian beam have special (polynomial) notations

$$\theta_1(t, q^\alpha, \sigma) = p_\alpha(t)\mathbf{Y}^\alpha + p_n(t)\mathbf{Y}^n, \tag{0.14}$$

where $p(t) := (p_\alpha(t), p_n(t))$ is ***an impulse*** of the Gaussian beam and

$$\theta_2(t, q^\alpha, \sigma) = \frac{1}{2}H_{\alpha\beta}(t)\mathbf{Y}^\alpha\mathbf{Y}^\beta + H_{\alpha n}(t)\mathbf{Y}^\alpha\mathbf{Y}^n + \frac{1}{2}H_{nn}(t)(\mathbf{Y}^n)^2, \tag{0.15}$$

where

$$H(t) := \begin{pmatrix} H_{\alpha\beta}(t) & H_{\alpha n}(t) \\ H_{n\beta}(t) & H_{nn}(t) \end{pmatrix}$$

is called ***the quadratic form*** which contains the divergence of rays and a form of the beam. Consequently $\theta_l(t, q^\alpha, \sigma)$ are homogeneous polynomials of order l with respect to $\mathbf{Y}$, terms of order $l > 2$ are of minor importance. Similarly we introduce notation for higher order terms of the expansion:

$$\theta_l = \frac{1}{l!}(Q_{\alpha_1\ldots\alpha_l}\mathbf{Y}^{\alpha_1}\ldots\mathbf{Y}^{\alpha_l} + C_l^1 Q_{\alpha_1\ldots\alpha_{l-1}n}\mathbf{Y}^{\alpha_1}\ldots\mathbf{Y}^{\alpha_{l-1}}\mathbf{Y}^n + ..),$$

where C_l^k are Bernoulli coefficients.

Definition: Formal Gaussian beam (ForGb) *is introduced by formal expansion:*

$$U(\varepsilon; t, \boldsymbol{q}, \sigma) \asymp M_\varepsilon \exp\{-(i\varepsilon)^{-1}\Theta(t, \boldsymbol{q}, \sigma)\}\sum_{l=0}^{\infty}(i\varepsilon)^l u_l(t, \boldsymbol{q}, \sigma), \tag{0.16}$$

where

$$\Theta(t, \mathbf{q}, \sigma) \asymp \sum_{l\geq 1}\theta_l(t) = \sum_{|\delta|\geq 1}\frac{\theta_\delta(t)}{\delta!}(\xi - \xi(t))^\delta. \tag{0.17}$$

Definition: Gaussian beam $\mathcal{U}_N$ of order N *is a solution of the following problem*

$$\begin{cases} \partial_t^2\mathcal{U}_N - \Delta_g\mathcal{U}_N = 0, \ (t, \mathbf{q}, \sigma) \in [-t_0, t_0] \times \mathcal{M}, \\ \mathcal{U}_N(-t_0, \mathbf{q}, \sigma) = U_N(\varepsilon; -t_0, \mathbf{q}, \sigma), \\ \partial_t\mathcal{U}_N(-t_0, \mathbf{q}, \sigma) = \partial_t U_N(\varepsilon; -t_0, \mathbf{q}, \sigma), \\ \mathcal{U}_N(t, \mathbf{q}, \sigma)|_{\Gamma_0} = U_N(\varepsilon; t, \mathbf{q}, \sigma)|_{\Gamma_0}. \end{cases} \tag{0.18}$$

We should mention that

$$|\partial_t^j\partial_{\mathbf{x}}^\kappa\left(\mathcal{U}_N(t, \mathbf{q}, \sigma) - \chi(t, \mathbf{q}, \sigma)U_N(\varepsilon; t, \mathbf{q}, \sigma)\right)| \leq CM_\varepsilon\varepsilon^{N-(j+|\kappa|)}, \ \forall j \geq 0, \tag{0.19}$$

here κ is a multi-index, $\partial_{\mathbf{x}}^\kappa$ denotes partial derivation over n spacial coordinates $\mathbf{x} = \{\mathbf{q}^\alpha, \sigma\}$, χ is a characteristic function equal to one in the vicinity of $\xi = \xi(t)$ and equal to zero outside of this vicinity.

1.2.1 Continuity conditions and a solution form.

For the wave equation to have unique solution we should supply it with condition on the smooth interface γ between Ω_- and Ω_+ (see, for example, Popov, M., [6]):

$$\mathcal{U}^-|_\gamma = \mathcal{U}^+|_\gamma, \qquad \sqrt{g^-}\frac{\partial \mathcal{U}^-}{\partial \sigma}|_\gamma = \sqrt{g^+}\frac{\partial \mathcal{U}^+}{\partial \sigma}|_\gamma, \tag{0.20}$$

where $\mathcal{U}^\pm$ is the field value in Ω_-, and Ω_+. We seek a solution in the form of ***incident and reflected waves*** in the "first" medium Ω_- and ***transmitted wave*** in the "second" medium Ω_+. Denote these waves by U^{in}, U^{ref}, U^{tr} correspondingly. Recalling the representation of ForGb (0.16) and assuming that it incidents the interface at the origin of semi-geodesic coordinates M_1 at time $t = 0$, we write conditions:

$$U^{in}(\varepsilon; t, \mathbf{q}, 0) + U^{ref}(\varepsilon; t, \mathbf{q}, 0) \asymp U^{tr}(\varepsilon; t, \mathbf{q}, 0), \tag{0.21}$$

$$\sqrt{g^-}(\mathbf{q}, 0)\Big[\frac{\partial U^{in}}{\partial \sigma^-}(\varepsilon; t, \mathbf{q}, 0) + \frac{\partial U^{ref}}{\partial \sigma^-}(\varepsilon; t, \mathbf{q}, 0)\Big] \asymp \sqrt{g^+}(\mathbf{q}, 0)\frac{\partial U^{tr}}{\partial \sigma^+}(\varepsilon; t, \mathbf{q}, 0). \tag{0.22}$$

Thus, we write solution in the form:

$$U^+(\varepsilon; t, \mathbf{q}, \sigma) = U^{tr}(\varepsilon; t, \mathbf{q}, \sigma),\ U^-(\varepsilon; t, \mathbf{q}, \sigma) = U^{in}(\varepsilon; t, \mathbf{q}, \sigma) + U^{ref}(\varepsilon; t, \mathbf{q}, \sigma).$$

1.3 Main results.

We will construct three formal series $U^{in}(\varepsilon, \cdot)$, $U^{ref}(\varepsilon, \cdot)$, $U^{tr}(\varepsilon, \cdot)$ and assume that all three can be smoothly continued into another domain $\Omega_- \bigcup \Omega_+$. We truncate these series at order N and construct finite series $U_N^{in}(\varepsilon, \cdot)$, $U_N^{ref}(\varepsilon, \cdot)$, $U_N^{tr}(\varepsilon, \cdot)$ in Ω_- and Ω_+. We will show that these solutions are close to the required solutions to the wave equation (Gaussian beams) in the H^1-norm. Let U^{in} propagate in Ω_- along the geodesic $\mu^{in}(t) := (q_{in}^\alpha(t), \sigma_{in}(t))$ and reaches the point $M_1 = (q^\alpha, \sigma) = (0, 0)$ on the interface γ at time $t = 0$.

Main Theorem: *Let ForGb U^{in} (0.16) start movement at time $-t_0 < 0$ at the point $M_0 = (\boldsymbol{z}_0, \tau_0) \in \Gamma_0$ and reached the interface γ transversally at point $M_1 \in \gamma$ at time $t = 0$. Assume that*

$$0 \le g_+^{\alpha\beta} p_\alpha^{in} p_\beta^{in} < 1. \tag{0.23}$$

Then

- *for* $t > 0$ *the solution to the wave equation can be presented by a sum of two ForGb* U^{ref} *and* U^{tr}*; the wave* U^{ref} *reflects from the interface inside* Ω_- *and* U^{tr} *refracts from the interface into* Ω_+*. Both beams* U^{ref} *and* U^{tr} *can be constructed if the incident beam* U^{in} *is known. Constructed by that procedure ForGb have all properties to be considered as asymptotic approximations to the required exact solutions* $\mathcal{U}^{ref}$, $\mathcal{U}^{tr}$.

- *There are anisotropic analogues of Frenel's and Snell's geometric optics laws for the incident, reflected and transmitted beams. The reflection and transmission angles can be represented in terms of the incidence angle.*

- *For any* N *one can construct the exact solution - a Gaussian beam. This solution* $\mathcal{U}^{in}$ *satisfies equation* $\Box\mathcal{U}^{in} = 0$ *and differs from constructed FGb* $U_N^{in} + U_N^{ref}$ *in* Ω_- *and* U_N^{tr} *in* Ω_+ *by function which is small enough with sufficient number of its derivatives. In other words, for any* N *there exist constants* $p(N)$ *and* $\widetilde{C}_N$ *such that the difference between the exact solution to the wave equation* $\mathcal{U}_N$ *and FGb is*
$$\|\mathcal{U}_N - \chi U_N\|_{C^{p(N)}([-t_0,t_0];\mathcal{D}(\Delta_g))} \leq \widetilde{C}_N \varepsilon^{p(N)}.$$

The first and the second statements of the theorem will be proved constructively in sections 2, 3, the third statement will be proved in section 4.

Remark: As $g^-_{\alpha\beta}(\mathbf{q},0) \neq g^+_{\alpha\beta}(\mathbf{q},0)$, the order of transmitted wave coincides with the order of incident wave. In the case of continuous metric tensor has discontinuous derivative(s), the order of the transmitted field is weaker, but this investigation is beyond the scope of this paper.

Remark: In the case when the condition (0.23) fails we have total internal reflection. The boundary case $g_+^{\alpha\beta} p_\alpha^{in} p_\beta^{in} = 1$ corresponds to the tangential to the interface direction of the transmitted wave U^{tr} propagation, we exclude this case from our considerations because the ray expansions fail to be valid as the interface becomes characteristic.

1.4 Formal series.

We follow procedures introduced in papers by Babich V., Ulin V., [1], Katchalov A., [3]. Consider the wave operator $\Box_g$ (0.4) applied to FGb $U_N(\varepsilon;\cdot)$. For the series to satisfy wave equation formally the eikonal Θ_N and u_l should be the solutions of Hamilton-Jacobi equations and transport equations correspondingly. We omit captions "in,ref,tr" as the following is valid for all three waves. We write out Hamilton-Jacobi equations for the terms of amplitude expansion:

$$[(\frac{\partial \Theta_N}{\partial t})^2 - g^{\alpha\beta}\frac{\partial \Theta_N}{\partial q^\alpha}\frac{\partial \Theta_N}{\partial q^\beta} - (\frac{\partial \Theta_N}{\partial \sigma})^2] \asymp 0, \tag{0.24}$$

thus, when they are formally satisfied for Θ_N, we write out equations for the amplitude functions. The operator $\mathcal{L}_{\Theta_N}$ is called ***the transport operator***:

$$\mathcal{L}_{\Theta_N} u = 2\frac{\partial \Theta_N}{\partial t}\frac{\partial u}{\partial t} - 2g^{\alpha\beta}\frac{\partial \Theta_N}{\partial q^\alpha}\frac{\partial u}{\partial q^\beta} - 2\frac{\partial \Theta_N}{\partial \sigma}\frac{\partial u}{\partial \sigma} + (\Box\Theta_N)\cdot u. \tag{0.25}$$

The first approximation to the amplitude functions follows from

$$[(\frac{\partial \Theta_N}{\partial t})^2 - g^{\alpha\beta}\frac{\partial \Theta_N}{\partial q^\alpha}\frac{\partial \Theta_N}{\partial q^\beta} - (\frac{\partial \Theta_N}{\partial \sigma})^2]u_0(t,q,\sigma) \asymp 0. \tag{0.26}$$

Equations for the next amplitude approximations, $l = 1, ..., N$ take forms of transport equations:

$$\mathcal{L}_{\Theta_N} u_l \asymp \Box_g u_{l-1},\ \ l = 0,..,N,\ \ u_{-2}(t,q,\sigma) \equiv u_{-1}(t,q,\sigma) \equiv 0. \tag{0.27}$$

2 Phase functions

We construct $\Theta_N^{ref}(t,\mathbf{q},\sigma)$, $\Theta_N^{tr}(t,\mathbf{q},\sigma)$, assuming that they have form (0.12) in this section. Firstly, we will study phase function of the incident field $\Theta_N^{in}(t,\mathbf{q},\sigma)$. Secondly, we will show how one can obtain FGb $\Theta_N^{ref}(t,\cdot)$, $\Theta_N^{tr}(t,\cdot)$ as a series of finite homogeneous polynomials in the terms of the incident field. We will investigate the differential equations, that are satisfied by the terms of those series.

2.1 Main equations.

Consider FGb $U_N^{in}(t,\mathbf{q},\sigma)$, propagating to the interface γ along the geodesic $\mu^{in}(t) := (q^\alpha_{in}(t), \sigma_{in}(t))$. Following the procedure introduced in, for instance, [5], consider $\Box_g U_N^{in}(t,\mathbf{q},\sigma)$.

We have already seen that the phase function must satisfy Hamilton-Jacobi equation (0.24). By construction, the waves must propagate with unit velocity, hence the ***eikonal equation*** for Θ_N^{in} :

$$g_-^{\alpha\beta}\frac{\partial\Theta_N^{in}}{\partial q^\alpha}\frac{\partial\Theta_N^{in}}{\partial q^\beta}+(\frac{\partial\Theta_N^{in}}{\partial\sigma^-})^2\asymp 1. \tag{0.28}$$

Introduce new notation and rewrite (0.12)(or (0.17)):

$$\Theta_N^{in}(t,\mathbf{q},\sigma)=\left(P^{in}(t)^t,\mathbf{Y}^{in}(t)\right)+\frac{1}{2}\left(H^{in}(t)\mathbf{Y}^{in}(t),\mathbf{Y}^{in}(t)\right)+...\,, \tag{0.29}$$

$$P^{in}(t)=\left(\begin{array}{cccc} p_1^{in}(t) & ... & p_{n-1}^{in}(t) & p_n^{in}(t)\end{array}\right)^t=\left(\begin{array}{c} p_\alpha^{in}(t)\\ p_n^{in}(t)\end{array}\right), \tag{0.30}$$

$$\mathbf{Y}^{in}(t)=\xi^{in}-\xi^{in}(t)=\left(\begin{array}{c} q^1-q_{in}^1(t)\\ \cdot\\ q^{n-1}-q_{in}^{n-1}(t)\\ \sigma-\sigma_{in}(t)\end{array}\right)=\left(\begin{array}{c} q^\alpha-q_{in}^\alpha(t)\\ \sigma-\sigma_{in}(t)\end{array}\right), \tag{0.31}$$

$$H^{in}(t)=\left(\begin{array}{cc} H_{\alpha\beta}^{in}(t) & H_{\alpha n}^{in}(t)\\ H_{n\beta}^{in}(t) & H_{nn}^{in}(t)\end{array}\right), \tag{0.32}$$

where $\xi(t)$ was introduced in (0.11). Continue standard procedure of Gaussian beams construction (see, for instance, [1], [3]), the eikonal equation (0.28) implies the equation for all terms in expansion (0.12), including Hamilton-Jacobi system of equations (0.36) for impulses (0.35) and Riccati equation (0.37) for the quadratic forms. Denote the ***hamiltonian*** of the system by h:

$$h^{in}(\mathbf{q},\sigma;\mathbf{p})=(\mathbf{p},\mathbf{p})_{g_-}^{1/2}=\sqrt{g_-^{\alpha\beta}p_\alpha^{in}(t)p_\beta^{in}(t)+(p_n^{in}(t))^2}\equiv 1, \tag{0.33}$$

$$(h^{in})^2=g_-^{\alpha\beta}\frac{\partial\Theta^{in}}{\partial q^\alpha}\frac{\partial\Theta^{in}}{\partial q^\beta}+(\frac{\partial\Theta^{in}}{\partial\sigma})^2=g_-^{\alpha\beta}p_\alpha^{in}p_\beta^{in}+(p_n^{in})^2=(\frac{\partial\Theta^{in}}{\partial t})^2, \tag{0.34}$$

then ***the impulse equation*** is

$$g_-^{\alpha\beta}p_\alpha^{in}(t)p_\beta^{in}(t)+(p_n^{in}(t))^2\equiv 1. \tag{0.35}$$

Hamilton equations (canonical equations) are:

$$\begin{cases}\dot q_{in}^\alpha(t)=\frac{\partial h^{in}}{\partial p_\alpha}=\frac{1}{h^{in}}g_-^{\alpha\beta}p_\beta^{in}(t),\\ \dot\sigma_{in}(t)=\frac{\partial h^{in}}{\partial p_n}=\frac{1}{h^{in}}p_n^{in}(t),\end{cases}\qquad \begin{cases}\dot p_\gamma^{in}(t)=-\frac{\partial h^{in}}{\partial q^\gamma}=-\frac{1}{2h^{in}}p_\alpha^{in}(t)p_\beta^{in}(t)\frac{\partial g_-^{\alpha\beta}}{\partial q^\gamma},\\ \dot p_n^{in}(t)=-\frac{\partial h^{in}}{\partial\sigma}=-\frac{1}{2h^{in}}p_\alpha^{in}(t)p_\beta^{in}(t)\frac{\partial g_-^{\alpha\beta}}{\partial\sigma}.\end{cases} \tag{0.36}$$

The solution of this system is a bi-characteristic $(q^\alpha_{in}(t), \sigma_{in}(t); p^{in}_\alpha(t), p^{in}_n(t))$. The third equation which is satisfied by the quadratic form of the Gaussian beam is ***the Riccati equation***:

$$\frac{d}{dt} H^{in} + D^{in} + (B^{in} H^{in} + H^{in} (B^{in})^t) + H^{in} C^{in} H^{in} = 0, \tag{0.37}$$

where the coefficients $(B^{in})^t = B^{in*}$, $(C^{in})^t = C^{in*} = C$, $(D^{in})^t = D^{in*} = D$ are $n \times n$ matrices of second derivatives of hamiltonian, taken at point $(\mathbf{q}, \sigma; \mathbf{p}) = (\mathbf{q}_{in}(t), \sigma_{in}(t); \mathbf{p}^{in}(t))$ on the bi-characteristic:

$$D^{in} = \begin{pmatrix} D^{in}_{\alpha\beta} & D^{in}_{\alpha n} \\ D^{in}_{n\beta} & D^{in}_{nn} \end{pmatrix} = \begin{pmatrix} \frac{\partial^2 h^{in}}{\partial q^\alpha \partial q^\beta} & \frac{\partial^2 h^{in}}{\partial q^\alpha \partial \sigma} \\ \frac{\partial^2 h^{in}}{\partial \sigma \partial q^\beta} & \frac{\partial^2 h^{in}}{\partial \sigma^2} \end{pmatrix}, \tag{0.38}$$

$$B^{in} = \begin{pmatrix} B^{\beta in}_\alpha & B^{n in}_\alpha \\ B^{\beta in}_n & B^{n in}_n \end{pmatrix} = \begin{pmatrix} \frac{\partial^2 h^{in}}{\partial q^\alpha \partial p_\beta} & \frac{\partial^2 h^{in}}{\partial q^\alpha \partial p_n} \\ \frac{\partial^2 h^{in}}{\partial q^\beta \partial p_n} & \frac{\partial^2 h^{in}}{\partial \sigma \partial p_n} \end{pmatrix}, \tag{0.39}$$

$$C^{in} = \begin{pmatrix} C^{\alpha\beta in} & C^{\alpha n in} \\ C^{n\beta in} & C^{n n in} \end{pmatrix} = \begin{pmatrix} \frac{\partial^2 h^{in}}{\partial p_\alpha \partial p_\beta} & \frac{\partial^2 h^{in}}{\partial p_\alpha \partial p_n} \\ \frac{\partial^2 h^{in}}{\partial p_n \partial p_\beta} & \frac{\partial^2 h^{in}}{\partial p_n^2} \end{pmatrix}. \tag{0.40}$$

Our next step is to obtain equations for the higher order terms. Note that all constructed equations are recurrent because each time the higher order terms vanish along bi-characteristics. The homogeneous term θ^{in}_l of the order l in the Tailor's expansion (0.24) depends on θ^{in}_m, where $m \le l + 1$. Recall that terms containing θ^{in}_{l+1} equal zero along bi-characteristics. Finally the obtained differential equations for the homogeneous polynomials θ^{in}_l, $l \ge 3$ are linear (see [5]),

$$\frac{\partial \theta^{in}_l}{\partial t} + \mathcal{N}^{(in)i}_j \frac{\partial \theta^{in}_l}{\partial (\mathbf{Y}^{in})^i} (\mathbf{Y}^{in})^j = \mathcal{F}_l(\theta^{in}_m), \quad l = 3, 4, .. \tag{0.41}$$

где $m < l$. The components $\mathcal{N}^{(tr)i}_j$ form $n \times n$ matrix:

$$\mathcal{N}^{(in)i}_j(t) = \frac{\partial^2 h^{in}}{\partial x^i \partial p_j} + \frac{\partial^2 h^{in}}{\partial p_i \partial p_k} H^{in}_{kj} = [B^t_{in} + C_{in} H_{in}]^i_j, \tag{0.42}$$

where matrices B^{in}, C^{in} were determined above in (0.39)-(0.40).

Remark: The obtained equations (0.41) can be reduced to the linear ODE with respect to t for $\theta^{in}_l(t)$. They require initial data to be given for the uniqueness of their solutions. Note once again that those equations obtained for the incident field can be written formally for the reflected and for the transmitted fields, as we used only general properties of Gaussian beams on obtaining them, as we equal coefficients of the homogeneous polynomials considered.

2.2 Preparation step.

Consider phase function expansion as a series (0.29) and re-expand it in the vicinity of $t=0$, i.e. we expand it into Tailor's series $P^{in}(t)$ (0.30), $H^{in}(t)$ (0.32) and $\mathbf{Y}^{in}(t)$ (0.31) and construct new forms, now with respect to $n+1$ variables $t, \mathbf{q}, \sigma$. Denote these new forms by $\widetilde{P}^{in}$, $\widetilde{H}^{in}$, i.e. tilde above the notation means it is a coefficient of the new form, obtained as a result of the expansion:

$$\mathbf{Y}^{in}(t) \asymp^2 \begin{pmatrix} q^\alpha - \dot{q}^\alpha_{in}(0)t - \frac{1}{2}\ddot{q}^\alpha_{in}(0)t^2 \\ \sigma - \dot{\sigma}_{in}(0)t - \frac{1}{2}\ddot{\sigma}_{in}(0)t^2 \end{pmatrix},$$

$$\begin{aligned} \Theta^{in}_N(t,\mathbf{q},\sigma) \asymp^2 \widetilde{p}_0 t + \widetilde{p}^{in}_\alpha q^\alpha + \widetilde{p}^{in}_n \sigma + \frac{1}{2}\widetilde{H}^{in}_{00} t^2 + \\ + \frac{1}{2}\widetilde{H}^{in}_{\alpha\beta} q^\alpha q^\beta + \frac{1}{2}\widetilde{H}^{in}_{nn}\sigma^2 + \widetilde{H}^{in}_{0\alpha} t q^\alpha + \widetilde{H}^{in}_{0n} t\sigma + \widetilde{H}^{in}_{\alpha n} q^\alpha \sigma = \\ = ((\widetilde{\mathbf{P}}^{in})^t, \begin{pmatrix} t \\ q^\alpha \\ \sigma \end{pmatrix}) + \frac{1}{2}(\widetilde{\mathbf{H}}^{in}\begin{pmatrix} t \\ q^\alpha \\ \sigma \end{pmatrix}, \begin{pmatrix} t \\ q^\alpha \\ \sigma \end{pmatrix}), \end{aligned} \tag{0.43}$$

where

$$\widetilde{\mathbf{H}}^{in} = \begin{pmatrix} \widetilde{H}^{in}_{00} & \widetilde{H}^{in}_{0\beta} & \widetilde{H}^{in}_{0n} \\ \widetilde{H}^{in}_{\alpha 0} & \widetilde{H}^{in}_{\alpha\beta} & \widetilde{H}^{in}_{\alpha n} \\ \widetilde{H}^{in}_{n0} & \widetilde{H}^{in}_{n\beta} & \widetilde{H}^{in}_{nn} \end{pmatrix}, \widetilde{\mathbf{P}}^{in} = \begin{pmatrix} \widetilde{p}^{in}_0 \\ \widetilde{p}^{in}_\alpha \\ \widetilde{p}^{in}_n \end{pmatrix} = \begin{pmatrix} -1 \\ p^{in}_\alpha(0) \\ p^{in}_n(0) \end{pmatrix}, \tag{0.44}$$

$$\begin{cases} \frac{1}{2}\widetilde{H}^{in}_{00} = -\dot{p}^{in}_\alpha(0)\dot{q}^\alpha_{in}(0) - \dot{p}^{in}_n(0)\dot{\sigma}_{in}(0) - \frac{1}{2}(p^{in}_\alpha(0)\ddot{q}^\alpha_{in}(0) + p^{in}_n(0)\ddot{\sigma}_{in}(0)) \\ \quad + \frac{1}{2}H^{in}_{nn}(0)(\dot{\sigma}_{in}(0))^2 + \frac{1}{2}H^{in}_{\alpha\beta}(0)\dot{q}^\alpha_{in}(0)\dot{q}^\beta_{in}(0) + H^{in}_{n\alpha}(0)\dot{q}^\alpha_{in}(0)\dot{\sigma}_{in}(0), \\ \widetilde{H}^{in}_{\alpha\beta} = H^{in}_{\alpha\beta}(0), \\ \widetilde{H}^{in}_{nn} = H^{in}_{nn}(0), \\ \widetilde{H}^{in}_{0\alpha} = \dot{p}^{in}_\alpha(0) - H^{in}_{\alpha\beta}(0)\dot{q}^\beta_{in}(0) - H^{in}_{\alpha n}(0)\dot{\sigma}_{in}(0), \\ \widetilde{H}^{in}_{n0} = \dot{p}^{in}_n(0) - H^{in}_{nn}(0)\dot{\sigma}_{in}(0) - H^{in}_{\alpha n}(0)\dot{q}^\alpha_{in}(0), \\ \widetilde{H}^{in}_{\alpha n} = H^{in}_{\alpha n}(0) \end{cases}. \tag{0.45}$$

The obtained expansion coefficients will be helpful in finding the initial data for the quadratic form. Note that similar expansions can be presented also for the reflected and transmitted fields.

2.3 Impulses $\mathbf{p}^{ref}$, $\mathbf{p}^{tr}$ construction.

The goal of this section is to construct the first linear terms in expansion of the reflected and transmitted phase functions Θ_N^{ref} and Θ_N^{tr}, i.e. to construct impulses $P^{ref}(t)$, $P^{tr}(t)$ (similar to (0.30)). We will find impulses by given initial point and unit velocity in metric.

2.3.1 Impulses initial data.

Consider expansion (0.14) with respect to $t, \mathbf{q}, \sigma$ (we omit similar expansion $\theta_1^{tr}(t)$ as it has similar form):

$$\theta_1^{ref}(t,\mathbf{q},\sigma) = p_\alpha^{ref}(0)q^\alpha + p_n^{ref}(0)\sigma - p_\alpha^{ref}(0)\dot{q}_{ref}^\alpha(0)t - p_n^{ref}(0)\dot{\sigma}_{ref}(0)t. \tag{0.46}$$

We want to find initial data using interface data (0.21) to this end, the first continuity condition implies equality

$$\theta_1^{in}(t,\mathbf{q},0) = \theta_1^{ref}(t,\mathbf{q},0) = \theta_1^{tr}(t,\mathbf{q},0), \tag{0.47}$$

hence it is clear that tangential components of impulses (corresponding to γ) are equal to

$$p_\alpha^{in}(t) = p_\alpha^{ref}(t) = p_\alpha^{tr}(t), \ \Rightarrow p_\alpha^{in}(0) = p_\alpha^{ref}(0) = p_\alpha^{tr}(0). \tag{0.48}$$

Then assuming that the value of the incident field of the Gaussian beam is given at $t = 0$, i.e. assuming that we know $P^{in}(0)$ and recalling that the velocity is unit, i.e. $|P^{tr,ref}(t)| = 1$ (impulse satisfies eikonal equation (0.28) for any t, thus for $t = 0$), we write

$$|P^{tr,ref}(0)| = h(0,0;P^{tr,ref}(0)) = \pm\sqrt{g_\pm^{\alpha\beta}(0,0)p_\alpha^{in}(0)p_\beta^{in}(0) + (p_n^{tr,ref}(0))^2} = 1.$$

Then $p_n^{tr,ref}(0)$ can be determined up to sign of the square root. As the transmitted Gaussian beam propagates inside Ω_+ we should take $\frac{\partial\theta^{tr}}{\partial\sigma^+} > 0$, i.e. $p_n^{tr}(0) > 0$, the reflected beam propagates inside Ω_-, thus $p_n^{in}(0) = -p_n^{ref}(0)$, i.e.

$$p_n^{tr}(0) = \sqrt{1 - g_+^{\alpha\beta}(0,0)p_\alpha^{in}(0)p_\beta^{in}(0)} > 0, \tag{0.49}$$

$$p_n^{ref}(0) = \sqrt{1 - g_-^{\alpha\beta}(0,0)p_\alpha^{in}(0)p_\beta^{in}(0)} = -p_n^{in}(0) > 0. \tag{0.50}$$

Remark: Condition $|p^{tr}| < 1$ guarantees that we do not have total internal reflection case.

2.3.2 Impulses construction.

We found initial values $P^{ref}(0)$ (and $P^{tr}(0)$). The initial point for $t = 0$ ($\mathbf{q}_{ref(tr)}(0)$, $\sigma_{ref(tr)}(0)) = (0,0)$ of the corresponding bi-characteristic $(\mathbf{q}_{ref(tr)}(t), \sigma_{ref(tr)}(t); \mathbf{p}_{ref(tr)}(t))$ is known, thus we can solve Hamilton system of equations (0.36). The corresponding hamiltonian is

$$(h^{ref(tr)})^2 = g_\pm^{\alpha\beta}(\mathbf{q},\sigma)p_\alpha^{ref(tr)}(t)p_\beta^{ref(tr)}(t) + (p_n^{ref(tr)}(t))^2 = 1.$$

2.4 Quadratic forms.

The goal of this section is to construct the second (quadratic) terms of the phase function expansion (0.29), namely, $H^{ref}(t,\cdot)$, $H^{tr}(t,\cdot)$ in assumption that $H^{in}(0)$ is given and that we have already constructed impulses. To this end we have to solve Riccati equation (0.37) after calculation initial data $H^{ref(tr)}(t=0)$.

2.4.1 Initial data $\theta_2^{ref}(0,\mathbf{q},\sigma)$, $\theta_2^{tr}(0,\mathbf{q},\sigma)$.

The goal of this subsection is to construct initial data with required properties using interface boundary data (given on γ), i.e. given $\widetilde{H}_{0\alpha}^{in}$, $\widetilde{H}_{\alpha\beta}^{in}$, $\widetilde{H}_{00}^{in}$, and assuming that we have already constructed impulses and found geodesics we have to express $H_{\alpha\beta}^{ref,tr}(t=0)$, $H_{\alpha n}^{ref,tr}(t=0)$, $H_{nn}^{ref,tr}(t=0)$ in terms of known values. This time we also use continuity conditions (0.21) recalling that $\Theta^{in}|_\gamma = \Theta^{tr}|_\gamma = \Theta^{ref}|_\gamma$. The latter we rewrite as

$$\widetilde{H}_{\alpha\beta}^{in} = \widetilde{H}_{\alpha\beta}^{ref} = \widetilde{H}_{\alpha\beta}^{tr} = H_{\alpha\beta}^{in}(0) = H_{\alpha\beta}^{ref}(0) = H_{\alpha\beta}^{tr}(0). \tag{0.51}$$

Then we get on the interface $\sigma = 0$ (0.43)

$$-t + p_\alpha^{in}q^\alpha + \frac{1}{2}\widetilde{H}_{00}^{in}t^2 + \frac{1}{2}\widetilde{H}_{\alpha\beta}^{in}q^\alpha q^\beta + \widetilde{H}_{0\alpha}^{in}tq^\alpha + \ldots =$$
$$-t + p_\alpha^{ref}q^\alpha + \frac{1}{2}\widetilde{H}_{00}^{ref}t^2 + \frac{1}{2}\widetilde{H}_{\alpha\beta}^{ref}q^\alpha q^\beta + \widetilde{H}_{0\alpha}^{ref}tq^\alpha + \ldots.$$

Consequently,

$$\begin{cases} \widetilde{H}^{in}_{00} = \widetilde{H}^{ref}_{00} = \widetilde{H}^{tr}_{00} \\ \widetilde{H}^{in}_{0\alpha} = \widetilde{H}^{ref}_{0\alpha} = \widetilde{H}^{tr}_{0\alpha} \end{cases} \tag{0.52}$$

Now we substitute coefficients (0.45) and similar expression for H^{ref} and H^{tr} into the second equation (0.52). Thus we get the representations for $H^{ref,tr}_{\alpha n}(0)$ in terms of the given incident field and known derivatives of the geodesics:

$$H^{ref}_{\alpha n}(0) = -H^{in}_{\alpha n}(0). \tag{0.53}$$

We will not present here formulae for reflected and transmitted quadratic forms at time zero as they are massive, we present only expression of their imaginary parts as we will need them later. We obtain following for the reflected field

$$\operatorname{Im} H^{tr}_{\alpha n}(0)\dot{\sigma}_{tr}(0) = -\operatorname{Im} H^{in}_{\alpha n}(0)\dot{\sigma}_{in}(0) - \operatorname{Im} H^{in}_{\alpha\beta}(0)[\dot{q}^{\alpha}_{in}(0) - \dot{q}^{\alpha}_{tr}(0)]. \tag{0.54}$$

Similarly $H^{ref}_{nn}(t=0)$:

$$\operatorname{Im}(H^{ref}_{nn}(0)(\dot{\sigma}_{ref}(0))^2) = \operatorname{Im}(H^{in}_{nn}(0)(\dot{\sigma}_{in}(0))^2), \tag{0.55}$$

$$\operatorname{Im}(H^{tr}_{nn}(0)(\dot{\sigma}_{tr}(0))^2) = \tag{0.56}$$

$$\operatorname{Im}\left(H^{in}_{nn}(0)(\dot{\sigma}_{in}(0))^2 + H^{in}_{\alpha\beta}(0)\left[\dot{q}^{\alpha}_{tr}(0)\dot{q}^{\beta}_{tr}(0) - \dot{q}^{\alpha}_{in}(0)\dot{q}^{\beta}_{in}(0)\right]\right)$$
$$-2\operatorname{Im}\left(H^{tr}_{n\alpha}(0)\dot{q}^{\alpha}_{tr}(0)\dot{\sigma}_{tr}(0) + H^{in}_{n\alpha}(0)\dot{q}^{\alpha}_{in}(0)\dot{\sigma}_{in}(0)\right).$$

Thus, we found initial data for the reflected and transmitted quadratic forms, using only given incident field and recently constructed impulses.

2.4.2 Initial boundary-value problem.

As we have already constructed initial data $H^{ref}(0)$, $H^{tr}(0)$, we can start solving Riccati equation (0.37). If we succeed to show that initial quadratic form $H(0) = H_0$ satisfies next lemma conditions, then the solution to the Riccati equation is required quadratic form of Gaussian beam. The next lemma is lemma 2.56 from [5] (see also Babich V., Ulin V., [1], or Babich V., Buldyrev V., Molotkov I., [2], Katchalov A., [3]):

Lemma 2.56 *Let H_0 be $n \times n$ complex-valued matrix such that*

$$H_0 = H_0^t, \tag{0.57}$$

$$\operatorname{Im} H_0 > 0. \tag{0.58}$$

Then (i) the initial boundary-value problem for the Riccati equation (0.37) with initial values

$$H|_{t=0} = H_0 \tag{0.59}$$

has a unique solution $H(t)$, $t \in \mathbf{R}$. *The derivatives are calculated at the point* $(\boldsymbol{q}, \sigma; \boldsymbol{p}) = (\boldsymbol{q}(t), \sigma(t); \boldsymbol{p}(t))$, *i.e. on the bi-characteristic, which is a solution to the Hamilton equation (0.36).*

(ii) The solution $H(t)$, $t \in \mathbf{R}$ *is symmetric* $H(t) = H(t)^t$, *and* $\operatorname{Im} H(t) > 0$.

(iii) Besides that for any Y_0, Z_0 *such that* $H_0 = Z_0 Y_0^{-1}$, *matrix* $H(t)$ *can be represented in the form* $H(t) = Z(t)Y(t)^{-1}$. *The pair of matrices* $(Z(t), Y(t))$ *is a solution to the initial boundary-value problem,*

$$\frac{d}{dt}Y(t) = B^t \cdot Y + C \cdot Z, \quad Y|_{t=0} = Y_0, \tag{0.60}$$

$$\frac{d}{dt}Z(t) = -D \cdot Y - B \cdot Z, \quad Z|_{t=0} = Z_0, \tag{0.61}$$

where matrix $Y(t)$ *is non-degenerate for any* $t \in \mathbf{R}$, $\det Y(t) \neq 0$.

Lemma *The determinant* $\det(\operatorname{Im} H(t)) \cdot |\det Y(t)|^2$ *is constant for any Gaussian beam.*

Both lemmas are proved in [5].

To use lemma 2.56 results we need to show that obtained $H^{ref}(0)$, $H^{tr}(0)$ satisfy its requirements (they should be symmetric and positive-definite).

Statement *Let us assume that we know that* $\operatorname{Im} \theta_2^{in}(t, q, 0) > 0$ *and let* $\theta_2^{in}(t, \boldsymbol{q}, 0)$ *be symmetric then* $\operatorname{Im} \theta_2^{ref,tr}(0, \boldsymbol{q}, \sigma) > 0$ *and* $\theta_2^{ref,tr}(0, \boldsymbol{q}, \sigma)$ *are also symmetric.*

We start statement proof with showing the symmetry of $H^{ref,tr}(0)$. In fact we cannot state that homogeneous polynomials $\theta_2^{ref,tr}(0, \mathbf{q}, \sigma)$ are determined by symmetric matrices $H^{ref,tr}(0)$, but we can always choose uniquely symmetric tensor which gives birth to the required polynomial $\theta^{ref,tr}(0, q, \sigma)$. Let us prove the second part of the statement. Assume we know that imaginary part $\operatorname{Im} H^{ref,tr}(t) > 0$, (0.32) for any t i.e.

$$\left(\begin{pmatrix} \operatorname{Im} H^{in}_{\alpha\beta}(0) & \operatorname{Im} H^{in}_{\alpha n}(0) \\ \operatorname{Im} H^{in}_{n\beta} & \operatorname{Im} H^{in}_{nn}(0) \end{pmatrix} \begin{pmatrix} q^\alpha \\ \sigma \end{pmatrix}, \begin{pmatrix} q^\beta \\ \sigma \end{pmatrix} \right) > 0.$$

Necessary and sufficient condition of positive definiteness by Silvester criteria is positiveness of all main minors. Note that all main minors of matrices $\operatorname{Im} H^{in,ref,tr}(0)$ excluding determinant are equal. Thus in order to show that $\operatorname{Im} H^{ref,tr}(0)$ is positive-definite one have to investigate the positiveness of their determinants only. Consider firstly the reflected quadratic form

$$\det\begin{pmatrix} \operatorname{Im} H^{ref}_{\alpha\beta}(0) & \operatorname{Im} H^{ref}_{\alpha n}(0) \\ \operatorname{Im} H^{ref}_{n\beta}(0) & \operatorname{Im} H^{ref}_{nn}(0) \end{pmatrix} = \det\begin{pmatrix} \operatorname{Im} H^{in}_{\alpha\beta}(0) & -\operatorname{Im} H^{in}_{\alpha n}(0) \\ -\operatorname{Im} H^{in}_{n\beta}(0) & \operatorname{Im} H^{in}_{nn}(0) \end{pmatrix}.$$

As we see the determinants are equal, and thus we proved that $\operatorname{Im} H^{ref}(0) > 0$. Consider now $\operatorname{Im} H^{tr}(0)$. Obviously, multiplying firstly the last column and secondly the last row by constant $\dot\sigma_{in,tr}(0)$ in matrices $\operatorname{Im} H^{in}(0)$, $\operatorname{Im} H^{tr}(0)$ correspondingly ((0.54),(0.56)), the sign of determinant does not change. We work with matrix

$$\begin{pmatrix} \operatorname{Im} H^{in}_{\alpha\beta}(0) & \operatorname{Im} H^{in}_{\alpha n}(0)\dot\sigma_{in}(0) \\ \operatorname{Im} H^{tr}_{n\beta}(0)\dot\sigma_{in}(0) & \operatorname{Im} H^{tr}_{nn}(0)\dot\sigma^2_{in}(0) \end{pmatrix},$$

its determinant does not change, it is positive. Next we use linear transformations (they do not change the determinant) and get matrix $\operatorname{Im} H^{tr}(0)$ with components ((0.51), (0.54), (0.56)), this will prove its positive-definiteness. The noted above transformation are: we add the linear combination of all right rows with a factor $-[\dot q^\beta_{in}(0) - \dot q^\beta_{tr}(0)]$ to the last column, then the right column of the obtained matrix has components $-\operatorname{Im} H^{in}_{\alpha n}\dot\sigma_{in}(0) - [\dot q^\beta_{in}(0) - \dot q^\beta_{tr}(0)]\operatorname{Im} H^{in}_{\alpha\beta}(0)$, which coincides with (0.54). The last component is $\operatorname{Im} H^{in}_{nn}(0)\dot\sigma_{in}(0) - \operatorname{Im} H^{in}_{n\beta}(0)\dot\sigma_{in}(0)[\dot q^\beta_{in}(0) - \dot q^\beta_{tr}(0)]$. Now we add linear combination of all upper rows with factor $-[\dot q^\alpha_{in}(0) - \dot q^\alpha_{tr}(0)]$ to the lowest row. The first $(n-1)$ components of the lowest row coincide now with (0.54), the last component coincides with (0.56), this proves positive definiteness of matrix $\operatorname{Im} H^{tr}(0) > 0$.

We proved the statement and now we can construct solutions to the Riccati equation with required properties

$$\begin{cases} H^{ref}(t) = Z^{ref}(t)(Y^{ref}(t))^{-1}, \\ H^{tr}(t) = Z^{tr}(t)(Y^{tr}(t))^{-1}. \end{cases} \tag{0.62}$$

Now ForGb (0.16) is concentrated in the vicinity of the point and propagates along the geodesic $(p^{ref,tr}(t); q^{ref,tr}(t), \sigma^{ref,tr}(t))$ with unit velocity on the manifold.

2.5 Higher order polynomials.

We use similar procedure to obtain homogeneous polynomials $\theta_l^{ref}(t, \mathbf{q}, \sigma)$, $\theta_l^{tr}(t, \mathbf{q}, \sigma)$, $l \geq 3$. Firstly, we introduce schematically recurrent procedure on finding the initial data. Next we present differential equation to find these data. Supplying these equations with initial data they can be solved uniquely.

2.5.1 Initial data $\theta_l^{ref}(0, \mathbf{q}, \sigma)$, $\theta_l^{tr}(0, \mathbf{q}, \sigma)$.

We will not present here any formulae for the initial data finding as they all are too massive, we present here only basic ideas of the method. Let us fix $l \geq 3$, and let us assume that we found all $\theta_l^{ref,tr}(t, \mathbf{q}, \sigma)$ of all lower orders. Again we assume the incident field to be known. Consider formal series (similarly to what we did for $l = 1, 2$ (0.14), (0.15)). Next we consider Tailor's expansions of these series in the vicinity of $t = 0$ (similarly to (0.43)). Continuity conditions on the interface imply the coefficients of forms $\theta_l^{ref,tr}(t, \mathbf{q}, \sigma)$ at $t = 0$ can be expressed in terms of coefficients of the incident field $\theta_l^{ref,tr}(t, \mathbf{q}, \sigma)$ of orders less than l. Indeed, the obtained by this procedure system of equations is triangular, i.e. all the required coefficients can be obtained consequently. That means that we can start by equaling coefficients of monomials that do not contain σ and contain combinations of l tangential and spacial variables. Those coefficients equal directly similar known coefficients of the incident field. Going on with this procedure we equal the next group of coefficients of monomials, that contain only one σ and $l - 1$ tangential and spacial variables, the coefficients that are obtained on the previous step, known incident field coefficients and polynomials of the lower orders already obtained before.

2.5.2 Higher order terms equations.

Assume once again that we know $\theta_1^{ref,tr}(t), .., \theta_{l-1}^{ref,tr}(t)$ and initial data $\theta_l^{ref,tr}(0)$. We write out here the initial boundary-value problem for $\theta_l^{ref,tr}(t)$. We follow the procedure suggested in, say, [5]. Recall that each term of the phase function should satisfy (0.41) for the initial data obtained above. To simplify (0.41) we use new coordinates

$$\hat{t} = t, \;\; \hat{y} = Y^{-1}(t)\mathbf{Y}(t), \tag{0.63}$$

where $Y(t)$ is defined in (0.62) and $\mathbf{Y}(t)$ is defined in (0.31) such that

$$\frac{\partial}{\partial \hat{t}} = \frac{\partial}{\partial t} + \left[(B)^t Y + CZ\right]^i_j \hat{y}^j \frac{\partial}{\partial \mathbf{Y}^i} = \frac{\partial}{\partial t} + \mathcal{N}^i_j \mathbf{Y}^j \frac{\partial}{\partial \mathbf{Y}^i},$$

where $\mathcal{N}^i_j$ are determined in (0.42). Let $\hat{\theta}_l(\hat{t}, \hat{y})$ be the representations of polynomials $\theta_l(t, \mathbf{Y})$ in the new coordinates with "tildes" then equations (0.41) take form

$$\frac{\partial}{\partial \widetilde{t}} \widetilde{\theta}_l = \widetilde{\mathcal{F}}_l, \;\; l = 3, 4, \dots .$$

Supplying these equations by initial data $\hat{\theta}_l(\hat{t}, \widetilde{y})|_{\hat{t}=0} = \theta_l(t, \mathbf{Y})|_{t=0}$ and $Y(0) = I$ one can find $\hat{\theta}_l(\hat{t}, \hat{y})$ for any $\hat{t}$, and, consequently, one can obtain $\theta_l(t, \mathbf{Y})$ for any t.

2.6 Phase functions $\Theta^{ref}_N(t, \mathbf{q}, \sigma)$, $\Theta^{tr}_N(t, \mathbf{q}, \sigma)$.

We presented the procedure of construction of any finite number of terms in the phase function expansion $\theta^{ref,tr}_1(t), \theta^{ref,tr}_2(t), ..., \theta^{ref,tr}_N(t)$. Let us write out lemma 2.61 conclusions form [5]. Suppose that Θ^{in}_N is given. Then we can construct $\Theta^{ref(tr)}_n$ such that constructed functions can be presented by series

$$\Theta^{ref(tr)}(t, \mathbf{q}, \sigma) \asymp^N \Theta^{ref(tr)}_N(t, \mathbf{q}, \sigma) = \sum_{l=1}^{N-1} \theta^{ref(tr)}_l(t) = \sum_{|\iota|=1}^{N-1} \frac{1}{\iota!} \theta^{ref(tr)}_\iota(t) \mathbf{Y}^\iota(t), \qquad (0.64)$$

and such that

$$\Theta^{ref(tr)}_1 = (P^{ref(tr)}(t), \mathbf{Y}(t)), \;\; |P^{ref(tr)}(t)| = 1$$

are real ($P^{ref(tr)}(t)$ are determined by (0.30), and $\mathbf{Y}(t)$ are determined by (0.31),

$$\Theta^{ref(tr)}_2 = \frac{1}{2}(H^{ref(tr)}(t)\mathbf{Y}(t), \mathbf{Y}(t)), \;\; \mathrm{Im}\, H(t) \geq 0,$$

$H^{ref,tr}(t)$ was constructed above. Moreover, the constructed phase functions satisfy conditions (0.8), (0.9) and estimates

$$|((\partial_t)^2 - g^{ij}\partial_i \partial_j)\Theta^{ref(tr)}_N| \leq C_N |\mathbf{Y}(t)|^N, \qquad (0.65)$$

that means that all requirements form these functions to be Gaussian beam phase functions are satisfied.

2.6.1 Reflection and transmission laws.

Consider cotangent bundle $T^*_{M_1\in\gamma}(\mathcal{M})$ and coplane $\pi^{in}(M_1)$ in it such that the point $M_1 = (\mathbf{0}, 0)$ is a point where the beam reaches the interface. The coplane $\pi^{in}(M_1)$ is a 2D coplane spanned by covector

$$P^{in}(0) = d\Theta^{in}|_\gamma(M_1) = (\frac{\partial\Theta^{in}}{\partial q^1}, ..., \frac{\partial\Theta^{in}}{\partial\sigma_-})|_{M_1}$$

and normal covector $d\sigma_- = (0, ..., 0, -1)$ at the point M_1 at time $t = 0$. ***An incidence angle*** $\varphi^-(M_1) \in \pi^{in}(M_1)$ in the cotangent bundle is an angle with which the incident wave U^{in} reaches the interface γ at time $t = 0$.

Let U^{tr} start from γ inside Ω_+ with a ***transmission angle*** φ^{tr} between the covector $P^{tr}(0) = d\Theta^{tr}|_\gamma(t = 0) = (\frac{\partial\Theta^{tr}}{\partial q^1}, ..., \frac{\partial\Theta^{tr}}{\partial\sigma_+})|_{t=0}$ and normal covector $d\sigma_+ = (0, ..., 0, 1)$, $\pi^{tr} \in T^*_{M_1\in\gamma}(\Omega_+)$, where the 2D coplane π^{tr} is spanned by covectors $d\sigma_+$ and $P^{tr}(0)$. The transmission angle φ^{tr} determine geodesic $\mu^{tr}(t)$.

Similarly we define ***a reflection angle*** φ^{ref} at the point M_1 between covectors $P^{ref}(0) = d\Theta^{ref}|_\gamma(t = 0) = (\frac{\partial\Theta^{ref}}{\partial q^1}, ..., \frac{\partial\Theta^{ref}}{\partial\sigma_-})|_{t=0}$ and $d\sigma_-$ in the coplane π^{ref}.

Proposition. *Coplanes π^{in},π^{ref},π^{tr} coincide. If the value φ^{in} of the angle is known then we can find the transmission angle φ^{tr} and the reflection angle φ^{ref}.*

Recalling continuity conditions and differentiating the first one in (0.21) with respect to q^α (we can not differentiate with respect to σ as this condition is valid only on γ) we obtain

$$\frac{\partial\Theta^{in}}{\partial q^\alpha}|_\gamma = \frac{\partial\Theta^{ref}}{\partial q^\alpha}|_\gamma = \frac{\partial\Theta^{tr}}{\partial q^\alpha}|_\gamma.$$

Eikonal equation (0.28) and coordinates normalization at the point M_1 imply

$$\frac{\partial\Theta^{in}_N}{\partial\sigma_-}(M_1) = \cos\varphi^{in} = p^{in}_n(0) < 0, \ \frac{\partial\Theta^{in}_N}{\partial\sigma_-}(M_1) = p^{ref}_n(0) = -p^{in}_n(0) > 0,$$

and

$$\sin\varphi^{in} = g^{\alpha\beta}_- \frac{\partial\Theta^{in}}{\partial q^\alpha}\frac{\partial\Theta^{in}}{\partial q^\beta}(M_1) = g^{\alpha\beta}_- p^{in}_\alpha(0)p^{in}_\beta(0) \le 1.$$

Consider two covectors $P^{in}(0)$, $d\sigma_-$ the coplane $\pi^{in}(M_1)$ is spanned by. Consider two covectors $P^{ref}(0)$ and $d\sigma_-$ that the coplane $\pi^{ref}(M_1)$ is spanned by. We have got the value $P^{ref}(0)$ above. One can see that all its tangential components coincide with

$P^{in}(0)$. Formula (0.49) implies that they have only the last component different. Thus the coplane $\pi^{ref}(M_1)$ coincides with the coplane $\pi^{in}(M_1)$, as the covector $P^{ref}(0)$ can be presented by linear combination $P^{in}(0)$ and $d\sigma_-$. Hence they belong to the same 2D coplane.

Similarly, we consider covectors $P^{in}(0)$ and $P^{tr}(0)$, they also differ in the last coordinate, as one can see that from (0.50). Compare now $d\sigma_-$ and $d\sigma_+$. Both covectors belong to the same straight line, we present $P^{tr}(0)$ in a form of linear combination of $P^{in}(0)$ and $d\sigma_-$. Thus coplanes $\pi^{in}(M_1)$ and $\pi^{tr}(M_1)$ coincide.

Introduce two covectors

$$b_- \in \pi^{in}(M_1) \bigcap T^*_{M_1\in\gamma}\gamma, \;\; b_+ \in \pi^{tr}(M_1) \bigcap T^*_{M_1\in\gamma}\gamma,$$

such that

$$(d\sigma_-, b_-)_{g^-} = 0, \;\; |b_-|_{g^-} = \sin\varphi^{in},$$

$$(d\sigma_+, b_+)_{g^+} = 0, \;\; |b_+|_{g^+} = \sin\varphi^{tr}.$$

Then in equal coplanes $\pi^{ref}(M_1)$ and $\pi^{in}(M_1)$ the following is true:

$$(d\Theta^{in}, d\sigma_-) = g_-^{\alpha\beta}(M_1)\frac{\partial\Theta^{in}}{\partial q^\alpha}b_\beta = \sin\varphi^{in} g_-^{\alpha\beta}(M_1)b_\alpha b_\beta = \sin\varphi^{ref} g_-^{\alpha\beta}(M_1)b_\alpha b_\beta$$

$$= (d\Theta^{ref}, d\sigma_-) = g_-^{\alpha\beta}(M_1)\frac{\partial\Theta^{ref}}{\partial q^\alpha}b_\beta.$$

Thus $\sin\varphi^{in} = \sin\varphi^{ref}$, and eikonal equation gives us a reflected cosine $\cos\varphi^{ref} = -\cos\varphi^{in}$. The latter corresponds to the Frenel's law of geometrical optics, i.e.

$$\begin{cases} \sin\varphi^{ref} = g_-^{\alpha\beta}\frac{\partial\Theta^{in}}{\partial q^\alpha}\frac{\partial\Theta^{in}}{\partial q^\beta}(M_1) = g_-^{\alpha\beta}(M_1)p_\alpha^{in}(0)p_\beta^{in}(0) = \sin\varphi^{in}, \\ \cos\varphi^{ref} = -\sqrt{1 - (g_-^{\alpha\beta}\frac{\partial\Theta^{in}}{\partial q^\alpha}\frac{\partial\Theta^{in}}{\partial q^\beta}(M_1))} = -\cos\varphi^{in}. \end{cases} \tag{0.66}$$

$$\frac{\partial\Theta^{in}}{\partial q^\alpha}|_\gamma = \sin\varphi^-\frac{b_-}{|b_-|_{g^-}} = \frac{\partial\Theta^{tr}}{\partial q^\alpha}|_\gamma = \sin\varphi^+\frac{b_+}{|b_+|_{g^+}}$$

The latter implies the formula for the cosine of the transmitted wave $\cos\varphi^{tr}$ expressed in terms of given function of the incident wave. For the Gaussian beam U^{tr} to propagate inside Ω_+ we should take the positive sign in the square root

$$\begin{cases} \cos\varphi^{tr} = +\sqrt{1 - \sin^2\varphi^{in}\frac{g_+^{\alpha\beta}b_\alpha b_\beta}{g_-^{\alpha\beta}b_\alpha b_\beta}} > 0 \\ \sin\varphi^{tr} = \sin\varphi^{in}\sqrt{\frac{g_+^{\alpha\beta}b_\alpha b_\beta}{g_-^{\alpha\beta}b_\alpha b_\beta}} \end{cases} \tag{0.67}$$

We found the values of the reflected and transmitted angles, determined by their cosines (0.66) and (0.67) in terms of the incident field.

Formula (0.67) rewritten in the form of sinuses ratio corresponds to the Snell's law. One can see that in the case of $\varphi^{tr} = \frac{\pi}{2}$, the angle φ^{in} becomes critical, the transmitted wave propagates in a tangential direction to the interface. In this case the ray expansions could not be used, but such a kind of propagation is beyond the scope of this work.

3 Amplitudes

3.1 Amplitude values on the interface

We rewrite interface conditions such that we are able to express boundary data of the reflected and transmitted waves in terms of incident field, as we did it for the phase function.

Lemma 2 *Let $u_N^{in}(t, \boldsymbol{q}, 0)$ be known trace of the incident amplitude function on the interface. We can find traces of the reflected and transmitted amplitudes $u_N^{ref}(t, \boldsymbol{q}, 0)$ and $u_N^{tr}(t, \boldsymbol{q}, 0)$ on the interface for any $N > 0$.*

We will prove this by giving the construction procedure.

3.1.1 Proof of Lemma 2

Consider continuity conditions (0.21) on the interface and substitute the formal Gaussian beam series into it. The first condition gives us the following equations

$$\begin{cases} \Theta^{in}(\mathbf{x}, t)|_\gamma \asymp \Theta^{ref}(\mathbf{x}, t)|_\gamma \asymp \Theta^{tr}(\mathbf{x}, t)|_\gamma, \\ u_k^{in}(\mathbf{x}, t)|_\gamma + u_k^{ref}(\mathbf{x}, t)|_\gamma = u_k^{tr}(\mathbf{x}, t)|_\gamma, \ \forall k. \end{cases} \tag{0.68}$$

The continuity condition for normal derivatives implies equations

$$(i\varepsilon)^{-1}\sum_{l=0}^{N}(i\varepsilon)^l \left(S^{in}u_l^{in} + S^{ref}u_l^{ref}\right)|_\gamma + \sqrt{g_-|_\gamma}\sum_{l=0}^{N}(i\varepsilon)^l \left(\frac{\partial u_l^{in}}{\partial \sigma_-} + \frac{\partial u_l^{ref}}{\partial \sigma_-}\right)|_\gamma = \tag{0.69}$$

$$= (i\varepsilon)^{-1}\sum_{l=0}^{N}(i\varepsilon)^l \left(S^{tr}u_l^{tr}\right)|_\gamma + \sqrt{g_+|_\gamma}\sum_{l=0}^{N}(i\varepsilon)^l \left(\frac{\partial u_l^{tr}}{\partial \sigma_+}\right)|_\gamma,$$

where we used some useful notations:

$$S^{in,ref} := \big(\sqrt{g_-(q,0)}\frac{\partial\Theta_N^{in,ref}}{\partial\sigma_-}(t,q,0)\big),\ S^{tr} := \big(\sqrt{g_+(q,0)}\frac{\partial\Theta_N^{tr}}{\partial\sigma_+}(t,q,0)\big).$$

Equating coefficients of similar orders of $(i\varepsilon)$, assuming that

$$u_0^{in,ref,tr} \not\equiv 0, \quad u_k^{in,ref,tr} \equiv 0,\ \forall k < 0,$$

we get the system of equations for $k = -1, 0, 1, ...$:

$$\mathbf{P}\begin{pmatrix} u_{k+1}^{tr}(t,q,0) \\ u_{k+1}^{ref}(t,q,0) \end{pmatrix} = \mathbf{Q}u_{k+1}^{in}(t,q,0) + \mathbf{R}_k,$$

where

$$\mathbf{P} := \begin{pmatrix} S^{tr} & -S^{ref} \\ -1 & 1 \end{pmatrix},\ \mathbf{Q} := \begin{pmatrix} S^{in} \\ -1 \end{pmatrix},\ \mathbf{R}_k := \begin{pmatrix} \mathcal{R}(u_k) \\ -1 \end{pmatrix}, \tag{0.70}$$

$$\mathcal{R}(u_k) := -\sqrt{g_-(q,0)}(\frac{\partial u_k^{in}}{\partial\sigma}(t,q,0) + \frac{\partial u_k^{ref}}{\partial\sigma}(t,q,0)) + \sqrt{g_+(q,0)}\frac{\partial u_k^{tr}}{\partial\sigma}(t,q,0). \tag{0.71}$$

We denote the determinant of the obtained system by

$$\mathcal{D} := (-S^{tr} + S^{ref}) \neq 0. \tag{0.72}$$

The obtained system for the polynomials on γ (0.70) has a unique solution for any RHS, as the determinant (0.72) is $\mathcal{D}_k \neq 0$ for any $k > 0$. We tackle the system of equations and obtain

$$\begin{pmatrix} u_{k+1}^{tr}(t,q,0) \\ u_{k+1}^{ref}(t,q,0) \end{pmatrix} = \mathbf{P}^{-1}\mathbf{Q}u_{k+1}^{in}(t,q,0) + \mathbf{P}^{-1}\mathbf{R}_k, \tag{0.73}$$

where

$$\mathbf{P}^{-1} := \frac{1}{\mathcal{D}}\begin{pmatrix} 1 & S^{ref} \\ 1 & S^{tr} \end{pmatrix}. \tag{0.74}$$

Thus we have found the recurrent formulae for any finite number of boundary values on γ for the reflected and transmitted amplitudes in terms of known incident field. Hence Lemma 2 is proved.

Additionally we present main terms $u_0^{ref}(t,q,0)$ and $u_0^{tr}(t,q,0)$ as following:

$$\begin{pmatrix} u_0^{tr}(t,q,0) \\ u_0^{ref}(t,q,0) \end{pmatrix} = \mathbf{P}^{-1}\mathbf{Q}u_0^{in}(t,q,0) = \begin{pmatrix} \mathcal{R} \\ \mathcal{T} \end{pmatrix} u_0^{in}(t,q,0), \tag{0.75}$$

where $\mathcal{R}$, $\mathcal{T}$ are reflection and transmission coefficients correspondingly, such that $1 + \mathcal{R} = \mathcal{T}$. Similarly,

$$\begin{pmatrix} u_1^{tr}(t,q,0) \\ u_1^{ref}(t,q,0) \end{pmatrix} = \mathbf{P}^{-1}\mathbf{Q}u_1^{in}(t,q,0) + \mathbf{P}^{-1}\mathbf{R}_0. \tag{0.76}$$

3.2 Amplitude equations

We write out ODE for the amplitude functions $u^{ref}(t,q,\sigma)$ and $u^{tr}(t,q,\sigma)$. To solve them uniquely we have to supply them with initial data. These initial data will be obtained later. The investigation of transport equations (0.27) on geodesic $\mu(t)$ is based on Tailor's expansion. For any l the expansion is

$$u_l(t,\mathbf{q},\sigma) \asymp \sum_{m\geq 0} u_{l(m)}(t, q-q(t), \sigma-\sigma(t)) \asymp \sum_{m\geq 0} \hat{u}_{l(m)}(\hat{t},\hat{\xi}), \tag{0.77}$$

here $u_{l(m)}$, $\hat{u}_{l(m)}$ are homogeneous polynomials of order m, $m = 0, 1, ...$ with respect to t, q, σ and $\hat{t}, \hat{\xi}$ (see (0.63)) correspondingly.

Operator $\mathcal{L}_{\Theta_N}$ is the linear differential operator of the first order, it depends only on such coefficients $\hat{u}_{l(m)}$ that $m \leq l+1$ in equation (0.27). Those terms that are contained in $u_{l(m+1)}$ are equal to zero along the bi-characteristics because of Hamilton system (0.36). Similarly, $\hat{u}_{l(m)}$ as a function of time t values in the space of homogeneous polynomials of order m with respect to $\hat{\xi}$. Then ODE for $\hat{u}_{l(m)}$ has a form

$$\frac{d}{dt}\hat{u}_{l(m)}(t) + r(t)\hat{u}_{l(m)}(t) = \hat{\mathcal{F}}_{l(m)}(t),\ l = 0, 1, \ldots, \tag{0.78}$$

$\hat{\mathcal{F}}_{l(m)}(t)$ are homogeneous polynomials of order m that depend on $\hat{u}_{s(p)}(t)$ and $\hat{\theta}_{N,p}$ only as $p \leq m+2$, $s < l$. As

$$r(t) = -\frac{1}{2}tr(B^t + CH) + \frac{1}{4}\frac{d}{dt}\ln g(t),\ H(t) = Z(t)Y^{-1}(t),$$

i.e.

$$r(t) = \frac{1}{2}\frac{d}{dt}\ln[\det Y(t)] + \frac{1}{4}\frac{d}{dt}\ln g(t),$$

then we can solve these equations once supplied them with initial data:

$$\hat{u}_{m,l}(\hat{t}) = \varrho(t)\big(\hat{u}_{m,l}(0) + \int_0^t \varrho^{-1}(t')\hat{\mathcal{F}}_{m,l}(t')dt',\big) \tag{0.79}$$

$$\varrho(t) = \sqrt{\frac{\det Y(0)}{\det Y(t)}} \sqrt[4]{\frac{g(0)}{g(t)}}. \qquad (0.80)$$

Note that $\det Y^{-1}(t)$ corresponds to the geometric divergence of ray field. In this section we used well-known procedure from, say, [1], see also [5] p.173.

3.3 Initial values

Consider now obtained above amplitude representations (we omit captions ref,tr, as the following is valid for both amplitudes):

$$\begin{cases} u(t,q,\sigma) \asymp^N \sum_{l=0}^{N} u_l (i\varepsilon)^l, \text{ see (0.13)}, \\ u_l(t,q,\sigma) = u_{l(0)} + ... + u_{l(L(N))}, \ L(N) = \frac{2(N+1-l)}{1-2\eta}, \text{ see (0.77)}, \\ u_l^{ref,tr}(t,q,0) \text{ are found , see (0.73)}, \\ u_l^{ref,tr}(0,q,\sigma) \text{ are the goals of this section.} \end{cases} \qquad (0.81)$$

Again as before we will use Tailor's expansions. Rewrite (0.13) similarly to the way we rewrote the phase expansions (0.29). Simultaneously we introduce some special notations for the homogeneous polynomials in series (0.13) in order to reduce the number of indices. We construct just several first terms, the rest terms can be obtained similarly. We get rid of index l assuming that the following is valid for any $l > 0$:

$$u_l(t,\mathbf{q},\sigma) = \mathrm{A}(t) + \mathrm{B}(t)Y(t) + \frac{1}{2}\mathrm{C}(t)Y(t), Y(t) + ..., \qquad (0.82)$$

где $\mathrm{A}(t) := u_{l(0)}(t)$, $\mathrm{B}(t) = (\mathrm{B}_\alpha(t) \ \mathrm{B}_n(t)) := u_{l(1)}(t)$,

$$\mathrm{C}(t) = \begin{pmatrix} \mathrm{C}_{\alpha\beta}(t) & \mathrm{C}_{n\beta}(t) \\ \mathrm{C}_{\alpha n}(t) & \mathrm{C}_{nn}(t) \end{pmatrix} := u_{l(2)}(t), ...$$

and so on. As we want to obtain $u_l(0,q,\sigma)$, or, more precisely, $u_{l(m)}^{ref,tr}(0,q,\sigma)$, then in our notations we need to obtain the values $A^{ref,tr}(0)$, $B^{ref,tr}(0)$, $C^{ref,tr}(0), ...$ for any $l > 0$. We will use method similar to the one we use in subsection 2.4.1 in order to obtain (0.45):

$$u_l(t,\mathbf{q},\sigma) \asymp \widetilde{\mathrm{A}} + \widetilde{\mathrm{B}}_0 t + \widetilde{\mathrm{B}}_\alpha q^\alpha + \widetilde{\mathrm{B}}_n \sigma + \tag{0.83}$$
$$+\frac{1}{2}\widetilde{\mathrm{C}}_{00}t^2 + \frac{1}{2}\widetilde{\mathrm{C}}_{\alpha\beta}q^\alpha q^\beta + \frac{1}{2}\widetilde{\mathrm{C}}_{nn}\sigma^2 + \widetilde{\mathrm{C}}_{0\alpha}tq^\alpha + \widetilde{\mathrm{C}}_{0n}t\sigma + \widetilde{\mathrm{C}}_{\alpha n}q^\alpha\sigma + \ldots =$$
$$= \widetilde{\mathrm{A}} + (\widetilde{\mathrm{B}})^t \begin{pmatrix} t \\ q^\alpha \\ \sigma \end{pmatrix} + \frac{1}{2}\widetilde{\mathrm{C}} \begin{pmatrix} t \\ q^\alpha \\ \sigma \end{pmatrix}, \begin{pmatrix} t \\ q^\alpha \\ \sigma \end{pmatrix} + \ldots,$$

where

$$\widetilde{\mathrm{A}} = \widetilde{\mathrm{A}}_0,\ \widetilde{\mathrm{B}} = \begin{pmatrix} \widetilde{\mathrm{B}}_0 \\ \widetilde{\mathrm{B}}_\alpha \\ \widetilde{\mathrm{B}}_n \end{pmatrix},\ \widetilde{\mathrm{C}} = \begin{pmatrix} \widetilde{\mathrm{C}}_{00} & \widetilde{\mathrm{C}}_{0\beta} & \widetilde{\mathrm{C}}_{0n} \\ \widetilde{\mathrm{C}}_{\alpha 0} & \widetilde{\mathrm{C}}_{\alpha\beta} & \widetilde{\mathrm{C}}_{\alpha n} \\ \widetilde{\mathrm{C}}_{n0} & \widetilde{\mathrm{C}}_{n\beta} & \widetilde{\mathrm{C}}_{nn} \end{pmatrix}. \tag{0.84}$$

Next we consider the following expression on the interface γ :

$$u_N(t,\mathbf{q},0) \asymp \widetilde{\mathrm{B}}_0 t + \widetilde{\mathrm{B}}_\alpha q^\alpha + \frac{1}{2}\widetilde{\mathrm{C}}_{00}t^2 + \frac{1}{2}\widetilde{\mathrm{C}}_{\alpha\beta}q^\alpha q^\beta + \widetilde{\mathrm{C}}_{0\alpha}tq^\alpha + \ldots = \tag{0.85}$$
$$= (\widetilde{\widetilde{\mathrm{B}}})^t \begin{pmatrix} t \\ q^\alpha \end{pmatrix} + \frac{1}{2}\widetilde{\widetilde{\mathrm{C}}}(\begin{pmatrix} t \\ q^\alpha \end{pmatrix}, \begin{pmatrix} t \\ q^\alpha \end{pmatrix}) + \ldots,$$

and

$$\widetilde{\widetilde{\mathrm{A}}} = \widetilde{\mathrm{A}}_0,\ \widetilde{\widetilde{\mathrm{B}}} = \begin{pmatrix} \widetilde{\mathrm{B}}_0 \\ \widetilde{\mathrm{B}}_\alpha \end{pmatrix},\ \widetilde{\widetilde{\mathrm{C}}} = \begin{pmatrix} \widetilde{\mathrm{C}}_{00} & \widetilde{\mathrm{C}}_{0\beta} \\ \widetilde{\mathrm{C}}_{\alpha 0} & \widetilde{\mathrm{C}}_{\alpha\beta} \end{pmatrix},\ \ldots \text{ are known matrices.} \tag{0.86}$$

Now assuming that the latter is known we obtain the following $\widetilde{\mathrm{B}}_n$, $\widetilde{\mathrm{C}}_{0n}$, $\widetilde{\mathrm{C}}_{\alpha n} = \mathrm{C}_{\alpha n}(0)$, $\widetilde{\mathrm{C}}_{nn}, \ldots$, i.e. we can construct B(0), C(0), ... The procedure is: firstly we find the initial value for u_0, i.e.

$$\widetilde{\mathrm{A}}_0 = \mathrm{A}(0), \tag{0.87}$$

next we solve ODE (0.13) for $u_{l(0)}(t)$. Now having found $A(t)$, we can construct $\dot{A}(0)$, and then we can find initial data for the second term:

$$\begin{cases} \widetilde{\mathrm{B}}_0 = \dot{\mathrm{A}}(0) - \mathrm{B}_\alpha(0)\dot{q}^\alpha(0) - \mathrm{B}_n(0)\dot{\sigma}(0), \\ \widetilde{\mathrm{B}}_\alpha = \mathrm{B}_\alpha(0), \quad \widetilde{\mathrm{B}}_n = \mathrm{B}_n(0). \end{cases} \tag{0.88}$$

Now assuming that we known the initial data for the second term we solve again the ODE (0.13) and find $B(t)$ and $\dot{B}(0)$, thus we can obtain

$$\begin{cases} \mathrm{C}_{\alpha n}(0)\dot{\sigma}(0) = -\widetilde{\mathrm{C}}_{0\alpha} - \dot{\mathrm{B}}_\alpha(0) + \mathrm{C}_{\alpha\beta}(0)\dot{q}^\beta(0), \\ \frac{1}{2}\widetilde{\mathrm{C}}_{00} = \frac{1}{2}\ddot{\mathrm{A}}(0) - \dot{\mathrm{B}}_\alpha(0)\dot{q}^\alpha(0) - \dot{\mathrm{B}}_n(0)\dot{\sigma}(0) - \frac{1}{2}(\mathrm{B}_\alpha(0)\ddot{q}^\alpha(0) + \mathrm{B}_n(0)\ddot{\sigma}(0)) \\ \quad +\frac{1}{2}\mathrm{C}_{nn}(0)(\dot{\sigma}(0))^2 + \frac{1}{2}\mathrm{C}_{\alpha\beta}(0)\dot{q}^\alpha(0)\dot{q}^\beta(0) + \mathrm{C}_{n\alpha}(0)\dot{q}^\alpha(0)\dot{\sigma}(0), \\ \widetilde{\mathrm{C}}_{\alpha\beta} = \mathrm{C}_{\alpha\beta}(0). \end{cases} \tag{0.89}$$

The last three equations imply the initial data C(0). The mentioned procedure can be continued for $u_{l(m)}$, $l = 0, 1, ...\ m = 0, 1, ..., L(l)$. Thus reflected and transmitted amplitudes initial data are expressed in terms of the incident field (see (0.85)), hence we can express required initial data in terms of the incident field. For instance, one can show that

$$\begin{cases} u^{ref}_{0(0)}(t) = \varrho(t)\mathcal{R}u^{in}_{0(0)}(0), \\ u^{tr}_{0(0)}(t) = \varrho(t)\mathcal{T}u^{in}_{0(0)}(0). \end{cases}$$

4 Exact solution

We have constructed all terms of FGb. Now we will show that for any N there exists an exact solution - Gaussian beam, corresponding to this FGb of order N. So, we have constructed ForGb $U(\varepsilon; t, q, \sigma)$, ForGb $U_N(\varepsilon; t, q, \sigma)$, corresponding to $U(\varepsilon; t, q, \sigma)$ and the required exact solution $\mathcal{U}_N(t, q, \sigma)$ to the following problem

$$\begin{cases} \Box\mathcal{U}_N = 0 \quad \text{in}\,\Omega_- \bigcup \Omega_+ \\ \mathcal{U}^-_N|_\gamma = \mathcal{U}^+_N|_\gamma, \\ \sqrt{g^-}\frac{\partial}{\partial\sigma_-}\mathcal{U}^-_N|_\gamma = \sqrt{g^+}\frac{\partial}{\partial\sigma_+}\mathcal{U}^+_N|_\gamma. \end{cases} \tag{0.90}$$

As we know only the approximation U_N to the required solution $\mathcal{U}_N$, we consider an approximation χU_N to the solution of the problem (0.90):

$$\begin{cases} \Box_{g_-}\chi(U^{in}_N + U^{ref}_N) = R^-_0 \quad \text{in}\,\Omega_- \\ \Box_{g_+}(\chi U^{tr}_N) = R^+_0 \quad \text{in}\,\Omega_+ \\ \chi(U^{in}_N + U^{ref}_N)|_\gamma = \chi U^{tr}_N|_\gamma + R_1, \\ \sqrt{g^-}\frac{\partial}{\partial\sigma_-}\chi(U^{in}_N + U^{ref}_N)|_\gamma = \sqrt{g^+}\frac{\partial}{\partial\sigma_+}\chi U^{tr}_N|_\gamma + R_2. \end{cases} \tag{0.91}$$

One can show that

$$R^-_0 < C^-_0(2t_0)\varepsilon^N M_\varepsilon,\ R^+_0 < C^+_0(t_0)\varepsilon^N M_\varepsilon,\ R_1 < C_1\varepsilon^{N+1} M_\varepsilon,\ R_2 < C_2\varepsilon^N M_\varepsilon.$$

Useful notation 3: We say that function ψ ***is polynomially small of order*** k, if $\|\psi\|_{C^k} \leq \varepsilon^k$.

The last formula implies now that R^-_0, R^+_0 are polynomially small of order $N - \frac{n}{4}$, R_1, R_2 are polynomially small of order $N + 1 - \frac{n}{4}$. The inhomogeneous interface conditions on γ are to be replaced by the homogeneous ones by introducing a new function F in $\Omega_-\bigcup\Omega_+$. Thus we get a new problem for F

$$\begin{cases}\Box F = s \quad \text{in}\,\Omega_+\\ F|_\gamma = R_1\\ \sqrt{g^+}\frac{\partial}{\partial\sigma_+}F|_\gamma = R_2.\end{cases} \tag{0.92}$$

Note that F is polynomially small of order $N+1-\frac{n}{4}$ in $\Omega_-\bigcup\Omega_+$. The RHS of (0.92), or s, is also polynomially small of order $N+1$. We can choose function F to be the following

$$F = R_1(t,q)\widetilde{\chi}(\sigma)\frac{R_2}{\sqrt{g_+}}|_{\sigma=0}\widetilde{\chi}(\sigma)\sigma.$$

Next we introduce new function

$$W_N = \begin{cases}W_N^- = \chi(U_N^{in}+U_N^{ref}) \ \text{ in } \Omega_-\\ W_N^+ = F + \chi U_N^{tr} \ \text{ in } \Omega_+.\end{cases}$$

Thus we get a problem for W_N and the corresponding continuity conditions on the interface are satisfied:

$$\begin{cases}\Box W_N = \mathcal{E} = \begin{cases}\mathcal{E}^-, \text{ in } \Omega_-\\ \mathcal{E}^+, \text{ in } \Omega_+\end{cases},\\ W_N^-|_\gamma = W_N^+|_\gamma,\\ \sqrt{g^-}\frac{\partial}{\partial\sigma}W_N^-|_\gamma = \sqrt{g^+}\frac{\partial}{\partial\sigma}W_N^+|_\gamma.\end{cases} \tag{0.93}$$

Here $\mathcal{E}^- < C_0^-(2t_0)\varepsilon^{(N-\frac{n}{4})}$, $\mathcal{E}^+ < C_0^+(t_0)\varepsilon^{(N-\frac{n}{4})}$, i.e. $\mathcal{E}^-$ and $\mathcal{E}^+$ are polynomially small of order $N-\frac{n}{4}$. The solution W_N to the problem (0.93) satisfies exactly the continuity conditions (0.20). Recall that $\mathcal{E}\in C^{\alpha(N)}(t; L^2(\Omega_-\bigcup\Omega_+))$, i.e. there is a big number of time derivatives of $\mathcal{E}$ with values from L^2 in the whole region. Compare now W_N with the exact formally written solution $\mathcal{U}_N$, as $\Box\mathcal{U}_N$ is determined in a sense of the Dirichlet form existence (it contains the continuity conditions)

$$\begin{cases}\Box\mathcal{U}_N = 0,\\ \mathcal{U}_N^-|_\gamma = \mathcal{U}_N^+|_\gamma,\\ \sqrt{g_-}\frac{\partial}{\partial\sigma_-}\mathcal{U}_N^-|_\gamma = \sqrt{g_+}\frac{\partial}{\partial\sigma_+}\mathcal{U}_N^+|_\gamma\\ \mathcal{U}_N|_{-t_0} = U_N^{in}|_{-t_0},\\ \partial_t\mathcal{U}_N|_{-t_0} = \partial_t U_N^{in}|_{-t_0},\end{cases}, \quad \begin{cases}\Box W_N = \mathcal{E},\\ W_N^-|_\gamma = W_N^+|_\gamma,\\ \sqrt{g_-}\frac{\partial}{\partial\sigma_-}W_N^-|_\gamma = \sqrt{g_+}\frac{\partial}{\partial\sigma_+}W_N^+|_\gamma\\ W_N|_{-t_0} = \chi U_N^{in}|_{-t_0},\\ \partial_t W_N|_{-t_0} = \partial_t(\chi U_N^{in})|_{-t_0}\end{cases}.$$

It is clear that $\Box(\mathcal{U}_N - W_N) = -\mathcal{E}$. Let us a give a notation to the difference of solutions, say $-\mathcal{E} = \Box V$. The initial data coincide (in the domain of the Laplace-Beltrami operator $\mathcal{D}((\Delta_g)^q)$, $q > 2$). Our problem is linear thus we consider solution:

$$V_N = \mathcal{U}_N - W_N : \quad \begin{cases} \Box V_N = \Box(\mathcal{U}_N - W_N) = -\mathcal{E}, \\ V_N|_{-t_0} = \partial_t V_N|_{-t_0} = 0, \\ V_N|_{(\partial\Omega_- \bigcup \partial\Omega_+) \bigcup [-t_0, t_0]} = 0. \end{cases} \tag{0.94}$$

Here we assumed that at time $t_0 > 0$ the beam has not yet reached the boundary $\partial\Omega_- \bigcup \partial\Omega_+$.

4.0.1 Convergence

This follows from the construction procedure that the RHS $\mathcal{E}$ of (0.94) is small on the time interval $[-t_0, t_0]$ with values from L^2. All time derivatives (there is a large number) are also from L^2 and are small.

The main idea is to estimate the difference V_N between the exact solution and the constructed FGb corresponding to it. We write out Fourier series of V_N and $\mathcal{E}$ with respect to their eigenfunctions $\Delta_g \varphi_k = \lambda_k \varphi_k$:

$$V_N(t, \mathbf{x}) = \sum v_k(t)\varphi_k(\mathbf{x}), \;\; v_k(-t_0) = \dot{v}_k(-t_0) = 0, \tag{0.95}$$

$$\mathcal{E}(t, \mathbf{x}) = \sum e_k(t)\varphi_k(\mathbf{x}). \tag{0.96}$$

Then

$$\Box V_N(t, \mathbf{x}) = \sum [\ddot{v}_k(t) + \lambda_k v_k(t)]\varphi_k(\mathbf{x}) = \sum e_k(t)\varphi_k(\mathbf{x}),$$

where as $e_k(t)$ are small and all its time derivatives are polynomially small, we has $\sum |\frac{\partial^p}{\partial t^p} e_k(t)|^2 < C_p(t_0)\varepsilon^L < \infty$, where $p \leq L = (N - p - \frac{n}{4})^2$, then

$$\ddot{v}_k(t) + \lambda_k v_k(t) = e_k(t),$$

and then

$$v_k(t) = \int\limits_{-\infty}^{t} \frac{sin\sqrt{\lambda_k}(t - t')}{\sqrt{\lambda_k}} e_k(t')dt', \tag{0.97}$$

We integrate the latter by parts, hence

$$\frac{-1}{\lambda_k}\cos\sqrt{\lambda_k}(t-t')e_k(t')|_{\infty}^{t}+\int\limits_{-\infty}^{t}\frac{1}{\lambda_k}\cos\sqrt{\lambda_k}(t-t')e_k'(t')dt'=$$

where $e_k(-\infty)=0$,

$$=\frac{1}{\lambda_k}e_k(t)|_{-\infty}^{t}+\frac{1}{\lambda_k}\int\limits_{-\infty}^{t}\cos\sqrt{\lambda_k}(t-t')e_k'(t')dt'.$$

Consider firstly the second summand - the integral by introducing new notation for the integrand:

$$\frac{1}{\lambda_k}\int\limits_{-\infty}^{t}\cos\sqrt{\lambda_k}(t-t')e_k'(t')dt'=\frac{1}{\lambda_k}\int\limits_{-\infty}^{t}r(t,t^{'})dt'.$$

We estimate a new function

$$\sum r_k(t)\varphi_k(\mathbf{x})=r(t),$$

using Parseval inequality: $\|r_k(t)\|^2=\sum|r_k(t)|^2$, and then the Cauchy-Shwartz inequality we get

$$|r_k(t)|^2\leq\int\limits_{-t_0}^{t}\cos^2\sqrt{\lambda_k}(t-t')\int\limits_{-t_0}^{t}|e_k'(t')|^2dt'<(t-t_0)\int\limits_{-t_0}^{t}|e_k'(t')|^2dt'.$$

The latter implies

$$\|r_k(t)^2\|\leq(t+t_0)\int\limits_{-t_0}^{t}\sum|e_k'(t')|^2dt'=$$

($\sum|e_k'(t')|^2$ is positive thus we change sign, the result is an L^2-norm on $\mathcal{E}$)

$$=(t+t_0)\int\limits_{-t_0}^{t}\|\mathcal{E}(t')\|'dt'\leq 2t_0\int\limits_{-t_0}^{t_0}\|\mathcal{E}(t')\|_{L^2}^2dt':=E,$$

where $\|\mathcal{E}(t')\|_{L^2}$ is polynomially small of order $(N-\frac{n}{4})$. Let us estimate the first summand, as $\frac{-1}{\lambda_k}e_k(t)$ is from $H^1\bigcap\mathcal{D}(\Delta)$, then for any t

$$\sum\lambda_k^2|v_k(t)|^2=\sum|e_k|^2+E<\infty. \tag{0.98}$$

We showed that $v_k(t)$ is small in the H^1-norm. Consider $\dot{v}_k(t)$:

$$\dot{v}_k = \int\limits_{-\infty}^{t} \cos\sqrt{\lambda_k}(t-t')e_k(t')dt' = -\frac{1}{\lambda_k}\int\limits_{-\infty}^{t} \sin\sqrt{\lambda_k}(t-t')e'_k(t')dt', \qquad (0.99)$$

$$\sum |\dot{v}_k(t)|^2 \asymp \widetilde{C}(t)\int \frac{|e'_k(t')|^2dt'}{\lambda_k} < \infty. \qquad (0.100)$$

Inequality (0.98) says that $\Delta V_N \in L^2$, and that $V_N \in \mathcal{D}(\Delta)$ correspondingly. Function $\mathcal{E}$ can be time differentiated; similarly to inequality (0.100) one can estimate time derivatives of V_N. In particular, (0.100) implies that $V'_t \in L^2$. The result can be rewritten as:

$$V_N \in C^{p(N)}([-t_0, t_0], \mathcal{D}(\Delta)),\ p \geq 1. \qquad (0.101)$$

We have proved the third statement of theorem 1.

4.1 Conclusion.

We constructed phase and amplitude functions of Gaussian beams of "quasiphoton" type for the incident, reflected and transmitted wave fields near the interface γ in sections (2), (3).

We assumed the incident field to be given ($f^0, V^0, \Gamma_0, \Theta^0, M_0, -t_0$ (0.5), (0.6)). We used its trace on the interface, continuity conditions and Hamilton-Jacobi equations for construction of the reflected and transmitted fields as a formal expansions, checking that all homogeneous polynomials in these expansion satisfy condition for these series to be ForGb. We truncated these formal series on order N and showed that for any N there exists an exact solution to the wave equation, which is asymptotically close to constructed FGb.

One can write out the analogue of theorem 3.19 (form [5])'s result for "quasiphotons" propagating from the interface along geodesics which are not normal to γ. The corresponding geodesics $\mu^{ref,tr}(t)$ are such that the directing cosine at M_0 is

$$\cos\varphi^{ref,tr}|_{t=0} = \dot{\sigma}_{ref,tr}(0) \geq 0,$$

Phase function $\Theta_N^{ref,tr}$ and amplitude function $\sum^{N} u_l^{ref,tr}$ satisfy (0.68) on γ. Each term of these functions as an homogeneous polynomial is a solution to the ODE (0.24), (0.27) with respect to time taking into account constructed initial data.

Author is grateful to Professor Kurylev Ya.V. for the formulation of the problem and for the attention and supervising during working on current paper as well. The author is also thankful to Professor Katchalov A.P. for series of valuable remarks.

References

1. Babich V., Ulin V., *The complex space-time ray method and "quasiphotons."* (Russian) Zap. Nauchn. Sem. LOMI, **117**, 1981, 5-12.

2. Babich V., Buldyrev V., Molotkov I., *The space-time Ray Method, Linear and Nonlinear waves.* (Russian), Leningrad Univ., Leningrad, 1985. 272 pp.

3. Katchalov A., *Gaussian beams, Hamilton-Jacobi equations and Finsler geometry*, Zap. Nauchn. Sem. POMI, **297**, 2003, 66-92.

4. Kathcalov A., *A system of coordinates for describing the "quasiphoton."* Zap. Nauchn. Sem LOMI, **140**, 1984, 73-76.

5. Katchalov A., Kurylev Ya., Lassas M., (2001) *Inverse Boundary Spectral Problems.* Chapman Hall / CRC **123**.

6. Popov M., *Ray Theory and Gaussian Beam Method for Geophysicists*, Edufba, Salvator-Bahia, 2002.

7. Ralston J., (1983) *Gaussian beams and propagation of singularities.* Studies in PDE, MAA Studies in Mathematics, **23**, Walter Littman ed.