arXiv:math-ph/0510070 v3 24 Oct 2005

The Variable Coefficient Hele-Shaw Problem, Integrability and Quadrature Identities

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Abstract

The theory of quadrature domains for harmonic functions and the Hele-Shaw problem of the fluid dynamics are related subjects of the complex variables and mathematical physics. We present results generalizing the above subjects for elliptic PDEs with variable coefficients, emerging in a class of the free-boundary problems for viscous flows in non-homogeneous media. Such flows posses an infinite number of conservation laws, whose special cases may be viewed as quadrature identities for solutions of variable-coefficient elliptic PDEs. If such PDEs are gauge equivalent to the Laplace equation (gauge-trivial case), a time-dependent conformal map technique, employed for description of the quadrature domains, leads to differential equations, known as "string" constraints in the theory of integrable systems. Although analogs of the string constraints have non-local forms for gauge-non-trivial equations, it is still possible to construct the quadrature domains explicitly, if the elliptic operator belongs to a class of the Calogero-Moser Hamiltonians.

1 Introduction

Recently, the constant-coefficient Hele-Shaw problem has received a good deal of attention due to its connection with the theory of integrable hierarchies in the dispersionless limit [6], [14]. These integrable structures turned out to have natural interpretations in the classical free-boundary problems.

Although concerned with various aspects of integrability, the present paper is primarily devoted to construction of solutions for a more general class of the variable-coefficient free-boundary problems. Namely, motivated by practical applications, we derive a class of rational solutions of the variable-coefficient Hele-Shaw problems, related to the dihedral Calogero-Moser systems. These solutions provide examples of quadrature identities for elliptic PDEs with variable coefficients. The theory of quadrature domains for such PDEs is highly reminiscent of that for harmonic functions.

The structure of the paper is as follows: A brief summary of the theory Hele-Shaw flows, as well as that of the quadrature domains, is given in the next three sections, followed by introduction of the time-dependent conformal map technique in Section 5. We digress into consideration of the gauge-trivial problems, considered in the above mentioned works on relationship between dispersionless integrable hierarchies and the Hele-Shaw problem, in Section 6. Sections 7 and 8 are devoted to derivation of explicit results for a class of the gauge-non-trivial problems. Concluding remarks are given in Section 9.

2 Variable Coefficient Hele-Shaw Problem

The Hele-Shaw flow governs the dynamics of the boundary $\partial\Omega = \partial\Omega(t)$ in the plane separating two disjoint, open regions $\Omega = \Omega(t)$ and $\mathbb{C}\backslash\bar{\Omega}$ in which scalar fields are defined. These may be interpreted as the pressure fields for two incompressible (viscous and non-viscous) immiscible fluids, trapped in a gap between two plates or propagating in a layer of the porous medium. If the size of the gap or the layer height is negligible, the velocity of the viscous liquid can be averaged across the direction perpendicular to the layer or the plates respectively, and the problem becomes effectively two dimensional. The velocity of the viscous liquid is then proportional to the two dimensional pressure gradient (the Darcy law)

$$v = -\nabla P \tag{1}$$

The flow is incompressible and the liquid velocity satisfies a continuity equation

$$\nabla \cdot v = 0 \tag{2}$$

The region Ω can be chosen to be bounded and occupied with a viscous liquid, so that (1) holds in that region. It will be referred as the "interior" region. It is surrounded by a non-viscous liquid occupying the "exterior" (unbounded) region $\mathbb{C}\backslash\bar{\Omega}$. Since the flow of the liquid occupying the exterior domain is inviscid, it must be driven by the pressure field with vanishing gradient and, therefore, the pressure is constant in that region. Without loss of generality the pressure can be set to zero in $\mathbb{C}\backslash\bar{\Omega}$. It is a continuous function across the moving boundary $\partial\Omega(t)$, so that

$$P(\partial\Omega) = 0 \tag{3}$$

and the normal velocity of the boundary coincides with that of the viscous flow at $\partial\Omega$

$$n \cdot \frac{d}{dt}\partial\Omega = n \cdot v \tag{4}$$

where n denotes the outward normal to the boundary. In the present and the next section we consider flows that are driven by a single point source located inside Ω. Without loss of generality we can locate it at the origin $z = 0$

$$P \to \frac{-1}{4}\log(z\bar{z}), \quad \text{as} \quad z \to 0$$

where $z := X + iY$, and X, Y stand for the Cartesian coordinates on the plane. The last equation, together with the Darcy law (1) and the continuity equation (2) leads to

$$\Delta P = -\pi\delta(X)\delta(Y)$$

where δ denotes the Dirac delta-function. In this paper we deal with a generalization of the above problem, namely the Hele-Shaw problem with coefficients depending spatial variables X and Y

$$v = -\kappa\nabla P, \quad \nabla \cdot (\eta v) = 0 \tag{5}$$

where $\kappa = \kappa(X, Y), \eta = \eta(X, Y)$ are arbitrary functions of X, Y, sufficiently regular in Ω. As in the constant coefficient case, the pressure has a logarithmic singularity at the origin, i.e.

$$P \to \frac{-1}{4\kappa(0)\eta(0)}\log(z\bar{z}), \quad \text{as} \quad z \to 0 \tag{6}$$

when the flow is driven by a single point source.

Note, that one has to modify the asymptotic condition (6) if $\kappa\eta$ vanishes or has a singularity at the point source, or if several point sources coalesce in a special way ("multipole" sources).

We consider a situation when the exterior region is occupied by a non-viscous liquid, so that (3) holds. Equations (3), (4), (5), as well as asymptotic conditions (e.g. (6), for a single point source) set the Hele-Shaw problem with variable coefficients, describing, for instance, propagation of the liquid in a thin horizontal layer of non-homogeneous porous medium, whose permeability and height depend on X, Y.

3 Conservation Laws

From (5), (6) it follows that the pressure satisfies the second order elliptic PDE

$$(\nabla\kappa\eta\nabla)P = -\pi\delta(X)\delta(Y) \tag{7}$$

that replaces the Laplace equation of the constant coefficient problem. Let $\phi(z, \bar{z})$ be any regular in the interior domain solution of the equation

$$(\nabla\kappa\eta\nabla)\phi = \left(\frac{\partial}{\partial z}\kappa\eta\frac{\partial}{\partial \bar{z}} + \frac{\partial}{\partial \bar{z}}\kappa\eta\frac{\partial}{\partial z}\right)\phi = 0 \tag{8}$$

Introduce the following quantities

$$M[\phi] = \int_{\Omega(t)} \eta\phi dXdY \tag{9}$$

and estimate their time derivatives. Considering an infinitesimal variation of the interior domain $\Omega(t) \to \Omega(t + dt)$ along the boundary $\partial\Omega(t)$, we get

$$\frac{dM[\phi]}{dt} = \oint_{\partial\Omega(t)} \eta\phi v_n dl$$

where dl is the boundary length element and v_n is the normal velocity of the boundary. From (4) and (5) it follows that $v_n = v \cdot n = -\kappa n \cdot \nabla P$. Since $P(\partial\Omega) = 0$

$$\frac{dM[\phi]}{dt} = \oint_{\partial\Omega(t)} (P\kappa\eta\nabla\phi - \phi\kappa\eta\nabla P) \cdot ndl$$

Applying the Stokes theorem and taking (7) and (8) into account, from the last equation, we get

$$\frac{dM[\phi]}{dt} = \oint_{z=0} (P\kappa\eta\nabla\phi - \phi\kappa\eta\nabla P) \cdot ndl$$

that, according to (6), becomes

$$\frac{dM[\phi]}{dt} = \pi\phi(0) \tag{10}$$

Therefore, the quantity $M[\phi]$ is conserved for any (regular in Ω) solution of (8), such that $\phi(0) = 0$.

The Richardson harmonic moments $\int_{\Omega(t)} z^k dXdY$ [8] of the constant coefficient Hele-Shaw problem correspond to the special case

$$\kappa = \eta = 1, \quad \phi(z, \bar{z}) = z^k, \quad k \geq 0$$

4 Quadrature Domains

The derivation of conservation laws can be easily generalized to the case of several sources. Consider the flow driven by N sources of the time-dependent powers $q_k(t)$ that are located at points $z_k, k = 1..N$ in Ω

$$P \to \frac{-q_k(t)}{2\kappa(x_k, y_k)\eta(x_k, y_k)} \log|z - z_k|, \quad \text{as} \quad z \to z_k, \quad z_k = x_k + iy_k \in \Omega$$

Then by arguments, similar to those used for the single point source problem we arrive at

$$\frac{dM[\phi]}{dt} = \pi \sum_{k=1}^{N} q_k(t)\phi(z_k, \bar{z}_k)$$

for any $\phi(z, \bar{z})$, regular in Ω and satisfying (8). It follows that

$$M[\phi](t) = M[\phi](0) + \pi \sum_{k=1}^{N} Q_k\phi(z_k, \bar{z}_k), \quad Q_k = Q_k(t) = \int_0^t q_k(t')dt'$$

From the last equation, we see that $M[\phi](t)$ does not depend on history of sources and is a function of total charges Q_i, produced by the time t. This fact reflects the integrability of the problem, where flows produced by different sources commute.

Consider the special case when $M[\phi](0) = 0$. It describes the injection of the fluid to an initially empty Hele-Shaw cell. Taking into account the definition (9) of $M[\phi]$ we obtain

$$\int_\Omega \eta\phi(z, \bar{z})dXdY = \pi \sum_{k=1}^{N} Q_k\phi(z_k, \bar{z}_k) \tag{11}$$

This is an identity expressing the integral over the domain Ω in the left-hand side as a sum of terms evaluated at a finite number of points, given on the right-hand side. The special case $\eta = \kappa = 1$ provides quadrature identities for harmonic functions. Special domains Ω, possessing the above property, are called quadrature domains [10], [11], [13]. The simplest example of a quadrature domain is a circular disc, produced by a single point source in the constant coefficient Hele-Shaw problem. The corresponding quadrature identity is a "mean value" theorem for harmonic functions. Equation (11) is a generalization of quadrature identities appearing in the theory of harmonic functions to the case of elliptic equations with variable coefficients. The quadrature domains for such PDEs are, thus, solutions to the variable coefficient, interior Hele-Shaw problems with zero initial conditions. In the sequel we will mainly deal with situation when groups of sources coalesce in such a way that (11) becomes

$$\int_\Omega \eta\phi(z, \bar{z})dXdY = \pi \sum_{k=1}^{N} \hat{Q}_k[\phi](z_k, \bar{z}_k) \tag{12}$$

Where $\hat{Q}_k$ is a finite-order, differential in $z, \bar{z}$ operator of the following form

$$\hat{Q}_k = Q_k^{(0)} + \sum_{i=1}^{i_k} \left(Q_k^{(i)} \frac{\partial^i}{\partial z^i} + \bar{Q}_k^{(i)} \frac{\partial^i}{\partial \bar{z}^i} \right), \quad Q_k^{(0)} = \bar{Q}_k^{(0)}, \tag{13}$$

For such combinations of multipole sources, the integral over Ω in the left-hand side of (12) is expressed as a sum of terms involving values of function ϕ as well as a finite number of its derivatives at a finite number of points inside Ω. The boundary $\partial\Omega(t)$ of the quadrature domain is a solution to the variable coefficient Hele-Shaw problem with pressure satisfying the following equation

$$\nabla\kappa\eta\nabla P = -\pi\sum_{k=1}^{N}\hat{q}_k[\delta(X-x_i)\delta(Y-y_i)], \quad \hat{q}_k = \frac{\partial Q_k^{(0)}}{\partial t} + \sum_{i=1}^{i_k}(-1)^i\left(\frac{\partial Q_k^{(i)}}{\partial t}\frac{\partial^i}{\partial z^i} + \frac{\partial \bar{Q}_k^{(i)}}{\partial t}\frac{\partial^i}{\partial \bar{z}^i}\right)$$

Note, that operators $\hat{Q}_k$ do not contain mixed derivatives $\frac{\partial^{n+m}}{\partial\bar{z}^n\partial z^m}$, since, due to (8), $\frac{\partial^2\phi}{\partial z\partial\bar{z}}$ is a linear combination of the first order derivatives of ϕ. The coefficients in front of $\frac{\partial^i}{\partial z^i}$ must be complex conjugates of that in front of $\frac{\partial^i}{\partial\bar{z}^i}$, since both $\phi(z,\bar{z})$ and its complex conjugate satisfy (8).

5 Time-Dependent Conformal Maps

Usually, a time dependent conformal map technique is implemented to find explicit solutions to the constant coefficient Hele-Shaw problem and, in the special case of zero initial conditions, it is also an efficient method of constructing quadrature domains. This technique is also applicable to the the variable coefficient case, leading to explicit solutions of some non-trivial problems.

Introduce a "mathematical" w-plane and denote by $z(w,t)$ a conformal map from the unit disc $|w| < 1$ in the w-plane to a simply-connected interior region $\Omega(t)$ in the physical z plane. According to the Riemann mapping theorem a one-to one analytic in $|w| \le 1$ map

$$z(w,t) = r(t)w + \sum_{i>0} u_i(t)w^{i+1} \tag{14}$$

exists, such that the unit circle in the w-plane is mapped to the (analytic) boundary contour in the z-plane

$$z(w,t) \in \partial\Omega(t), \quad \text{if} \quad |w| = 1 \tag{15}$$

Alternatively to the derivation of Section 2, one can estimate the time derivatives of $M[\phi]$, transforming the two-dimensional integrals (9), taken over Ω, to line integrals along the unit circle in the w-plane. Introducing a function $\xi(z,\bar{z})$ such that

$$\eta(X,Y)\phi(z,\bar{z}) = \frac{\partial}{\partial\bar{z}}\xi(z,\bar{z}) \tag{16}$$

by the Green theorem and (15), we may rewrite (9) as

$$M[\phi] = \frac{1}{2i}\oint_{\partial\Omega}\xi dz = \frac{1}{2i}\oint_{|w|=1}\xi\frac{\partial z}{\partial w}dw \tag{17}$$

Note that $r(t)$ in (14) can be made to be real and

$$\bar{w} = 1/w, \quad \bar{z} = \bar{z}(1/w,t) = \frac{r(t)}{w} + \sum_{i>0}\frac{\bar{u}_i(t)}{w^{i+1}} \tag{18}$$

along the boundary. In the sequel we mainly deal with $z(w,t)$ evaluated at the boundary $\bar{w} = 1/w$ and therefore we use $\bar{z}$ to denote $\bar{z}(1/w,t)$ (if not otherwise specified). It follows from (16) and (17) that

$$\frac{dM[\phi]}{dt} = \frac{1}{2i}\oint_{|w|=1} \eta\phi\left(\frac{\partial z}{\partial w}\frac{\partial \bar{z}}{\partial t} - \frac{\partial z}{\partial t}\frac{\partial \bar{z}}{\partial w}\right)dw + \frac{1}{2i}\oint_{|w|=1}\frac{\partial}{\partial w}\left(\xi\frac{\partial z}{\partial t}\right)dw$$

which equals

$$\frac{dM[\phi]}{dt} = \frac{1}{2i}\oint_{|w|=1}\phi\eta\frac{\{z(w,t),\bar{z}(1/w,t)\}}{w}dw \tag{19}$$

provided ξ in (16) is univalent at $\partial\Omega$. In (19), $\{,\}$ denotes the Poisson bracket

$$\{f(w,t), g(w,t)\} := w\frac{\partial f}{\partial w}\frac{\partial g}{\partial t} - w\frac{\partial g}{\partial w}\frac{\partial f}{\partial t} \tag{20}$$

defined on the cylinder parameterized by coordinates ($w = e^{i\theta}, t$) with real θ and t. By virtue of (19) the evolution equation (10) may be rewritten as

$$\frac{1}{2\pi i}\oint_{|w|=1}\phi\eta\frac{\{z,\bar{z}\}}{w}dw = \phi(0) \tag{21}$$

6 Gauge Trivial Problems, String Equations

Consider equation (8). Its solution ϕ also satisfies

$$L[\phi] = 0, \quad L := \psi^{-2}\frac{\partial}{\partial z}\psi^2\frac{\partial}{\partial \bar{z}} + \psi^{-2}\frac{\partial}{\partial \bar{z}}\psi^2\frac{\partial}{\partial \bar{z}}, \quad \psi := \sqrt{\kappa\eta}, \quad \psi(z,\bar{z}) = \bar{\psi}(\bar{z},z) \tag{22}$$

The elliptic second order differential operator L is amenable, by a gauge transformation, to the two-dimensional zero-magnetic field Schroedinger operator

$$H := -\psi L\frac{1}{\psi} = -\frac{1}{2}\Delta + V(X,Y), \quad V = \frac{\Delta\psi}{2\psi}, \quad H[\psi\phi] = 0, \quad \text{if} \quad L[\phi] = 0 \tag{23}$$

The potential V vanishes when $\psi = h$, where h stands for a harmonic function

$$L = \mathcal{L}_0 := h^{-1}\Delta h, \quad \mathcal{L}_0\left[\frac{z^k}{h}\right] = 0, \quad k \geq 0, \quad h := \Psi(z) + \bar{\Psi}(\bar{z}) \tag{24}$$

The elliptic operator $\mathcal{L}_0$ is gauge equivalent to the Laplace operators. We call such operators and corresponding Hele-Shaw problems "gauge-trivial". According to (9), (24) the following quantities

$$I_0 = \int_{\Omega(t)}\frac{\eta}{h}dXdY, \quad I_k = \int_{\Omega(t)}\frac{\eta}{h}z^k dXdY, \quad k > 0$$

are linear and constant in time.

Using (24), from (21) we see, that in the gauge-trivial case

$$\oint_{|w|=1} \frac{z^k}{w}\frac{\eta}{h}\{z,\bar z\}dw = \oint_{|w|=1} \frac{\bar z^k}{w}\frac{\eta}{h}\{z,\bar z\}dw = 2\pi i\delta_{k0}, \quad k\geq 0$$

or, taking (14), (18) into account

$$\oint_{|w|=1} \left(\frac{\eta}{h}\{z,\bar z\}-1\right)\frac{z^k}{w}dw = \oint_{|w|=1} \left(\frac{\eta}{h}\{z,\bar z\}-1\right)\frac{\bar z^k}{w}dw = 0$$

The expression in the parentheses of the last equation must vanish, since $z^k/w = r^k w^{k-1} + ...$ and $\bar z^k = r^k w^{-k-1} + ...$, $k \geq 0$ form a basis of an arbitrary Laurent series in w. Therefore, $z, \bar z$ satisfy the following differential equation

$$\{z(w,t), \bar z(1/w,t)\} = h/\eta$$

or equivalently

$$\{q(z,\bar z), \bar q(\bar z, z)\} = 1, \quad (\partial_z q)(\partial_{\bar z}\bar q) - (\partial_z \bar q)(\partial_{\bar z} q) = h/\eta \tag{25}$$

known as the "string" constraint in the theory of the dispersionless integrable hierarchies [4], [6], [12], [14]. In the special case $h = 1, \eta = 1$, or equivalently $q(z,\bar z) = z$, corresponding to the constant coefficient Hele-Shaw problem, the conformal maps satisfy the following constraint

$$\{z(w,t), \bar z(1/w,t)\} = 1$$

known as a Galin-Polubarinova equation [3], [7] in the theory of the Hele-Shaw flows.

The string equation (25) is preserved by the Lax-Hamilton flows

$$\frac{dz}{d\tau} = \{z, \mathcal{H}\}, \quad \frac{d\bar z}{d\tau} = \{\bar z, \mathcal{H}\}, \quad \mathcal{H} = \mathcal{H}(w,t,\tau)$$

Indeed,

$$\frac{d}{d\tau}\{q(z,\bar z), \bar q(\bar z, z)\} = \left\{\frac{dq(z,\bar z)}{d\tau}, \bar q(\bar z, z)\right\} + \left\{q(z,\bar z), \frac{d\bar q(\bar z, z)}{d\tau}\right\} = \{\mathcal{H}, \{q(z,\bar z), \bar q(\bar z, z)\}\} = 0$$

i.e. functions $z(w,t), \bar z(1/w,t)$, satisfying (25) belong to invariant, under the action of the Lax-Hamilton vector fields $\{\mathcal{H}, \cdot\}$, subspace of space of functions of w, t. It is also necessary to satisfy the condition of the form-invariance of $z(w,t)$ along the Lax-Hamilton flow lines, i.e. such Lax-Hamilton functions must be chosen, that $z(w,t)$ will remain the Taylor series (14) along the corresponding flows. So selected Lax-Hamilton functions, generate symmetry transformations of the Hele-Shaw problem, mapping continuously one solution of the problem into the others. An ableian subset of these transformations, forms a parametrizable set of deformations, leaving ivariant equations governing the Hele-Shaw flow. The Lax-Hamilton functions generating such abelian subset can be conveniently chosen as

$$H_k = (z^{-k})_{<0} + 1/2(z^{-k})_0, \quad \bar H_k = (\bar z^{-k})_{>0} + 1/2(\bar z^{-k})_0, \quad k > 0$$

where $()_{>0}, ()_{<0}, ()_0$ stand for negative, positive and zero parts of the Laurent expansions

$$(f)_{>0} := \sum_{k>0} f_k w^k, \quad (f)_0 := f_0, \quad (f)_{<0} := \sum_{k<0} f_k w^k, \quad \text{if} \quad f = \sum_{k\in\mathbb{Z}} f_k w^k$$

The flow equations

$$\begin{array}{ll} \frac{\partial z}{\partial \tau_k} = \{\mathcal{H}_k, z\} & \frac{\partial z}{\partial \bar{\tau}_k} = \{\bar{\mathcal{H}}_k, z\} \\ \frac{\partial \bar{z}}{\partial \tau_k} = \{\mathcal{H}_k, \bar{z}\} & \frac{\partial \bar{z}}{\partial \bar{\tau}_k} = \{\bar{\mathcal{H}}_k, \bar{z}\} \end{array}, \quad k > 0 \tag{26}$$

constitute the two-dimensional Toda hierarchy in the dispersionless limit (2dToda) or Sdiff(2) hierarchy [12]. Due to commutativity of the 2dToda vector fields, the maps (14), (18) are functions of the deformation parameters $\tau_1, \tau_2, \ldots.\ \bar{\tau}_1, \bar{\tau}_2, \ldots$ (the 2dToda "times"). The 2dToda system is an integrable Hamiltonian system of PDEs for the coefficients $r(t, \tau_1, \ldots, \bar{\tau}_1 \ldots)$, $u_k(t, \tau_1, \ldots, \bar{\tau}_1 \ldots)$, $\bar{u}_k(t, \tau_1, \ldots, \bar{\tau}_1 \ldots), k > 0$ of the series (14), (18), obtained by equating (26) as Laurent series in the dummy variable w. The connection between the 2dToda hierarchy and the Hele-Shaw problem was first found in [6], [14].

In the context of the constant-coefficient Hele-Shaw problem, the 2dToda "times" τ_k can be naturally interpreted as harmonic moments of the domain Ω. More precisely, the kth harmonic moment $\int_{\Omega(t,\tau_1,\ldots,\bar{\tau}_1,\ldots)} z^k dXdY$ evolves linearly in τ_k and is constant along the other 2dToda flows [4], [6], [14].

7 Gauge Non-trivial Problems, Quantum Integrable Systems on Plane

In the gauge trivial cases, equation (21) can be transformed into the differential "string" equation (25) for time dependent conformal maps. In contrast to the gauge-trivial case, similar differential representations of the gauge-non-trivial integral relations (21) seem to be generically impossible. Nevertheless, we can still construct explicitly the quadrature domains for special gauge-non-trivial elliptic PDEs. In this section, we derive sets of solutions to (8) and related conserved quantities that will be used for construction of such quadrature domains.

An explicit evaluation of conserved quantities is possible when a gauge-non-trivial elliptic operator L in (22) is equal, up to a gauge transformation, (or (23) is equal) to a Hamiltonian of a quantum integrable system on the plane.

We consider problems (5) with such $\kappa\eta$ that the corresponding second-order elliptic differential operator L (22) can be related to the Laplace operator by a differential operator T

$$T\Delta = LT \tag{27}$$

T is usually called the intertwining operator and (27) is an intertwining identity. The simplest intertwining identity corresponds to the gauge trivial case (22) when $T = T_0 := h^{-1}$ is a zero-order differential operator and $L = \mathcal{L}_0 := \frac{1}{h}\Delta h$. If, however, T is a differential operator of a non-zero order, the corresponding L equals, up to a gauge transformation, a Hamiltonian of a non-trivial integrable quantum system on the plane. Indeed, from (27) it follows that any eigenfuction $\Psi(z, \bar{z}, \lambda)$ of Δ

$$\Delta\left[\Psi(z, \bar{z}, \lambda)\right] = \lambda\Psi(z, \bar{z}, \lambda)$$

is transformed by the action of T, $\Psi(z, \bar{z}, \lambda) \to T[\Psi(z, \bar{z}, \lambda)]$, to an eigenfunction of L having the same eigenvalue λ or to zero.

We start with a class of examples, in which κ, η vary only in one direction (say X-direction) and equal

$$\kappa = \frac{1}{X^{2n}}, \quad \eta = 1, \quad n = 0, 1, 2, \ldots \tag{28}$$

The corresponding elliptic operator (22)

$$L = \mathcal{L}_n := X^{2n} \frac{\partial}{\partial X} \frac{1}{X^{2n}} \frac{\partial}{\partial X} + \frac{\partial^2}{\partial Y^2} \tag{29}$$

is amenable, by a gauge transformation, to the two-dimensional Schrodinger operator

$$\mathcal{H}_n = X^n \mathcal{L}_n X^{-n} = \frac{\partial}{\partial Y^2} + \frac{\partial^2}{\partial X^2} - \frac{n(n+1)}{X^2} \tag{30}$$

which , when rewritten down in the polar coordinates, has the following form

$$\mathcal{H}_n = \frac{\partial^2}{\partial \rho^2} + \frac{1}{\rho} \frac{\partial}{\partial \rho} + \frac{S_n}{\rho^2}, \quad S_n = \frac{\partial^2}{\partial \theta^2} - \frac{n(n+1)}{\cos(\theta)^2}, \quad z = X + iY = \rho e^{i\theta} \tag{31}$$

The operator S_n admits the following alternative factorisations

$$S_n = \left(\tfrac{\partial}{\partial \theta} + n \tan(\theta)\right) \left(\tfrac{\partial}{\partial \theta} - n \tan(\theta)\right) - n^2 = \left(\tfrac{\partial}{\partial \theta} - (n+1) \tan(\theta)\right) \left(\tfrac{\partial}{\partial \theta} + (n+1) \tan(\theta)\right) - (n+1)^2$$

and therefore

$$\left(\frac{\partial}{\partial \theta} + (n+1) \tan(\theta)\right) S_n = S_{n+1} \left(\frac{\partial}{\partial \theta} + (n+1) \tan(\theta)\right)$$

In view of (29), (31), this leads to the intertwining identity

$$T_n \Delta = \mathcal{L}_n T_n, \quad T_n = X^n \left(X \frac{\partial}{\partial Y} - Y \frac{\partial}{\partial X} + (n+1) \frac{Y}{X}\right) \cdots \left(X \frac{\partial}{\partial Y} - Y \frac{\partial}{\partial X} + \frac{Y}{X}\right) \tag{32}$$

Note, that the intertwining operator is not unique. To see it on the example of equation (29), we can use its translational invariance along the Y-direction, shifting Y in T_n by a constant λ, and so obtaining a linear combination of operators

$$T_n + \sum_{i=1}^{n} \lambda^i T_n^{(i)}$$

$T_n^{(i)}$ as well as T_n, are all intertwining operators of the nth order. They are homogenous polynomials in $X, Y, \frac{\partial}{\partial X}, \frac{\partial}{\partial Y}$. For instance, in the simplest $n = 1$ case

$$T_1 = X \left(X \frac{\partial}{\partial Y} - Y \frac{\partial}{\partial X}\right) - Y, \quad T_1^{(1)} = X \frac{\partial}{\partial X} - 1 \tag{33}$$

It is easy to see that operators $T_n^{(n)}$ (e.g. $T_1^{(1)}$ in the above example) are X-dependent only, i.e.

$$T_n^{(n)} = X^n \frac{\partial^n}{\partial X} + \sum_{k=0}^{n-1} a_{k;n} X^k \frac{\partial^k}{\partial X^k} \tag{34}$$

(34) can be also obtained alternatively by exploiting separation of (29) in Cartezian coordinates X, Y, through a chain of factorizations leading from $\frac{\partial^2}{\partial X^2}$ to $X^{2n}\frac{\partial}{\partial X}\frac{1}{X^{2n}}\frac{\partial}{\partial X}$.

Images of all intertwining operators, acting on harmonic functions, coincide for a given n. Therefore, one can choose any operator from $T_n, T_n^{(i)}, i = 1..n$ (or their linear combination) to construct same set of solutions to (29). In so doing, we return to the $n = 1$ example (33), choosing $T_1^{(1)}$. In this example, it is convenient to make a shift of z by the distance x_1 along the X direction, displacing the singular line $X = 0$ of the problem (29) to $X = -x_1$. Then, without loss of generality we can locate a source at point $z = 0$. It follows that the functions

$$\phi_{k;1} = 2T_1^{(1)}[z^{k+1}] = (k+1)\,(z + \bar{z} + 2x_1)\,z^k - 2z^{k+1}, \quad k = 0, 1, 2, ...$$

and their complex conjugates, form a set of solutions of equation

$$\left((X + x_1)^2 \frac{\partial}{\partial X} \frac{1}{(X + x_1)^2} \frac{\partial}{\partial X} + \frac{\partial^2}{\partial Y^2}\right)[\phi_k] = 0$$

related to the variable-coefficient Hele-Shaw problem with $\kappa = 1/(X + x_1)^2, \quad \eta = 1$. Plugging this set into (9), by (10) we get an infinite number of quantities

$$M_{k;1} = M[\phi_{k;1}], \tag{35}$$

that are linear and constant in time. They form a complete set of local coordinates. A simple way to see the latter is to tend x_1 to infinity

$$\frac{\phi_{k;1}}{2(k+1)x_1} = z^k + \frac{(k+1)(z+\bar{z})z^k - 2z^{k+1}}{2(k+1)x_1} \to z^k, \quad \text{as} \quad x_1 \to \infty$$

observing that the set $\phi_{k;1}$ tends continuously to a basis $z^k, k = 0, 1, ..$ of functions analytic in a neighborhood of $z = 0$. The linearizing coordinates (35) are, therefore in one to one correspondence with harmonic moments $\int_\Omega z^k dXdY$, at least in some neighborhood of infinity. The corresponding variable-coefficient Hele-Shaw problem also transforms continuously into the constant coefficient one. Since harmonic moments are local coordinates for a generic set of simply-connected domains Ω (see e.g [13] and references therein), so are $M[\phi_k], k = 0, 1, ...$

A similar argument about completeness may be applied to the rest of examples considered below. For instance, it is convenient to use the following set of regular solutions to (29) when dealing with an arbitrary n problem for a flow, driven by a finite combination of multipole sources located at $z = z_1$

$$\phi_0^{(n)} = 1, \quad \phi_k^{(n)}(z, \bar{z}) = T_n^{(n)}[(z - z_1)^{n+k}], \quad \bar{\phi}_k^{(n)}(\bar{z}, z) = T_n^{(n)}[(\bar{z} - \bar{z}_1)^{n+k}], \quad k \geq 1 \tag{36}$$

The following class of variable coefficient Hele-Shaw problem, whose conserved quantities write in terms of polynomials in $\bar{z}, z$, generalizes (28)

$$\kappa = \frac{1}{(z^m + \bar{z}^m)^{2n}(z^m - \bar{z}^m)^{2l}}, \quad \eta = 1, \quad n > l \geq 0 \tag{37}$$

An elliptic operator (8), corresponding to (37)

$$L=\mathcal{L}_{n,l;m}:=2\frac{\partial^2}{\partial z\partial\bar z}-\frac{2nm}{z^m+\bar z^m}\left(z^{m-1}\frac{\partial}{\partial\bar z}+\bar z^{m-1}\frac{\partial}{\partial z}\right)-\frac{2lm}{z^m-\bar z^m}\left(z^{m-1}\frac{\partial}{\partial\bar z}-\bar z^{m-1}\frac{\partial}{\partial z}\right)\quad(38)$$

equals, up to a gauge transformation, the Schrodinger operator of the Calogero-Moser system related to (dihedral) group of symmetries of a regular $4m$-polygon ($2m$-polygon if $l=0$).

$$\mathcal{H}_{n,l;m}=(z^m+\bar z^m)^n(z^m-\bar z^m)^l\mathcal{L}_{n,l;m}(z^m+\bar z^m)^{-n}(z^m-\bar z^m)^{-l}=\frac{\partial^2}{\partial\rho^2}+\frac{1}{r}\frac{\partial}{\partial\rho}+\frac{S_{n,l;m}}{\rho^2},$$
$$S_{n,l;m}=\frac{\partial^2}{\partial\theta^2}-\frac{m^2n(n+1)}{\cos(m\theta)^2}-\frac{m^2l(l+1)}{\sin(m\theta)^2}$$

By analogy with S_n in (31), $S_{n,l;m}$ admits factorizations leading to the intertwining identity

$$T_{n,l;m}\Delta=\mathcal{L}_{n,l;m}T_{n,l;m}$$

where the intertwining operator $T_{n,l;m}$ can be represented in a form of the Wronskian [1]

$$T_{n,l;m}[f]=\rho^{m(n+l)}\frac{W[\psi_1,\psi_2,...,\psi_n,f]}{\cos(m\theta)^{n(n-1)/2}\sin(m\theta)^{l(l-1)/2}},\quad W[f_1,..,f_k]:=\det\left[\frac{\partial^{j-1}f_i}{\partial\theta^{j-1}}\right]_{1\le i,j\le k}\quad(39)$$

with

$$\psi_k=\begin{cases}\sin\left(k(m\theta+\pi/2)\right), & k=1,2,..n-l\\ \cos\left((2k+l-n)(m\theta+\pi/2)\right), & k=n-l+1,n-l+2,..n\end{cases}$$

$T_{n,l;m}$ is a differential operator of the nth order and is a homogenous polynomial in $z,\bar z,\frac{\partial}{\partial z},\frac{\partial}{\partial\bar z}$. It transforms any holomorphic function $f(z)$ into a polynomial in $\bar z$, which is crucial for construction of quadrature domains.

It is noteworthy that the intertwining operator $T_{n,l;m}$ is also not unique. Although, in difference from the $l=0,m=1$ case we cannot apply a simple argument connected with the translational invariance of the system along the Y-direction, another intertwining operator, constructed by the Dunkl method [2], exists. Various operators connecting Δ with $\mathcal{L}_{n,l;m}$, that are compositions of the intermediate intertwining operators constructed by the Dunkl method and those constructed through factorizations, can be also obtained. Similarly to the $l=0,m=1$ case, images of all so obtained intertwining operators, acting on harmonic functions, coincide for fixed n,l, and to construct a complete set of solutions to (38) one may use just one of them, say (39).

8 Construction of Gauge-non-trivial Quadrature Domains

Let us start from gauge-non-trivial examples of the Hele-Shaw flows (5), (28), that are driven by a multipole source located at $z=z_1$. Since the quadrature domain Ω can be viewed as a solution to the Hele-Shaw problem that develops continuously in time, starting from zero initial data, its form is defined by the condition that the quadrature identity (12) holds for any solution of (29), regular in Ω.

We show first, that curves, parametrized by polynomial maps of any non-negative degree $s\ge 0$

$$z(w)=z_1+rw+\sum_{k=2}^{s}u_{k-1}w^k,\quad z(e^{i\theta})\in\partial\Omega,\quad(40)$$

where r can be made to be real, are boundaries of the quadrature domains for solutions of (29). As in the constant-coefficient problem, forms, sizes and positions of these domains are functions of $s+1$ free complex parameters $z_1, r, u_k, k = 2..s$.

As follows from the previous section, any solution to (29), that is regular in Ω can be represented as (34)

$$\phi = 2T_n^{(n)}[f(z)] = \sum_{k=0}^{n} \frac{a_{k;n}}{2^k}(z+\bar{z})^k \frac{\partial^k f(z)}{\partial z^k}, \quad a_{n;n} = 1$$

where $f(z)$ is analytic in vicinity of $z = z_1$. Substituting this solution into (12) and using the Green theorem, we can transform the integral over Ω in the left-hand side of (12) to the line integral

$$\int_\Omega \phi dXdY = \frac{1}{2i} \oint_{|w|=1} \sum_{k=0}^{n} \frac{a_{k;n}}{2^k(k+1)} \left((z(w) + \bar{z}(1/w))^{k+1} \left(\frac{1}{\frac{\partial z}{\partial w}} \frac{\partial}{\partial w} \right)^k [f(z(w))] \right) dw$$

Using (40) and taking analyticity of $f(z)$ into account, we get

$$\int_\Omega \phi dXdY = \sum_{i=0}^{(n+1)(s+1)-2} U_i(z_1, r, u_1, .., u_s, \bar{u}_1, .., \bar{u}_s) \left(\frac{\partial^i f(z)}{\partial z^i} \right)_{z=z_1} \tag{41}$$

Comparing (41) with the right-hand side of (12), we see that the latter must contain $s(n+1)-1$ derivatives of ϕ and is equal to

$$\sum_{i=0}^{(s+1)(n+1)-2} V_i \left(\frac{\partial^i f(z)}{\partial z^i} \right)_{z=z_1} \tag{42}$$

where, V_i are linear functions of $Q_1^{(0)}, ..Q_1^{(s(n+1)-1)}, \bar{Q}_1^{(0)}, ..\bar{Q}_1^{(s(n+1)-1)}$. Eqs. (41) and (42) lead to the non-homogenous over-determined linear system of $2(s+1)(n+1)-1$ equations

$$U_i = V_i, \quad \bar{U}_i = \bar{V}_i, \quad i = 0..(s+1)(n+1)-2, \quad Q_1^{(0)} = \bar{Q}_1^{(0)} \tag{43}$$

for $2s(n+1)$ unknowns $Q_1^{(k)}, \bar{Q}_1^{(k)}, k = 0..s(n+1)-1$. Although the number of equations exceeds the number of unknowns by $2n+1$, not all these equations are independent. For instance, solving (43) for circular domains, $s = 1, z = z_1 + rw$, we get the simplest gauge-non-trivial quadrature identities

$$\int_{|z-z_1|\leq r} \phi dXdY = \pi r^2 \phi(z_1) + \frac{\pi r^4}{4x_1} \left(\frac{\partial \phi}{\partial z} + \frac{\partial \phi}{\partial \bar{z}} \right)_{z=z_1}, \quad z_1 = x_1 + iy_1$$

$$\int_{|z-z_1|\leq r} \phi dXdY = \pi r^2 \phi(z_1) + \frac{\pi r^4 (r^2 + 12x_1^2)}{24x_1^3} \left(\frac{\partial \phi}{\partial z} + \frac{\partial \phi}{\partial \bar{z}} \right)_{z=z_1} + \frac{\pi r^6}{24x_1^2} \left(\frac{\partial^2 \phi}{\partial z^2} + \frac{\partial^2 \phi}{\partial \bar{z}^2} \right)_{z=z_1}$$

for solutions of (29) with $n = 1$ and $n = 2$ respectively. Moreover, a circular solution to the arbitrary n variable-coefficient Hele-Shaw problem (28) exists, if the flow is driven by a special combination of a monopole, dipole ... $n+1$-pole sources located at the same point z_1.

To show that the polynomial solutions of an arbitrary non-negative degree exist for any n in (28), (i.e that the number of independent equations in arbitrary n, s case equals the number of unknowns

$Q_1^{(i)}, \bar{Q}_1^{(i)}$ and these equations are compatible), it is more convenient to check the quadrature identity (12) on a complete set (36) of solutions to (29).

According to (41), (42) and (36), we have to evaluate

$$\sum_{j=0}^{(s+1)(n+1)-2} (V_j - U_j) \frac{\partial^j f(z)}{\partial z}, \qquad \sum_{j=0}^{(s+1)(n+1)-2} \left(\bar{V}_j - \bar{U}_j\right) \frac{\partial^j f(\bar{z})}{\partial \bar{z}^j} \tag{44}$$

on $f(z) = (z - z_1)^{n+k}$ (and $f(\bar{z}) = (\bar{z} - \bar{z}_1)^{n+k}$ respectively) at point $z = z_1$ for $k \geq 1$. Also there is one more equation, obtained by substituting $\phi = \phi_{0;1}^{(n)} = 1$ in (12). The latter is equation for $Q_1^{(0)}$, that, as seen from (12), has a real solution, and equation $Q_1^{(0)} = \bar{Q}_1^{(0)}$ in (43) is satisfied. So, there remains $2s(n+1) - 2$ unknowns $Q_1^{(j)}, \bar{Q}_1^{(j)}, j = 1..s(n+1) - 1$.

The highest derivatives in (44) are of the order $(s+1)(n+1) - 2$, so that $(z - z_1)^j$ (or $(\bar{z} - \bar{z}_1)^j$ for the second equation in (44)) is annihilated at $z = z_1$ if $j > (s+1)(n+1) - 2$. As a consequence, (44) is identically satisfied for $k > s(n+1) - 1$ in (36), and there remains $2s(n+1) - 2$ equations and the equal number of unknowns $Q_1^{(j)}, \bar{Q}_1^{(j)}, j = 1..s(n+1) - 1$.

We now have to prove the compatibility of remaining equations. (44) is a non-homogenous system of linear equations for unknowns $Q_1^{(0)}, Q_1^{(k)}, \bar{Q}_1^{(k)}$, fixing their dependence on parameters $z_1, \bar{z}_1, r, u_1, ..., u_s, \bar{u}_1, ..., \bar{u}_s$. Such a system is compatible if its determinant does not vanish i.e. its homogenous part does not have nontrivial solutions. Let us suppose that it does. Remind that the homogenous part of the system has been obtained by the action of the operator

$$\hat{Q}_1 = Q_1^{(0)} + \sum_{j=1}^{s(n+1)-1} \left(Q_1^{(i)} \frac{\partial^i}{\partial z^i} + \bar{Q}_1^{(i)} \frac{\partial^i}{\partial \bar{z}^i} \right) \tag{45}$$

in the right hand-side of (12), to an arbitrary solution of (29) at point $z = z_1$. So, if the determinant of the system vanished, then there would exist such $Q_1^{(0)}, Q_1^{(k)}, \bar{Q}_1^{(k)}, k = 1..(s+1)n-1$, that the operator (45) would annihilate any solution of (29) at $z = z_1$, i.e.

$$\hat{Q}_1 T_n^{(n)}[f(z)] = 0, \quad \text{at} \quad z = z_1 \tag{46}$$

where $f(z)$ is any analytic in Ω function. If the above were true, then changing continuously the position $z = z_1$ of the source, we could pick such coefficients $Q_1^{(0)} = Q_1^{(0)}(z, \bar{z}), Q_1^{(i)} = Q_1^{(i)}(z, \bar{z}), \bar{Q}_1^{(i)} = \bar{Q}_1^{(i)}(z, \bar{z})$, that (46) would hold at each point in some region of the plane. In so doing we could construct a such operator $\hat{Q}_1$, with coefficients depending on z, that $\hat{Q}_1 T_n^{(n)}[f(z)] = 0$ in some region of the plane for an arbitrary $f(z)$. But this is evidently impossible, since the highest symbols of $\hat{Q}_1$ and $T_n^{(n)}$ contain pure derivatives in z, so does their composition.

Thus, the system of equations (43) is compatible and has a unique solution

$$Q_1^{(0)} = Q_1^{(0)}(r, u_1, .., u_s, \bar{u}_1, ..\bar{u}_s), \quad Q_1^{(k)} = Q_1^{(k)}(x_1, r, u_1, .., u_s, \bar{u}_1, ..\bar{u}_s), \quad k = 1..s(n+1) - 1$$

defining quadrature domains with boundaries parametrized by polynomial conformal maps (40) of an arbitrary non-negative degree $s \geq 0$. These domains are solutions to the Hele-Shaw problems, describing the free-boundary flows driven by a multipole (combination of a monopole, dipole, ..., $s(n+1)$-pole) source located at point $z = z_1$.

Analogous analysis can be applied to the problems related to general dihedral systems (37), (38). It also leads to the conclusion about existence of polynomial solutions (40) that depend on $s+1$ free parameters. The flow, in this case, is driven by a combination of a monopole, dipole, ..$s(m(n+l)+1)$-pole sources located at $z = z_1$.

The polynomial solutions (40) to the Hele-Shaw problems are limiting cases of the rational conformal maps, analytic in the unit disc $|w| \leq 1$

$$z(w) = z_0 + \sum_{k=1}^{N} \frac{\alpha_k}{w^{-1} - \bar{w}_k}, \quad |w_i| < 1 \tag{47}$$

We will now show that (47) also defines solutions of the variable-coefficient problems under consideration.

By evaluating the left hand side of the quadrature identity (12) for $\phi = T_n^{(n)}[f(z)]$ in the domain defined by (47), we easily come to the conclusion that it equals a linear combination $\sum_{k=1}^{N} \sum_{i=0}^{2n} V_i^{(k)} \left(\frac{\partial^i f(z)}{\partial z^i}\right)_{z=z_k}$ of derivatives of $f(z)$ at points $z(w_k)$. Comparing it to the linear combinations of derivatives of ϕ, evaluated at

$$z_k = z(w_k) \tag{48}$$

in the right-hand side of (12), similarly to the polynomial case, we obtain a system of equations

$$\sum_{i=0}^{2n} (V_{i;k} - U_{i;k}) \left(\frac{\partial^i f(z)}{\partial z^i}\right)_{z=z_k} = 0, \quad \sum_{i=0}^{2n} (\bar{V}_{i;k} - \bar{U}_{i;k}) \left(\frac{\partial^i f(\bar{z})}{\partial \bar{z}^i}\right)_{z=z_k} = 0, \quad Q_k^{(0)} = \bar{Q}_k^{(0)}, \quad k = 1,..N \tag{49}$$

Let us now select $4n+3$ equations of the $k=1$ subset of above $N(4n+3)$ equations. Evaluation of the quadrature identity (12) on the basis

$$\phi_i^{(n)} = T_n^{(n)}[(z - z_1)^{n+i}], \quad \bar{\phi}_i^{(n)} = T_n^{(n)}[(\bar{z} - \bar{z}_1)^{n+i}], \quad i = 0, 1, 2, ..$$

of solutions to (29) is equivalent to substituting $f(z) = (z - z_1)^{n+i}, i = 0, 1, 2, ..$ into (49), which yields at most $2n+3$ independent equations in the $k=1$ subset of (49). Since (49) is invariant with respect to the permutation of sources, the same argument is applicable to any $k = 1, ..N$ subset of (49), and the total number of independent equations equals at most $(2n+3)N$. Choosing $(2n+3)N$ unknowns

$$\alpha_k, \bar{\alpha}_k, \quad Q_k^{(i)}, \bar{Q}_k^{(i)}, \quad i = 1..n, \quad \mathrm{Im} Q_k^{(0)}, \quad k = 1, ..N,$$

we can express their dependence on N real parameters $Q_k^{(0)} = \bar{Q}_k^{(0)}, k = 1..N$. The latter are variables that, together with $w_k, k = 1..N$, parametrize rational solutions. A similar set of variables parametrize the rational solutions in constant coefficient case [8], [9]. They define magnitudes of charges produced by sources at $z = z_1, ...z_k$ by time t and positions (48) of the sources respectively. The flow, corresponding to the above rational solutions to the problems (29), (28), is driven by a special combination of monopoles, dipoles, ... $n+1$-poles located at $z = z_1, ...z_N$.

An analogous result can be, without difficulty, extended to the problems (37), related to arbitrary dihedral Calogero-Moser systems, where the flow is driven by a combination of monopoles, dipoles, ...$m(n+l)+1$ poles.

9 Concluding Remarks

We have shown that, similarly to the constant coefficient Hele-Shaw problem, a class of the variable coefficient problems, connected with the dihedral quantum Calogero-Moser systems, admits rational solutions for the flows driven by a finite number of multipole sources. In particular, the spaces of rational solutions have same dimensions in both the constant and variable coefficient cases.

More generally, it would be of interest to extend the above class of solutions to analogs of so-called logarithmic "finger" solutions [5] and consider corresponding "abelian" domains [13]. In contrast to the rational solutions, the logarithmic solutions do not develop finite-time singularities when flows are driven by sinks.

It is also of interest to understand significance of the integrable variable coefficient Hele-Shaw systems from the analytic point of view as well as from that of the inverse-potential problem.

Finally, one can consider the variable-coefficient problems involving multiply-connected fluid regions. In the constant coefficient case, the Cauchy transform of the domains is a convenient tool for solving such problems [9]. For the variable coefficient problems considered here, the following integral transforms of the domains

$$C_{n,l;m}[\Omega](X',Y') = \frac{1}{\pi}\int_{\Omega} T_{n,l;m}[(z-z')^{-1}]dXdY, \quad z' = X' + iY' \tag{50}$$

rather than Cauchy transform should be used. Similarly to the Cauchy transform it is analytic outside Ω. Although (50) is not continuous at $\partial\Omega$, its jump across the boundary can be easily evaluated by elementary methods of the complex variables and the form of the domain can be found exploiting properties of the analytic continuation of (50) into Ω.

Acknowledgement

The author would like to acknowledge useful information and help received from S. Howison and J. Ockendon. This work is supported by the European Community IIF MIF1-CT-2005-007323.

References

[1] Yuri Yu. Berest, Igor M. Loutsenko, Huygens' principle in Minkowski spaces and soliton solutions of the Korteveg-de Vries Equation, Comm.Math.Phys. 190, 113-132 (1997), solv-int/9704012

[2] Dunkl C.F., Differential-difference operators associated to reflection groups, Trans.Amer.Math.Soc. 311 (1989), 181-191

[3] Galin, L. A. Unsteady filtration with a free surface. C. R. (Doklady) Acad. Sci. URSS (N.S.) 47, (1945). 246–249

[4] J. Harnad, I. Loutsenko, O. Yermolayeva, Constrained reductions of the 2dToda hierarchy, Hamiltonian structure and interface dynamics, to appear in J.Math.Phys, math-ph/0312058

[5] Howison, S. D. , Fingering in Hele-Shaw cells, J.Fluid Mech, 167 (1986)

[6] M. Mineev-Weinstein, P. Wiegmann, A. Zabrodin, Integrable Structure of Interface Dynamics, Phys. Rev. Lett. 84, (2000)5106-5109, nlin.SI/0001007

[7] Polubarinova-Kotschina, P. J. On the displacement of the oil-bearing contour. C. R. (Doklady) Acad. Sci. URSS (N. S.) 47, (1945). 250–254

[8] Richardson S. Hele Shaw flows with a free boundary produced by the injection of fluid into a narrow channel. J.Fluid Mech., 56, part 4, (1972), pp.609-618

[9] Richardson S. Hele-Shaw flows with time-dependent free boundaries involving a multiply- connecetd fluid region, Euro.J.Appl.Math., 12 (2001) 571-599

[10] Sakai M. Quadrature domains, Lecture Notes in Mathematics, (1978) 934, Springer-Verlag

[11] H.S. Shapiro, The Schwartz function and its generalization to higher dimension, Wiley, new York, (1992)

[12] K.Takasaki, T.Takebe, Integrable hierarchies and dispersionless limit. Rev. Math. Phys. 7 (1995), no. 5, 743–808, hep-th/9405096

[13] A.N. Varchenko and P.I. Etingof, Why the boundary of a round drop becomes a curve of order four, American Mathematical Society, University Lecture Series, 3, (1994)

[14] Wiegmann, P. B.; Zabrodin, A., Conformal maps and integrable hierarchies, Comm. Math. Phys. 213 (2000), no. 3, 523–538, hep-th/9909147