IOP FTC

Institute of Physics Publishing — Journal of Physics A: Mathematical and Theoretical

J. Phys. A: Math. Theor. **40** (2007) F1–F8 doi:10.1088/1751-8113/40/1/F01

FAST TRACK COMMUNICATION

Seed and soliton solutions for Adler's lattice equation

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Received 27 September 2006, in final form 27 October 2006
Published 6 December 2006
Online at stacks.iop.org/JPhysA/40/F1

Abstract

Adler's lattice equation has acquired the status of a master equation among 2D discrete integrable systems. In this paper we derive what we believe are the first explicit solutions of this equation. In particular it turns out to be necessary to establish a non-trivial seed solution from which soliton solutions can subsequently be constructed using the Bäcklund transformation. As a corollary we find the corresponding solutions of the Krichever–Novikov equation which is obtained from Adler's equation in a continuum limit.

PACS numbers: 02.30.Ik, 05.45.Yv

1. Introduction

In [1] Adler derived a quadrilateral lattice equation as the nonlinear superposition principle for Bäcklund transformations (BTs) of the Krichever–Novikov (KN) equation [2, 3]. This equation can be written as follows (cf [4]):

$$\begin{aligned} A[(u-b)(\widehat{u}-b)-(a-b)(c-b)][(\widetilde{u}-b)(\widehat{\widetilde{u}}-b)-(a-b)(c-b)] \\ +B[(u-a)(\widetilde{u}-a)-(b-a)(c-a)][(\widehat{u}-a)(\widehat{\widetilde{u}}-a)-(b-a)(c-a)] \\ = ABC(a-b), \end{aligned} \tag{1.1}$$

where $u=u_{n,m}$, $\widetilde{u}=u_{n+1,m}$, $\widehat{u}=u_{n,m+1}$, $\widehat{\widetilde{u}}=u_{n+1,m+1}$ denote the values of a scalar-dependent variable defined as a function of the independent variables $n,m\in\mathbb{Z}^2$. The parameters (a,A), (b,B) and (c,C) in (1.1) are related points on the Weierstrass elliptic curve and can be written in terms of the Weierstrass $\wp$-function:

$$\begin{aligned} &(a,A)=(\wp(\alpha),\wp'(\alpha)), \qquad (b,B)=(\wp(\beta),\wp'(\beta)), \\ &(c,C)=(\wp(\beta-\alpha),\wp'(\beta-\alpha)). \end{aligned} \tag{1.2}$$

The main integrability property of this equation is that of multidimensional consistency, cf [5, 6], which implies that solutions of (1.1) can be embedded in a multidimensional lattice such that they obey simultaneously equations of a similar form (albeit with different choices of the lattice parameters) in all two-dimensional sublattices. Equation (1.1) emerged as the most general equation in a classification of scalar quadrilateral lattice equations integrable in this sense, [7], which includes the previously known cases of lattice equations of Korteweg-de Vries type, cf [8, 9]. Furthermore, connections have been established between Adler's equation and other so-called elliptic integrable systems [10], in particular the elliptic Toda Lattice of Krichever [11] and the elliptic Ruijsenaars–Toda lattice [12]. Also in [13] integrable mappings of a generalized QRT type were found as periodic reductions of this equation. Among 2D scalar discrete integrable equations, Adler's equation has thus been revealed as remarkably pervasive, making its study of singular importance.

One well-established procedure to obtain explicit solutions is the application of BTs on known, perhaps elementary, solutions of the lattice equation. This is particularly natural here as the multidimensional consistency implies that the BT is inherent in the lattice equation itself. However, in order to implement the BT one needs an initial solution: the *seed* solution, and the establishment of elementary solutions that qualify as seed solutions turns out already to be a delicate problem in the case of Adler's equation. The resolution of this problem, and the subsequent construction of soliton solutions using the BT, is the main achievement of this paper.

2. The Jacobi form of Adler's equation

We will study Adler's equation in a different form from (1.1), which was found in [14], namely

$$p(u\widetilde{u}+\widehat{u}\widehat{\widetilde{u}})-q(u\widehat{u}+\widetilde{u}\widehat{\widetilde{u}})-r(u\widehat{\widetilde{u}}+\widetilde{u}\widehat{u})+pqr(1+u\widetilde{u}\widehat{u}\widehat{\widetilde{u}})=0, \tag{2.1}$$

where the parameters p, q and r are related to each other and can be expressed in terms of Jacobi elliptic functions with modulus k by introducing the points

$$\begin{aligned}(p,P)&=(\sqrt{k}\,\mathrm{sn}(\alpha;k),\mathrm{sn}'(\alpha;k)), &\quad (q,Q)&=(\sqrt{k}\,\mathrm{sn}(\beta;k),\mathrm{sn}'(\beta;k)),\\ (r,R)&=(\sqrt{k}\,\mathrm{sn}(\gamma;k),\mathrm{sn}'(\gamma;k)), &\quad \gamma&=\alpha-\beta\end{aligned} \tag{2.2}$$

(the primes denoting derivatives w.r.t. the first arguments of the Jacobi functions) on the elliptic curve

$$\Gamma=\{(x,X):X^2=x^4+1-(k+1/k)x^2\}. \tag{2.3}$$

It was pointed out by Adler and Suris to one of us that the *Weierstrass form* (1.1) and the *Jacobi form* (2.1) of Adler's equation are equivalent in the sense that there exists a Möbius transformation of the variables together with a bi-rational transformation of the parameters that takes one form into the other [15].

We adopt here the Jacobi form because the analysis leading to the solutions is simpler than for the Weierstrass form. For that purpose it is useful to view the relation between the parameters in terms of the Abelian group structure on the elliptic curve (2.3) which has the following rational representation:

$$\mathfrak{p}=\mathfrak{q}*\mathfrak{r}=\left(\frac{qR+rQ}{1-q^2r^2},\frac{Qq(r^4-1)-Rr(q^4-1)}{(1-q^2r^2)(rQ-qR)}\right), \tag{2.4}$$

where $\mathfrak{p}=(p,P)$, $\mathfrak{q}=(q,Q)$ and $\mathfrak{r}=(r,R)$. It can be verified by direct computation that the product in the group $*$ defined in this way is associative, it is obviously commutative, furthermore the identity is $\mathfrak{e}=(0,1)$ and the inverse of a point $\mathfrak{p}$ is $\mathfrak{p}^{-1}=(-p,P)$. We consider the lattice parameters of (2.1) to be the points $\mathfrak{p}=(p,P)$ and $\mathfrak{q}=(q,Q)$ on Γ, while r is found from the formula $\mathfrak{r}=\mathfrak{p}*\mathfrak{q}^{-1}$, which expresses the original relation between the uniformizing variables, $\gamma=\alpha-\beta$ introduced in (2.2). In fact, this relation encodes the addition formulae for the relevant Jacobi functions, where $\mathfrak{r}$ can be seen as the point obtained by a shift of $\mathfrak{p}$ on the elliptic curve defined by the point $\mathfrak{q}$. Alternatively, it is well known that a symmetric biquadratic equation defines a shift on an elliptic curve, see for example [16], and this can be made explicit by introducing the symmetric biquadratic

$$\mathcal{H}_{\mathfrak{r}}(p,q) = \frac{1}{2r}\left(p^2+q^2-(1+p^2q^2)r^2-2pqR\right). \tag{2.5}$$

From the factorization

$$\left(p-\frac{qR+rQ}{1-q^2r^2}\right)\left(p-\frac{qR-rQ}{1-q^2r^2}\right) = \frac{2r}{1-q^2r^2}\mathcal{H}_{\mathfrak{r}}(p,q) \tag{2.6}$$

we see that $\mathcal{H}_{\mathfrak{r}}(p,q)=0$ is satisfied if $\mathfrak{p}=\mathfrak{q}*\mathfrak{r}$ or $\mathfrak{p}=\mathfrak{q}*\mathfrak{r}^{-1}$, i.e., if $\alpha=\beta\pm\gamma$ in terms of the uniformizing variables, and in this way $\mathcal{H}=0$ defines an addition formula for Jacobi elliptic functions. The biquadratic $\mathcal{H}$ is connected to a symmetric *tri*quadratic where the three variables p, q, r, now appear on an equal footing,

$$H(p,q,r) = \frac{1}{2\sqrt{k}}(p^2+q^2+r^2+p^2q^2r^2) - \frac{\sqrt{k}}{2}(1+p^2q^2+q^2r^2+r^2p^2) + \left(k-\frac{1}{k}\right)pqr. \tag{2.7}$$

$H=0$ defines a shift on the same curve by the statement that $H(p,q,r)=0$ if $\mathfrak{p}=\mathfrak{q}*\mathfrak{r}*\mathfrak{s}^{-1}$ or $\mathfrak{p}=\mathfrak{q}*\mathfrak{r}^{-1}*\mathfrak{s}$ where $\mathfrak{s}=(\sqrt{k},0)$, (the latter being a branch point of the curve Γ). This triquadratic is Möbius related to a similar triquadratic expression for Weierstrass elliptic functions which was used originally by Adler [1] in the construction of (1.1). In the construction of (2.1) an equivalent role is played by $\mathcal{H}$.

In terms of the points $\mathfrak{p}, \mathfrak{q}$ on the curve Γ it is now useful to introduce the quadrilateral expression

$$\mathcal{Q}_{\mathfrak{p},\mathfrak{q}}(u,\widetilde{u},\widehat{u},\widehat{\widetilde{u}}) = p(u\widetilde{u}+\widehat{u}\widehat{\widetilde{u}}) - q(u\widehat{u}+\widetilde{u}\widehat{\widetilde{u}}) - \frac{pQ-qP}{1-p^2q^2}(u\widehat{\widetilde{u}}+\widetilde{u}\widehat{u}-pq(1+u\widetilde{u}\widehat{u}\widehat{\widetilde{u}})) \tag{2.8}$$

in terms of which (2.1) takes the form

$$\mathcal{Q}_{\mathfrak{p},\mathfrak{q}}(u,\widetilde{u},\widehat{u},\widehat{\widetilde{u}}) = 0. \tag{2.9}$$

The relation between $\mathcal{Q}$ and $\mathcal{H}$ can then be given by the equations

$$\mathcal{Q}_{\mathfrak{p},\mathfrak{q}}(u,\widetilde{u},\widehat{u},\widehat{\widetilde{u}})\mathcal{Q}_{\mathfrak{p},\mathfrak{q}^{-1}}(u,\widetilde{u},\widehat{u},\widehat{\widetilde{u}}) = \frac{4p^2q^2}{1-p^2q^2}(\mathcal{H}_{\mathfrak{p}}(u,\widetilde{u})\mathcal{H}_{\mathfrak{p}}(\widehat{u},\widehat{\widetilde{u}}) - \mathcal{H}_{\mathfrak{q}}(u,\widehat{u})\mathcal{H}_{\mathfrak{q}}(\widetilde{u},\widehat{\widetilde{u}})) \tag{2.10}$$

and

$$\mathcal{Q}_{\widehat{u}}\mathcal{Q}_{\widehat{\widetilde{u}}} - \mathcal{Q}\mathcal{Q}_{\widehat{u}\widehat{\widetilde{u}}} = 2pqr\mathcal{H}_{\mathfrak{p}}(u,\widetilde{u}), \tag{2.11}$$

cf [7], where in (2.11) we have suppressed the arguments and subscripts of $\mathcal{Q}_{\mathfrak{p},\mathfrak{q}}(u,\widetilde{u},\widehat{u},\widehat{\widetilde{u}})$ in order to make room for partial derivatives with respect to the arguments.

3. Seed solution

The cubic consistency of Adler's equation [5, 6] means that given one solution, u, of (2.9) the pair of ordinary difference equations for v

$$\mathcal{Q}_{\mathfrak{p},\mathfrak{l}}(u,\widetilde{u},v,\widetilde{v}) = 0, \qquad \mathcal{Q}_{\mathfrak{q},\mathfrak{l}}(u,\widehat{u},v,\widehat{v}) = 0, \tag{3.1}$$

are compatible and, moreover, if v satisfies this system it is also a solution of (2.9). We say that (3.1) forms an auto-Bäcklund transformation (BT) for (2.9). The new solution v may depend not only on u but also on the Bäcklund parameter $\mathfrak{l}$ and on one integration constant.

For Adler's equation in the Jacobi form (2.1) we can easily verify the solution

$$u = \sqrt{k}\,\mathrm{sn}(\xi_0 + n\alpha + m\beta; k), \tag{3.2}$$

where ξ_0 is an arbitrary constant and α, β are as defined in (2.2). It turns out, however, that this solution is not suitable as a *seed* for constructing soliton solutions using the BT. Specifically, for (3.2) the equations (3.1) for v are reducible and have only two solutions which we label $\overline{u}$ and $\underline{u}$, given by

$$\begin{aligned} \overline{u} &= \sqrt{k}\,\mathrm{sn}(\xi_0 + n\alpha + m\beta + \lambda; k), \\ \underline{u} &= \sqrt{k}\,\mathrm{sn}(\xi_0 + n\alpha + m\beta - \lambda; k), \end{aligned} \tag{3.3}$$

where λ is the uniformizing variable associated with $\mathfrak{l}$, i.e., $\mathfrak{l} = (l, L) = (\sqrt{k}\,\mathrm{sn}(\lambda; k), \mathrm{sn}'(\lambda; k))$. In fact (3.3) is nothing more than (3.2) extended into a third lattice direction associated with the BT (3.1) (hence the notation with $\overline{u}$ interpreted as forward shift on u in the direction of the BT, and $\underline{u}$ as a backward shift). This is an idea which we will use later and refer to as a covariant extension of the solution. Since the action of the BT only trivially extends (3.2), application of the BT on the solution (3.2) clearly does not lead to essentially new solutions. Therefore, in order to find soliton solutions for Adler's equation we require a *germinating* seed solution, i.e., a solution on which the action of the BT is non-trivial.

We proceed by making the hypothesis that there exists a solution of (2.9) which is a fixed point of its BT (3.1), i.e., which is constant in a third lattice direction associated with a particular Bäcklund parameter $\mathfrak{t} = (t, T) = (\sqrt{k}\,\mathrm{sn}(\theta; k), \mathrm{sn}'(\theta; k))$. In that case the BT reduces to the system

$$\mathcal{Q}_{\mathfrak{p},\mathfrak{t}}(u, \widetilde{u}, u, \widetilde{u}) = 0, \qquad \mathcal{Q}_{\mathfrak{q},\mathfrak{t}}(u, \widehat{u}, u, \widehat{u}) = 0. \tag{3.4}$$

Although it is not obvious that such fixed points exist, i.e., that the system (3.4) is compatible, we will show that for Adler's equation (2.1) such a solution indeed arises and provides a germinating seed.

To find such solutions we begin by considering first the equation of (3.4) involving the shift $u \mapsto \widetilde{u}$. From the expression (2.8) this is quadratic and symmetric in u and $\widetilde{u}$, and hence can be viewed as a quadratic correspondence having $n+1$ images on the nth iteration. What is not obvious is that as a multivalued map this correspondence commutes with its counterpart, the map $u \mapsto \widehat{u}$ defined by the second equation of (3.4), but this will become apparent below. From (2.8) the first equation of the system (3.4) is

$$\mathcal{Q}_{\mathfrak{p},\mathfrak{t}}(u, \widetilde{u}, u, \widetilde{u}) = 2pu\widetilde{u} - t(u^2 + \widetilde{u}^2) - \frac{pT - tP}{1 - p^2t^2}(2u\widetilde{u} - pt(1 + u^2\widetilde{u}^2)) = 0. \tag{3.5}$$

We introduce a new parameter $\mathfrak{p}_\theta = (p_\theta, P_\theta)$ defined by

$$p_\theta^2 = p\frac{pT - tP}{1 - p^2t^2}, \qquad P_\theta = \frac{1}{t}\left(p - \frac{pT - tP}{1 - p^2t^2}\right). \tag{3.6}$$

For later convenience we use the symbolic notation $\delta_\theta(\mathfrak{p}, \mathfrak{p}_\theta)$ to denote the correspondence between the original parameter $\mathfrak{p}$ and the θ-deformed parameter $\mathfrak{p}_\theta$, given by the relations (3.6). Thus, we can now write

$$\mathcal{Q}_{\mathfrak{p},\mathfrak{t}}(u, \widetilde{u}, u, \widetilde{u}) = t\left(2u\widetilde{u}P_\theta - u^2 - \widetilde{u}^2 + (1 + u^2\widetilde{u}^2)p_\theta^2\right) = -2p_\theta t\mathcal{H}_{\mathfrak{p}_\theta}(u, \widetilde{u}) = 0 \tag{3.7}$$

bringing (3.5) to the standard form of the biquadratic addition formula (2.5) for the Jacobi elliptic functions. As before, this defines a shift on an elliptic curve of the same type as (2.3) but now with a new modulus k_θ, namely

$$\begin{aligned} \Gamma_\theta &= \{(x, X) : X^2 = x^4 + 1 - (k_\theta + 1/k_\theta)x^2\}, \\ k_\theta + \frac{1}{k_\theta} &= 2\frac{1 - T}{t^2} = 2\frac{1 - \mathrm{sn}'(\theta; k)}{k\,\mathrm{sn}^2(\theta; k)}. \end{aligned} \tag{3.8}$$

The deformed parameter $\mathfrak{p}_\theta$ is a point on this deformed curve, $\mathfrak{p}_\theta \in \Gamma_\theta$, as can be verified by direct computation using the fact that the seed parameter and the original lattice parameter lie on the original curve, $\mathfrak{t}, \mathfrak{p} \in \Gamma$. What (3.7) tells us is that the solution of the first equation of (3.4) can be parametrized in terms of Jacobi elliptic functions associated with the deformed curve Γ_θ, and thus by introducing uniformizing variables on Γ_θ we can write the solution explicitly. Setting $\mathfrak{p}_\theta = (\sqrt{k_\theta}\,\mathrm{sn}(\alpha_\theta; k_\theta), \mathrm{sn}'(\alpha_\theta; k_\theta))$, $u = \sqrt{k_\theta}\,\mathrm{sn}(\xi_\theta; k_\theta)$ it follows that

$$\widetilde{\xi}_\theta = \xi_\theta \pm \alpha_\theta \quad \Longrightarrow \quad \mathcal{H}_{\mathfrak{p}_\theta}(u, \widetilde{u}) = 0 \quad \Longrightarrow \quad \mathcal{Q}_{\mathfrak{p},\mathfrak{t}}(u, \widetilde{u}, u, \widetilde{u}) = 0. \tag{3.9}$$

Here the deformed lattice parameter α_θ is related in a transcendental way to the original lattice parameter α through either of the equivalent relations:

$$k_\theta\,\mathrm{sn}^2(\alpha_\theta; k_\theta) = k\,\mathrm{sn}(\alpha; k)\,\mathrm{sn}(\alpha - \theta; k), \qquad \mathrm{sn}'(\alpha_\theta; k_\theta) = \frac{\mathrm{sn}(\alpha; k) - \mathrm{sn}(\alpha - \theta; k)}{\mathrm{sn}(\theta; k)}, \tag{3.10}$$

which follow from (3.6).

However, so far we have only dealt with the first part of (3.4); we must also solve simultaneously the second part. The crucial observation is that the deformed curve which emerges in the solution of the first part is *independent of the lattice parameter* $\mathfrak{p}$ *characterising the direction in the lattice*. Thus, if we proceed in the same way with solving the second equation in (3.4) we get exactly the same curve Γ_θ and hence the solutions of the latter are given by the shifts on the curve as follows:

$$\widehat{\xi}_\theta = \xi_\theta \pm \beta_\theta \Longrightarrow \mathcal{H}_{\mathfrak{q}_\theta}(u, \widehat{u}) = 0 \Longrightarrow \mathcal{Q}_{\mathfrak{q},\mathfrak{t}}(u, \widehat{u}, u, \widehat{u}) = 0, \tag{3.11}$$

where we have introduced the deformed variable $\mathfrak{q}_\theta$, again through the correspondence $\delta_\theta(\mathfrak{q}, \mathfrak{q}_\theta)$, and uniformizing variable β_θ by the relation $\mathfrak{q}_\theta = (\sqrt{k_\theta}\,\mathrm{sn}(\beta_\theta; k_\theta), \mathrm{sn}'(\beta_\theta; k_\theta))$. β_θ is related to β through the relation (3.10) with α replaced by β, but involving the same modulus k_θ.

Now, clearly on the curve Γ_θ the maps given in (3.9) and (3.11) commute, and this implies the compatibility of the solutions of both members of (3.4), i.e., we have a simultaneous solution which is the required seed solution we are looking for. In explicit form this new seed solution can be expressed as

$$u_\theta(n, m) = \sqrt{k_\theta}\,\mathrm{sn}(\xi_\theta(n, m); k_\theta), \qquad \xi_\theta(n, m) = \xi_{\theta,0} + n\alpha_\theta + m\beta_\theta, \tag{3.12}$$

where $\xi_{\theta,0}$ is an arbitrary integration constant. We note that the seed solution u_θ, distinguished by the label θ, reduces to the non-germinating seed (3.2) in the limit $\theta \longrightarrow 0$, i.e., $\mathfrak{t} \longrightarrow \mathfrak{e} = (0, 1)$, the identity on the curve. The equations (3.9) and (3.11) clearly have other solutions as well, corresponding to the choice of sign at each iteration, and these lead to different seeds. As a particular example, an *alternating seed solution* can be obtained by choosing a flip of sign at each iteration step of the seed map. We will refer to (3.12) as the canonical seed solution.

4. One-soliton solution

We will now show that the canonical seed solution germinates by applying the BT to it, i.e., by computing the one-soliton solution. We need to solve the simultaneous ordinary difference equations in v

$$\mathcal{Q}_{\mathfrak{p},\mathfrak{l}}(u_\theta, \widetilde{u}_\theta, v, \widetilde{v}) = 0, \qquad \mathcal{Q}_{\mathfrak{q},\mathfrak{l}}(u_\theta, \widehat{u}_\theta, v, \widehat{v}) = 0, \tag{4.1}$$

which define the BT $u_\theta \mapsto v$ with the Bäcklund parameter $\mathfrak{l}$. The seed itself can be covariantly extended in the lattice direction associated with this BT, that is we may complement (3.12) with the equation $\bar{\xi}_\theta = \xi_\theta + \lambda_\theta$, where the $^{-}$ denotes a shift in this new direction which is associated with the parameter $\mathfrak{l}$ and as before λ_θ is the uniformizing variable for $\mathfrak{l}_\theta$ defined by the relation $\delta_\theta(\mathfrak{l}, \mathfrak{l}_\theta)$. The problem of solving the system (4.1) can be simplified because this covariantly extended seed provides two particular solutions, i.e.,

$$\begin{aligned} &\mathcal{Q}_{\mathfrak{p},\mathfrak{l}}(u_\theta, \widetilde{u}_\theta, \overline{u}_\theta, \overline{\widetilde{u}}_\theta) = 0, \qquad &\mathcal{Q}_{\mathfrak{q},\mathfrak{l}}(u_\theta, \widehat{u}_\theta, \overline{u}_\theta, \overline{\widehat{u}}_\theta) = 0, \\ &\mathcal{Q}_{\mathfrak{p},\mathfrak{l}}(u_\theta, \widetilde{u}_\theta, \underline{u}_\theta, \underline{\widetilde{u}}_\theta) = 0, \qquad &\mathcal{Q}_{\mathfrak{q},\mathfrak{l}}(u_\theta, \widehat{u}_\theta, \underline{u}_\theta, \underline{\widehat{u}}_\theta) = 0. \end{aligned} \tag{4.2}$$

(Compare with the non-germinating seed for which these were the only solutions.) From the multilinearity of (2.8) equations (4.1) are discrete Riccati equations for v. The key observation is that since these equations share two solutions (4.2) they can be simultaneously reduced to homogeneous linear equations for a new variable ρ by the substitution

$$v = \frac{1}{1-\rho}\underline{u}_\theta - \frac{\rho}{1-\rho}\overline{u}_\theta. \tag{4.3}$$

After some manipulation the system for ρ found by substituting (4.3) into (4.1) can be written as

$$\widetilde{\rho} = \left(\frac{p_\theta l - l_\theta p}{p_\theta l + l_\theta p}\right)\left(\frac{1 - l_\theta \overline{p}_\theta u_\theta \widetilde{u}_\theta}{1 + l_\theta \underline{p}_\theta u_\theta \widetilde{u}_\theta}\right)\rho, \qquad \widehat{\rho} = \left(\frac{q_\theta l - l_\theta q}{q_\theta l + l_\theta q}\right)\left(\frac{1 - l_\theta \overline{q}_\theta u_\theta \widehat{u}_\theta}{1 + l_\theta \underline{q}_\theta u_\theta \widehat{u}_\theta}\right)\rho, \tag{4.4}$$

where we mildly abuse notation by introducing the modified parameters

$$\begin{aligned} \overline{p}_\theta &= \sqrt{k_\theta}\,\mathrm{sn}(\alpha_\theta + \lambda_\theta; k_\theta), \qquad & \underline{p}_\theta &= \sqrt{k_\theta}\,\mathrm{sn}(\alpha_\theta - \lambda_\theta; k_\theta), \\ \overline{q}_\theta &= \sqrt{k_\theta}\,\mathrm{sn}(\beta_\theta + \lambda_\theta; k_\theta), \qquad & \underline{q}_\theta &= \sqrt{k_\theta}\,\mathrm{sn}(\beta_\theta - \lambda_\theta; k_\theta) \end{aligned} \tag{4.5}$$

(p_θ and q_θ do not depend on lattice shifts). We take (4.4) as the defining equations for ρ, which we refer to as the plane-wave factor. The compatibility of this system for ρ can be verified directly, specifically $\widehat{\widetilde{\rho}} = \widetilde{\widehat{\rho}}$ as a consequence of the identity for the Jacobi sn function

$$\begin{aligned} &\left(\frac{1 - k^2\,\mathrm{sn}(\lambda)\,\mathrm{sn}(\alpha+\lambda)\,\mathrm{sn}(\xi)\,\mathrm{sn}(\xi+\alpha)}{1 + k^2\,\mathrm{sn}(\lambda)\,\mathrm{sn}(\alpha-\lambda)\,\mathrm{sn}(\xi)\,\mathrm{sn}(\xi+\alpha)}\right)\left(\frac{1 - k^2\,\mathrm{sn}(\lambda)\,\mathrm{sn}(\beta+\lambda)\,\mathrm{sn}(\xi+\alpha)\,\mathrm{sn}(\xi+\alpha+\beta)}{1 + k^2\,\mathrm{sn}(\lambda)\,\mathrm{sn}(\beta-\lambda)\,\mathrm{sn}(\xi+\alpha)\,\mathrm{sn}(\xi+\alpha+\beta)}\right) \\ &\quad = \left(\frac{1 - k^2\,\mathrm{sn}(\lambda)\,\mathrm{sn}(\beta+\lambda)\,\mathrm{sn}(\xi)\,\mathrm{sn}(\xi+\beta)}{1 + k^2\,\mathrm{sn}(\lambda)\,\mathrm{sn}(\beta-\lambda)\,\mathrm{sn}(\xi)\,\mathrm{sn}(\xi+\beta)}\right) \\ &\quad\quad \times \left(\frac{1 - k^2\,\mathrm{sn}(\lambda)\,\mathrm{sn}(\alpha+\lambda)\,\mathrm{sn}(\xi+\beta)\,\mathrm{sn}(\xi+\alpha+\beta)}{1 + k^2\,\mathrm{sn}(\lambda)\,\mathrm{sn}(\alpha-\lambda)\,\mathrm{sn}(\xi+\beta)\,\mathrm{sn}(\xi+\alpha+\beta)}\right). \end{aligned}$$

The one-soliton solution for the Jacobi form of Adler's equation, which we denote by $u^{(1)}$, is thus given by

$$u^{(1)}(n,m) = \frac{\sqrt{k_\theta}}{1-\rho(n,m)}\big(\mathrm{sn}\big(\xi_\theta(n,m) - \lambda_\theta; k_\theta\big) - \rho(n,m)\,\mathrm{sn}\big(\xi_\theta(n,m) + \lambda_\theta; k_\theta\big)\big), \tag{4.6}$$

with $\xi_\theta(n,m)$ as in (3.12) and $\rho(n,m)$ defined by (4.4).

5. Compatible continuous systems

As was pointed out in [7] Adler's equation goes to the Krichever–Novikov (KN) equation in a particular continuum limit. We present this limit for the Jacobi form of Adler's equation by first introducing continuous variables x and y through the truncated vertex operator

$$C_p = \mathrm{e}^{\sqrt{2p}(\partial_x + \frac{p}{6}\partial_y)}. \tag{5.1}$$

By writing $\widetilde{u} = C_p u$ and $\widehat{u} = C_q u$ the variable u on the lattice can be reinterpreted as a sampling of u at points on the (x, y) plane. In the limit as $\mathfrak{p}, \mathfrak{q} \longrightarrow (0, 1), (0, 1)$ we find

$$\begin{aligned} &\mathcal{Q}_{\mathfrak{p},\mathfrak{q}}(u, C_p u, C_q u, C_p C_q u) = 0 \quad \longrightarrow \\ &u_y - u_{xxx} + \frac{3}{2u_x}\left(u_{xx}^2 - u^4 - 1 + \left(k + \frac{1}{k}\right) u^2\right) = 0, \end{aligned} \tag{5.2}$$

i.e., Adler's equation goes to the KN equation. In the same formal limit the equations (3.1) and (3.4) go to

$$u_x v_x - \mathcal{H}_{\mathfrak{t}}(u, v) = 0 \qquad \text{and} \tag{5.3}$$

$$u_x^2 - \mathcal{H}_{\mathfrak{t}}(u, u) = 0 \tag{5.4}$$

respectively, that is the auto-BT for the KN equation, cf [1], and the equation for its seed solution. We note that other compatible differential-difference equations can also be obtained in this way, for example by writing $\widehat{u} = C_q C_p^{-1} \widetilde{u}$ and taking the limit $\mathfrak{q} \longrightarrow \mathfrak{p}$, Adler's equation (2.1) goes to

$$p(\widetilde{u} - \underset{\sim}{u})u_z - P(\widetilde{u} + \underset{\sim}{u})u + \widetilde{u}\,\underset{\sim}{u} + u^2 - p^2(1 + \widetilde{u}\,\underset{\sim}{u}\,u^2) = 0 \tag{5.5}$$

where $z = \sqrt{2p}(x + 2y/p)/P$. Interestingly, the vector field associated with the continuous flow of equation (5.5), which commutes with the lattice shifts, appears also in the form of the master symmetries of Adler's equation, cf [17].

To compute the seed solution of (5.2) one can solve (5.4) coupled with (5.2) itself, or alternatively one can take a continuum limit of the seed solution for Adler's equation (3.12). Either way the calculation is straightforward and we find

$$\begin{aligned} &u_\theta(x, y) = \sqrt{k_\theta}\,\mathrm{sn}(\xi_\theta(x, y); k_\theta), \\ &\xi_\theta(x, y) = \sqrt{-t/2}(x_0 + x - (2 + T)y/t)/\sqrt{k_\theta}, \end{aligned} \tag{5.6}$$

where x_0 is an arbitrary integration constant. This can be verified as a solution of (5.2) directly (and for fixed y as the general solution of (5.4)).

Calculation of the one-soliton solution can proceed by the continuum limit of (4.6) and (4.4), or by substitution of the continuous seed solution (5.6) into (5.3) followed by the identification of two particular solutions (this time from an extension of the continuous seed solution defined by (5.6) together with $\overline{\xi}_\theta(x, y) = \xi_\theta(x, y) + \lambda_\theta$). This calculation gives the one-soliton solution for (5.2) as

$$u^{(1)}(x, y) = \frac{\sqrt{k_\theta}}{1 - \rho(x, y)}(\mathrm{sn}(\xi_\theta(x, y) - \lambda_\theta; k_\theta) - \rho(x, y)\,\mathrm{sn}(\xi_\theta(x, y) + \lambda_\theta; k_\theta)) \tag{5.7}$$

with $\xi_\theta(x, y)$ as in (5.6). Here $\rho(x, y)$ is the continuous plane wave satisfying the following ODEs in terms of x and y:

$$\begin{aligned} &\rho_x(x, y) = \frac{-l_\theta}{l\sqrt{-t/2}}\left(\frac{1 - l^2 u_\theta^2(x, y)}{1 - l_\theta^2 u_\theta^2(x, y)}\right)\rho(x, y), \\ &\rho_y(x, y) = \frac{-2l_\theta}{lt\sqrt{-t/2}}\left(2T + \left(1 - l_\theta^2/l^2\right)(1 - l^2 t^2)\right)\rho(x, y) - \frac{2 + T}{t}\rho_x(x, y). \end{aligned} \tag{5.8}$$

Note that in the above equations the square roots in $\sqrt{k_\theta}$ and $\sqrt{-t/2}$ refer to the same choice of branch wherever they appear.

6. Concluding remarks

In this communication we have given solutions to Adler's lattice equation in its Jacobi form. The seed solution is found as a fixed point of the auto-Bäcklund transformation (BT) for this equation, and application of the BT (with different Bäcklund parameter) to the seed solution yields the one-soliton solution. The construction of the seed solution requires a deformation of the original elliptic curve in terms of which the lattice parameters of the equation were given. It seems that there are interesting modular transformations that arise through this construction. The one-soliton solution involves some functions which are solutions of a consistent set of first order homogeneous, linear, but non-autonomous equations involving the seed solution.

In order to calculate higher soliton solutions it suffices to apply the permutability condition of the BTs, which is implicit in the original lattice equation. In particular this means that higher soliton solutions will be rational functions of the seed and the one-soliton solution given here. These results and other details will be given in a separate publication [18]. With one exception (the lattice Schwarzian KdV equation) the methodology presented here works for all equations in the classification of Adler, Bobenko and Suris [7], and a list of these solutions has been obtained which will be included in [18]. The results presented here form, as far as we are aware, the first examples of explicit solutions for Adler's equation.

Acknowledgments

JA was supported by the UK Engineering and Physical Sciences Research Council (EPSRC), and FWN would like to thank the Isaac Newton Institute for mathematical sciences for its hospitality during the finalization of this communication.

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