Global Existence and Finite-Time Blow-up for a Class of Nonlocal Parabolic Problems

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Research supported by Nato Collaborative
Research Grant No. CRG 930070

ABSTRACT

An analysis of the nonlocal parabolic equation

$$\text{(P)} \qquad u_t - \Delta u = \frac{\delta f(u)}{(\int_\Omega f(u)\,dx)^p} \quad , \quad x \in \Omega, \quad t > 0,$$

and its associated steady state equation

$$\text{(S)} \qquad -\Delta u = \frac{\delta f(u)}{(\int_\Omega f(u)\,dx)^p} \quad x \in \Omega,$$

with Dirichlet boundary conditons is given, assuming $\Omega \subset I\!R^n$ is a smooth bounded domain, $\delta > 0$, $p \geq 0$, and f is positive and locally Lipschitz continuous.

Existence-nonexistence results are proven for (S) when $f(u) = e^u$ or e^{-u} and Ω is a ball or a star-shaped domain.

For $f(u) \geq c > 0$, $n = 1$, and $p \geq 1$, we prove that (P) has a global bounded solution for any nonnegative initial data $u_0(x)$ and any $\delta > 0$.

For $f(u) = e^u$, $n = 2$, $\Omega = B_1(0)$, $u_0(x)$ radially symmetric, non-negative, if $p > 1$, (P) has a unique, globally bounded solution for any $\delta > 0$. If $p = 1$, $0 < \delta < 8\pi$, (P) again has a bounded global solution.

For $f(u) = e^u$, $n = 1$ or 2, if $p < 1$ and $\delta > \delta^*$ where δ^* is critical value for existence-nonexistence for (S), then the solution u of (P) blows up in finite time.

AMS(MOS) Subject Classification: 35K40, 35K57, 35K60

Key Words and Phrases: parabolic-elliptic systems, nonlocal, stationary solutions, blow-up, global existence

1. Introduction

In this paper we consider nonlocal problems of the form

$$(1) \qquad \begin{cases} u_t - \Delta u = \dfrac{\delta f(u)}{(\int_\Omega f(u))^p} \quad , & \Omega \times (0, \infty), \\ u(x,t) = 0, \qquad x \in \partial\Omega \quad , & t > 0, \\ u(x,0) = u_0(x) \quad , \qquad \Omega & \end{cases}$$

and associated nonlocal stationary problems

$$(2) \qquad \begin{cases} -\Delta u = \dfrac{\delta f(u)}{(\int_\Omega f(u))^p} \quad , & \Omega, \\ u(x) = 0 \quad , & \partial\Omega, \end{cases}$$

where f is Lipschitz continuous and positive, $p \geq 0$, and $\delta > 0$. The set $\Omega \subset I\!\!R^n$ is a smooth bounded domain, and the initial condition $u_0(x)$ is nonnegative and assumed to be in $L^2(\Omega)$. (The last condition is easily relaxed.)

The following facts follow from the classical theory of partial differential equations:

1. For $u_0 \in L^2(\Omega)$, $\sup u_0 < \infty$, the initial boundary value problem IBVP (1) has a unique, nonextendable, classical solution on $\Omega \times [0, T)$ where either $T = +\infty$ or $T < +\infty$ and $\lim_{t \to T} \sup_\Omega u(x,t) = +\infty$.
2. Any solution of IBVP(1) or BVP(2) is positive for $x \in \Omega$ ($t > 0$) with outer normal derivative $\dfrac{\partial u}{\partial N} \leq 0$ for $x \in \partial\Omega$.
3. For $\Omega = B_1(0) = \{x : |x| < 1\}$,
   (a) any solution of BVP(2) is radially symmetric and radially decreasing,
   (b) if $u_0(x)$ is radially symmetric and radially decreasing, the solution of IBVP(1) is also.

Such nonlocal problems arise, for example, in the analytical study of phenomena associated with the occurrence of shear bands in metals being deformed under high strain rates (Burns [6], Olmstead [21], Bebernes and Talaga [3]), in modelling the phenomena of Ohmic heating (Lacey [18, 19]), in the investigation of the fully turbulent behaviour of a real flow, using invariant measures for the Euler equation (Caglioti, Lions, Marchioro, and Pulvirenti [8]), and in the theory of gravitational equilibrium of polytropic stars (Krzywicki and Nadzieja [17]). The one-dimensional version of the model considered here can, for $p > 1$, be thought of as representing one-dimensional fluid flows with rate of strain proportional to a power of stress multiplied by a function of temperature.

Our goal here is to show the interplay of p, the exponent of the nonlocal term, the parameter δ, and the spatial dimension n, both in determining the existence and non-existence for the boundary-value problem (2), and in differentiating between global existence and finite-time blow-up for the initial-boundary value problem (1).

Major emphasis will be placed on cases where the spatial dimension n is 1 or 2, the nonlinearity is $f(u) = e^u$ or e^{-u}, and the boundary conditions are Dirichlet. Extensions to higher dimensions, more general nonlinearities, and Robin boundary conditions will be discussed at the end of each section. We note immediately that for Neumann boundary data $\dfrac{\partial u}{\partial N} = 0$ on $\partial\Omega$ there can be no solution of BVP(2) and any solution to IBVP(1) must be unbounded.

The layout of the paper is as follows. In Section 2 we show that solutions for BVP(2) on any star-shaped domain cannot exist for spatial dimensions $n \geq 2$ for $\delta > \delta^*$ where δ^* depends on p, n, the volume of Ω, and a constant which "measures" the shape of the star-shaped domain Ω. In Section 3, we obtain explicit existence-nonexistence results with multiplicity for the case when Ω is a unit ball in $I\!R^n$. In Section 4, for $n = 1$ and 2 with $\Omega = B_1(0)$, conditions are found under which global bounded solutions for IBVP(1) exist. The question of finite-time blow-up for IBVP(1) is addressed in Section 5.

Our results when applied to the one-dimensional shear band formation problem give the following interesting information. The model developed by Burns in [7] is IBVP(1) with $f(u) = e^u$, $p = 1$, and spatial dimension $n = 1$. By Theorem 4.1, we have global existence of a unique bounded solution for all $\delta > 0$ and $p \geq 1$. This implies that no shear banding occurs in the Burns model. If however, the model is given with $p < 1$ and such a model is easily justified, then for all $\delta > \delta^*$, δ^* given in Theorem 3.1, the solution blows up in finite time by Theorem 5.6. This does predict shear banding as observed in the experiments of Marchand and Duffy [20].

2. Existence and Nonexistence of Steady States

First we consider BVP(2) where Ω is a *strictly star-shaped* domain containing 0, meaning Ω is an open connected set with smooth boundary with the property that there is a positive constant a such that

$$x \cdot N \geq a \int_{\partial\Omega} ds \qquad \text{for all } x \in \partial\Omega$$

where N is the unique outer normal to Ω at $x \in \partial\Omega$. Note that for $\Omega = B_1(0)$, then $a = (n\omega_n)^{-1}$ where $\omega_n = |B_1(0)|$.

Theorem 2.1

1) Let $f(u) = e^u$. If $\delta > \frac{2n}{a}|\Omega|^{p-1}$ where $p \leq 1$ and $n \geq 2$, then BVP(2) has no solution.

2) Let $f(u) = e^{-u}$. If $\delta > \frac{2n}{a}|\Omega|^{p-1}$ where $p \geq 2$ and $n \geq 2$, then BVP(2) has no solution.

Proof. Recall the well-known Pohozaev identity. If $n \geq 2$ and u is a solution of

$$\begin{cases} \Delta u + \delta g(u) = 0 & , \quad \Omega \\ u = 0 & , \quad \partial\Omega \end{cases}$$

then

$$n\delta \int_\Omega G(u)dx - \frac{n-2}{2}\delta \int ug(u)dx = \tfrac{1}{2} \int_{\partial\Omega} (x \cdot N) \left(\frac{\partial u}{\partial N}\right)^2 ds \tag{3}$$

where $G(u) = \int_0^u g(z)dz$.

For $g(u) = \frac{e^u}{K^p}$, where $K = \int_\Omega e^u dx$, $G(u) = \frac{e^u - 1}{K^p}$ and (3) becomes

$$\begin{aligned} n\frac{\delta}{K^p} \int_\Omega (e^u - 1)dx \quad & - \frac{n-2}{2}\frac{\delta}{K^p} \int ue^u dx = \tfrac{1}{2} \int_{\partial\Omega} (x \cdot N) \left(\frac{\partial u}{\partial N}\right)^2 ds \\ & \geq \frac{a}{2} \left(\int_{\partial\Omega} \left(-\frac{\partial u}{\partial N} \right) ds \right)^2 \\ & = \frac{a}{2} \left(\int_\Omega (-\Delta u) dx \right)^2 \end{aligned} \tag{4}$$

using the Divergence Theorem and Hölder's inequality. Since

$$\int_\Omega (-\Delta u)dx = \frac{\delta}{K^p} \int_\Omega e^u dx = \frac{\delta}{K^{p-1}},$$

$$\tfrac{1}{2} \int_{\partial\Omega} (x \cdot N) \left(\frac{\partial u}{\partial N}\right)^2 ds \geq \frac{a}{2} \frac{\delta^2}{K^{2(p-1)}}$$

¿From the left-hand side of (4), dropping negative terms, we have

$$\frac{n\delta}{K^p}\int_\Omega e^u dx \geq \frac{a}{2}\frac{\delta^2}{K^{2(p-1)}}.$$

Thus, if $u(x)$ is a solution of BVP(2), then $\frac{2n}{a}K^{p-1} \geq \delta$. Also $K \geq |\Omega|$ for any $u \geq 0$. Thus, if $p \leq 1$, $K^{p-1} \leq |\Omega|^{p-1}$. Hence, we have that if $f(u) = e^u$, $p \leq 1$, and $\delta > \frac{2n}{a}|\Omega|^{p-1}$, then BVP(2) has no solution.

For $f(u) = e^{-u}$, Pohozaev gives

$$(5) \quad \frac{n\delta}{K^p}\int_\Omega (1-e^{-u})dx - \frac{n-2}{2}\frac{\delta}{K^p}\int_\Omega ue^{-u}dx = \tfrac{1}{2}\int_{\partial\Omega}(x\cdot N)\left(\frac{\partial u}{\partial N}\right)^2 ds.$$

Thus

$$\frac{n\delta}{K^p}\int_\Omega e^{-u}dx \leq \frac{n\delta}{K^p}|\Omega| - \frac{a\delta^2}{2K^{2(p-1)}}$$

and hence

$$\frac{n\delta}{K^{p-1}} \leq \frac{n\delta}{K^p}|\Omega| - \frac{a\delta^2}{2K^{2(p-1)}} \quad ,$$

giving

$$\delta \leq \frac{2n}{a}K^{p-2}(|\Omega| - K) \leq \frac{2n}{a}K^{p-2}|\Omega|$$

since

$$0 \leq K = \int_\Omega e^{-u}dx \leq \int_\Omega e^0 dx = |\Omega|.$$

If $p \geq 2$, $f(u) = e^{-u}$, and

$$\delta > \frac{2n}{a}|\Omega|^{p-1}$$

then BVP(2) has no solution.

We now extend part 2 of Theorem 2.1 to non-star-shaped regions and more general functions. The proof is rather more involved.

Theorem 2.2

1) Suppose that $\int_0^\infty f(u)\,du = 1$. Then for $p < 2$ there is a solution of BVP(2) for all $\delta > 0$. For $p = 2$ there is a solution for all $\delta < 2|\partial\Omega|^2$.

2) Suppose additionally that f is decreasing. Then for $p \geq 2$ there is some δ^* such that BVP(2) has no solution if $\delta > \delta^*$. For $p = 2$, $\delta^* \geq 2|\partial\Omega|^2$.

It should be observed that for any case where $\int_0^\infty f(u)\,du < \infty$ a rescaling is permissible to make this integral 1.

Proof. We let $V(y;\mu)$ be defined for $0 \leq y \leq \mu$ and solve

$$V'' + f(V) = 0 \text{ for } 0 < y < \mu, \text{ with } V(0) = V'(\mu) = 0.$$

We first use V to obtain a lower bound on the solution of

$$\Delta u + \lambda f(u) = 0 \text{ in } \Omega,\ u = 0 \text{ on } \partial\Omega,$$

when λ is large. The comparison function will be

$$v(x;\mu,\nu,\lambda) = \begin{cases} V(\sqrt{\nu\lambda}\, d(x,\partial\Omega);\mu) & \text{for } d(x,\partial\Omega) \le \mu/\sqrt{\nu\lambda} \\ V(\mu;\mu) & \text{for } d(x,\partial\Omega) \ge \mu/\sqrt{\nu\lambda} \end{cases},$$

where $d(x,\partial\Omega)$ is the distance between $x \in \Omega$ and the boundary $\partial\Omega$; d is smooth and $|\Delta d| < K$ for some K in a neighbourhood of the boundary if $\partial\Omega$ is smooth, in particular, for

$$d \le \frac{\mu}{\sqrt{\nu\lambda}} < \rho$$

with ρ being smaller than the infimum, over $x \in \partial\Omega$, of the radii of the largest interior ball touching the boundary at x.

Clearly v satisfies the correct boundary condition and is C^1 while

$$\begin{aligned} \Delta v + \lambda f(v) &= \lambda f(V(\mu)) > 0 \quad \text{for} \quad d(x,\partial\Omega) > \mu/\sqrt{\nu\lambda}, \\ \Delta v + \lambda f(v) &= \nu\lambda|\nabla d|^2 V'' + \sqrt{\nu\lambda}\, V'\Delta d + \lambda f(V) \\ &= (1-\nu)\lambda f(V) + \sqrt{\nu\lambda}\, V'\Delta d \quad \text{where} \quad d(x,\partial\Omega) < \mu/\sqrt{\nu\lambda} \\ &> (1-\nu)\lambda f(V) - \sqrt{\lambda} K V' \ge 0 \quad \text{for} \quad \nu < 1, \end{aligned}$$

provided that

$$\nu \le 1 - \frac{KG(\mu)}{\sqrt{\lambda}} \quad \text{where} \quad G(\mu) = \sup_{y\in(0,\mu)} \frac{V'}{f(V)}.$$

On choosing $\nu = 1 - KG(\mu)/\sqrt{\lambda}$, v is a lower solution for sufficiently large λ. Then $u \ge v$ and, in particular,

$$-\frac{\partial u}{\partial N} \ge -\frac{\partial v}{\partial N} = \sqrt{\nu\lambda} V'(0;\mu) \text{ for } x \text{ on } \partial\Omega$$

since $u = v$ on the boundary. Now $V'^2 = 2\int_V^M f(s)\,ds = 2\int_V^\infty f(s)\,ds - 2\int_M^\infty f(s)\,ds = 2(F(V) - F(M))$, where $F(u) = 2\int_u^\infty f(s)\,ds$ and $M(\mu) = \max V = V(\mu;\mu)$, so $\mu = \dfrac{1}{\sqrt{2}}\displaystyle\int_0^M [F(s) - F(M)]^{-\frac{1}{2}} ds \to \infty$ as $M \to \infty$.

It follows that as $\mu \to \infty$, $M \to \infty$, $V \to W$, which is given by

$$y = \frac{1}{\sqrt{2}} \int_0^W F(s)^{-\frac{1}{2}} ds,$$

$V' \to W' = \sqrt{2F(W)}$, and, in particular, $V'(0;\mu) \to \sqrt{2F(0)} = \sqrt{2}$.

Now choose some $\Lambda(\lambda) \le C\sqrt{\lambda}$, for a fixed positive $C < \rho$, such that $G(\Lambda)/\sqrt{\lambda} \to 0$ as $\lambda \to \infty$.

[It can be observed that fixing μ means that $G(\mu)/\sqrt{\lambda} \to 0$ as $\lambda \to \infty$ but $V'(0;\mu) = \sqrt{2(F(0) - F(M))} < \sqrt{2}$. On the other hand $G(\mu) \to \infty$ as $\mu \to \infty$. The latter observation follows from the unboundedness of $W'/f(W)$: If $W'/f(W)$ were bounded there would be some K such that $\sqrt{2F(W)}/f(W) \leq K$ for all y and hence W. Then $f(W)/\sqrt{2F(W)} = -\frac{d}{dW}F(W)/\sqrt{2F(W)} \geq 1/K$ and $-\sqrt{2F(W_2)} + \sqrt{2F(W_1)} \geq (W_2 - W_1)/K$ for $0 \leq W_1 \leq W_2$. Taking $W_2 \to \infty$ so that $F(W_2) \to 0$ produces a contradiction.]

The unboundedness of $G(\mu)$ might put an additional restriction on $\Lambda(\lambda)$ but for our present purposes this is of no importance.

We now have: for $\lambda \to \infty$, $-\frac{\partial u}{\partial N} \geq \sqrt{2\nu\lambda[1 - F(M(\Lambda(\lambda)))]}$ where $\nu \to 1$ and $F(M) \to 0$.

However the δ corresponding to such a λ is

$$\lambda\left(\int_\Omega f(u)\,dx\right)^p = \lambda^{1-p}\left(\int_{\partial\Omega}\left|\frac{\partial u}{\partial N}\right|\,dx\right)^p, \quad \text{so}$$

$$\delta \geq 2^{p/2}|\partial\Omega|^p\lambda^{1-p/2}[\nu(1 - F(M))]^{p/2} \sim 2^{p/2}|\partial\Omega|^p\lambda^{1-p/2} \text{ for } \lambda \to \infty.$$

Since: (I) $\delta \to 0$ as $\lambda \to 0$; (II) δ can be made arbitrarily large for $p < 2$; (III) δ can be made arbitrarily close to $2|\partial\Omega|^2$ for $p = 2$; the first part of the theorem is proved.

Let us next consider the case of an annular region $\mathcal{A} = B_R(x_0)\backslash B_\rho(x_0)$ with $0 < \rho < R$. The above proof of the first part of the theorem allows us to construct a lower solution $v \leq u$ for given large λ and having a boundary layer structure:

$$v = \begin{cases} V(\sqrt{\nu\lambda}(r-\rho)) & \text{for} \quad \rho \leq r \leq \rho + \mu/\sqrt{\nu\lambda} \\ M & \text{for} \quad \rho + \mu/\sqrt{\nu\lambda} \leq r \leq R - \mu/\sqrt{\nu\lambda} \\ V(\sqrt{\nu\lambda}(r-\rho)) & \text{for} \quad R - \mu/\sqrt{\nu\lambda} \leq r \leq R \end{cases},$$

where $r = |x - x_0|$.

A new comparison function z is given by the solution to

$$\Delta z + \lambda f(v) = 0 \text{ in } \mathcal{A},\ z = 0 \text{ on } \partial\mathcal{A}.$$

Since f is now decreasing

$$\Delta u + \lambda f(v) = \lambda(f(v) - f(u)) \geq 0$$

so u is a lower solution for z and $u \leq z$.

Taking such a z for a region $\mathcal{A} \supset \Omega$ (*i.e.* R is large enough) with the inner boundary touching $\partial\Omega$ at some point x_1, $\{x : |x - x_1| = \rho\} \subset \Omega^c$, then $u(\cdot;\Omega) \leq u(\cdot;\mathcal{A}) \leq z$ and $-\frac{\partial u}{\partial N}(x_1;\Omega) \leq \frac{dz}{dr}|_{r=\rho}$.

¿From the radial symmetry of the z-problem z can be determined in terms of a Green's function for the two-point boundary-value problem

$$(r^{n-1}z')' + \lambda r^{n-1} f(v) = 0 \quad \rho < r < R, \quad z(\rho) = z(R) = 0.$$

For $n = 2$

$$z(r) = \frac{\lambda}{\ln(R/\rho)} \left\{ \ln(R/r) \int_\rho^r f(v(s)) s \ln(s/\rho)\, ds + \ln(r/\rho) \int_r^R f(v(s)) s \ln(R/s)\, ds \right\}$$

$$\text{and} \quad z'(\rho) = \frac{\lambda}{\rho \ln(R/\rho)} \int_\rho^R f(v(s)) s \ln(R/s)\, ds.$$

For $n \geq 3$

$$\begin{aligned} z(r) \;&= \frac{\lambda}{(n-2)(R^{n-2} - \rho^{n-2})} \left\{ (1 - \rho^{n-2}/r^{n-2}) \int_r^R f(v(s)) s (R^{n-2} - s^{n-2})\, ds \right. \\ &\left. + (R^{n-2}/r^{n-2} - 1) \int_\rho^r f(v(s)) s (s^{n-2} - \rho^{n-2})\, ds \right\} \end{aligned}$$

$$\text{and} \quad z'(\rho) = \frac{\lambda}{\rho(R^{n-2} - \rho^{n-2})} \int_\rho^R f(v(s)) s (R^{n-2} - s^{n-2})\, ds.$$

Concentrating for the moment on $n = 2$,

$$\begin{aligned} z'(\rho)/\sqrt{\lambda} \;&= \frac{\sqrt{\lambda}}{\rho \ln(R/\rho)} \left[\int_\rho^{\rho + \mu/\sqrt{\nu\lambda}} f(v(s)) s \ln(R/s)\, ds + \int_{\rho+\mu/\sqrt{\nu\lambda}}^{R - \mu/\sqrt{\nu\lambda}} f(v(s)) s \ln(R/s)\, ds \right. \\ &\left. + \int_{R-\mu/\sqrt{\nu\lambda}}^R f(v(s)) s \ln(R/s)\, ds \right] \\ &= I_1 + I_2 + I_3 \end{aligned}$$

$$\begin{aligned} \text{where :} \quad I_1 \;&= \frac{1}{\sqrt{\nu}} \int_0^\mu f(V(y)) \left(1 + \frac{y}{\rho\sqrt{\nu\lambda}}\right) \left(1 - \frac{\ln(1 + y/\rho\sqrt{\nu\lambda})}{\ln(R/\rho)}\right) dy \\ &\to \int_0^\infty f(W(y))\, dy = W'(1) = \sqrt{2} \text{ as } \lambda \to \infty \end{aligned}$$

since $\nu \to 1$, taking $\mu = \Lambda(\lambda)$, and using dominated convergence;

$$\begin{aligned} I_2 \;&= \frac{\sqrt{\lambda}}{\rho \ln(R/\rho)} \int_{\rho+\mu/\sqrt{\nu\lambda}}^{R-\mu/\sqrt{\nu\lambda}} f(v(s)) s \ln(R/s) ds \\ &\sim \sqrt{\lambda} f(M) \int_\rho^R s \ln(R/s) ds / \rho \ln(R/\rho) \quad \text{as } \lambda \to \infty; \end{aligned}$$

and

$$\begin{aligned} I_3 \;&= \frac{-1}{\rho\sqrt{\nu}\ln(R/\rho)} \int_0^\mu f(V(y))(R - y/\sqrt{\nu\lambda}) \ln(1 - y/R\sqrt{\nu\lambda})\, dy \\ &\to 0 \text{ as } \lambda \to \infty. \end{aligned}$$

We now want $I_2 \to 0$ as λ (and μ) $\to \infty$. This will be true provided that we choose $\mu = \Lambda(\lambda)$ such that $\sqrt{\lambda} f(M(\mu)) \to 0$ as well as the earlier requirements of $\mu/\sqrt{\lambda} \to 0$ and $G(\mu)/\sqrt{\lambda} \to 0$. This can be done as long as

$$\mu f(M) \to 0 \text{ and } G(\mu) f(M) \to 0 \text{ as } \mu \to \infty$$

since it will then be possible to have

$$\mu \ll \sqrt{\lambda} \ll 1/f(M(\mu)) \text{ and } G(\mu) \ll \sqrt{\lambda} \ll 1/f(M(\mu))$$

for μ, $\lambda \to \infty$.

The former follows from the relation between μ and M:

$$\begin{aligned}
\sqrt{2}\mu f(M) &= f(M) \int_0^M \frac{ds}{\sqrt{F(s) - F(M)}} \\
&= \int_0^{1-F(M)} \frac{f(M)}{f(s)} \frac{d\sigma}{\sqrt{\sigma}} \\
&= \int_0^{1-F(M)} \frac{f(M)}{f(F^{-1}[F(M)+\sigma])} \frac{d\sigma}{\sqrt{\sigma}}.
\end{aligned}$$

The integrand is less than $1/\sqrt{\sigma}$ while $f(M)/f(F^{-1}[F(M)+\sigma]) \sim f(M)/f(F^{-1}[\sigma]) \to 0$ as M and $\mu \to \infty$ (for fixed σ). Thus $\mu f(M) \to 0$ as $\mu \to \infty$.

The latter holds since

$$f(M(\mu))G(\mu)/\sqrt{2} = \sup_{V \in (0,M)} \frac{f(M)\sqrt{F(V) - F(M)}}{f(V)}$$

and $f(M)^2(F(V) - F(M))/f(V)^2 \le (M - V) f(M)^2/f(V) \le (M - V) f(M)$ (because f is decreasing) $\le M f(M) \le 2 \int_{M/2}^M f(s)\,ds \to 0$ as μ (and M) $\to \infty$.

With such a choice of $\Lambda(\lambda)$, I_2 and $I_3 \to 0$, $I_1 \to \sqrt{2}$, and

$$z'(\rho)/\sqrt{\lambda} \to \sqrt{2} \text{ as } \lambda \to \infty.$$

The calculations for higher dimensions follow in the same way.

Then

$$\begin{aligned}
\delta &= \lambda^{1-p} \left(\int_{\partial\Omega} \left| \frac{\partial u}{\partial N} \right| dx \right)^p \\
&\le \lambda^{1-p/2} |\partial\Omega|^p (z'(\rho)/\sqrt{\lambda})^p \\
&\sim 2^{p/2} |\partial\Omega|^p \lambda^{1-p/2}
\end{aligned}$$

In particular, for $p > 2$, $\delta \to 0$ as $\lambda \to \infty$ and there is then a maximum value δ^* for δ. For $p = 2$ the bound on δ approaches $2|\partial\Omega|^2$ and there is again a critical value. Using the first part of the theorem, $\delta \to 2|\partial\Omega|^2$ as $\lambda \to \infty$ for $p = 2$ and it is seen that $\delta^* \ge 2$.

Formal asymptotics for $p = 2$ and large λ suggest that the boundary layer gives rise to the behaviour

$$\delta \sim 2|\partial\Omega|^2 - C\lambda^{-1/2} \int_{\partial\Omega} \kappa\, dx$$

where κ is the mean curvature (positive if Ω is convex) and C is some constant depending upon f and n. We conjecture that for $\Omega \subset \mathbb{R}^2$ simply connected, so $\int_{\partial\Omega} \kappa \, dx = 2\pi > 0$, or whenever Ω is convex, so $\kappa > 0$, then $\delta < \delta^* = 2|\partial\Omega|^2$ for $p = 2$. Note that if $f(u) \sim au^{-q}$ as $u \to \infty$ then the one-dimensional result, $\delta = 8(1 - F(M))$ (see [19]) means that $\delta \sim \delta^* - AM^{1-q}$. Also $\lambda \sim BM^{1+q}$ (see end of the following section) and $\delta \sim \delta^* - D\lambda^{\frac{1-q}{q+1}}$ for $\lambda \to \infty$. This suggests that for the higher-dimensional problem the boundary-layer correction only dominates for rapidly decaying f, $f \to 0$ faster than u^{-3}. For f decaying more slowly, like u^{-q} with $1 < q < 3$, we might conjecture that really $\delta \sim 2|\partial\Omega|^2 - D\lambda^{\frac{1-q}{q+1}}$. It could then be possible to have $\delta^* < |\partial\Omega|^2$ for more general regions.

Carrillo [9] considered the special case of $f(u) = e^{-u}$ (motivated by the Poisson-Boltzman equation which has $p = 1$). He found that for $p < 2$ there was a solution of BVP(2) for all δ and that the solution was unique for $p \leq 1$. With $p \geq 2$ it was found that a critical δ^* existed and for $p > 2$ there were at least two solutions if $\delta < \delta^*$ and at least one if $\delta = \delta^*$.

If f behaves approximately as a power, say $f(u) \sim Au^q$ for $u \to \infty$, with $q > -1$, then the critical value of p appearing in Theorem 2.2 is changed to $(q-1)/q$. If the Dirichlet condition is replaced by a Robin one, $\dfrac{\partial u}{\partial N} + \alpha u = 0$ on $\partial\Omega$, then it appears that: for f which decay faster than any power of u the critical value of p is 1 (in which case large solutions correspond to δ determined by a boundary layer and δ is then proportional to $|\partial\Omega|^2$); if $f \sim Au^q$ for large u then the critical value is $p = (q-1)/q$; if f grows faster than any power of u the special value is again $p = 1$. (See the comments appearing after Theorems 3.3 and 3.4.)

3. Existence and Explicit Steady-State Solutions on the Unit Ball

When Ω is the unit ball in $I\!R^n$, $\Omega = B_1(0) = \{x \in I\!R^n : |x| < 1\}$, then any solution of (2) is radially symmetric and $u = u(r)$, $r = |x|$ is a solution of

$$(6) \qquad \begin{cases} \dfrac{1}{r^{n-1}}(r^{n-1}u_r)_r + \dfrac{\delta f(u)}{(\int_\Omega f(u)dx)^p} = 0 \quad , \quad r \in (0,1) \\ u'(0) = 0 \quad , \quad u(1) = 0 \end{cases}$$

By the maximum principle, a solution of (6) is radially decreasing.

By integrating (6) on the interval [0,1], we have

$$(7) \qquad u'(1) = u_r(1) = -\frac{\delta}{n\omega_n K^{p-1}},$$

where $K \equiv \displaystyle\int_\Omega f(u)dx = n\omega_n \int_0^1 r^{n-1} f(u(r))dr$ and ω_n is volume of the n-dimensional ball. Set $\alpha = u(0)$.

Theorem 3.1 Let $n = 1$, $f(u) = e^u$.

a) If $p \geq 1$, BVP(2) has a unique solution for all $\delta > 0$.

b) If $0 < p < 1$, then there exists $\delta^* > 0$ such that

i) BVP(2) has two solutions for $\delta < \delta^*$,

ii) BVP(2) has one solution for $\delta = \delta^*$,

iii) BVP(2) has no solution for $\delta > \delta^*$.

Proof. BVP(2) can be written as

$$(8) \qquad \begin{cases} u'' + \dfrac{\delta}{K^p}e^u = 0 \\ u'(0) = 0 \qquad u(1) = 0 \end{cases}$$

where $K = 2\displaystyle\int_0^1 e^u dx$. Multiplying (8) by $2u'$ and integrating gives

$$(9) \qquad (u')^2 + \frac{2\delta}{K^p}e^u = \frac{2\delta}{K^p}e^{u(0)} = \frac{2\delta}{K^p}e^\alpha.$$

¿From (7), we have $u'(1) = \dfrac{-\delta}{2K^{p-1}}$; hence (9) with $r = 1$ can be solved for K giving

$$(10) \qquad K^{2-p} = \frac{8}{\delta}(e^\alpha - 1).$$

Since $u'(r) < 0$ on (0,1), we have

$$(11) \qquad \left(\frac{2\delta}{K^p}\right)^{\frac{1}{2}} r = -\int \frac{du}{(e^\alpha - e^u)^{\frac{1}{2}}}.$$

Integrating and evaluating at $r = 1$ gives

$$(12) \qquad \delta = \delta(\alpha) = 2^{2p-1} e^{\frac{-\alpha(2-p)}{2}} (e^{\alpha} - 1)^{\frac{p}{2}} \ln^{2-p} \left[\frac{1 + (1 - e^{-\alpha})^{\frac{1}{2}}}{1 - (1 - e^{-\alpha})^{\frac{1}{2}}} \right].$$

For $\alpha \in [0, \infty)$, we observe that $\delta = \delta(\alpha)$ defined by (12) satisfies $\delta(0) = 0$, $\delta(\alpha) > 0$ for $\alpha \in (0, \infty)$. If $p \geq 1$, $\delta'(\alpha) > 0$ for all $\alpha \in (0, \infty)$. If $p < 1$, $\delta(\alpha)$ has a unique maximum at $\alpha^* > 0$ and $\delta(\alpha) \to 0$ as $\alpha \to \infty$.

Theorem 3.2 Let $n = 2$, $f(u) = e^u$.

a) If $p > 1$, BVP(2) has a unique solution for all $\delta > 0$.

b) If $p = 1$, BVP(2) has a unique solution for all $\delta < 8\pi$ but no solution for $\delta \geq 8\pi$.

c) If $0 < p < 1$, then there exists $\delta^* > 0$ such that BVP(2) has:

i) two solutions for $\delta < \delta^*$

ii) a unique solution for $\delta = \delta^*$

iii) no solution for $\delta > \delta^*$.

Proof. For $n = 2$, BVP(2) can be expressed as

$$(13) \qquad \begin{cases} \frac{1}{r}(ru_r)_r = -\frac{\delta e^u}{K^p} \quad , \quad r \in (0,1) \\ u'(0) = 0 \quad , \quad u(1) = 0 \end{cases}$$

where

$$K = 2\pi \int_0^1 e^{u(r)} dr.$$

This in turn can be considered as

$$(14) \qquad \begin{cases} \frac{1}{r}(ru_r)_r + \nu e^u = 0 \\ u'(0) = 0 \quad , \quad u(1) = 0 \end{cases}$$

where $\delta = \nu K^p$ and, by (7), $\delta = -2\pi K^{p-1} u'(1)$.

Following Joseph & Lundgren [15] (or see Galaktionov & Lacey [12])

$$(15) \qquad w(r) = -2 \ln(1 - a + ar^2) \quad ,$$

then $-\frac{(rw')'}{r} = \frac{8a(1-a)}{(1-a+ar^2)^2}$ and $w(r)$ is a solution of (14) provided by $\nu = 8a(1-a)$.

Since $\int_0^1 e^{w(r)} r dr = \int_0^1 \frac{r dr}{(1-a+ar^2)^2} = \frac{1}{2(a-1)}$,

$$\begin{aligned} \delta = \nu K^p &= 8a(1-a)(2\pi)^p [2(1-a)]^{-p} \\ &= 8\pi^p a(1-a)^{1-p}. \end{aligned}$$

Thus $w(r)$ given by (15) is a solution of BVP(13) provided

$$(16) \qquad \begin{cases} \delta = 8\pi^p a(1-a)^{1-p} \\ \alpha \equiv w(0) = -2\ln(1-a) \quad , \quad 0 \leq a < 1. \end{cases}$$

If $p > 1$, $\delta(a)$ is increasing on $[0,1)$ with $\delta(a) \to \infty$ as $a \to 1-$, and $\alpha(a)$ is increasing on $[0,1)$ with $\alpha(a) \to \infty$ as $a \to 1-$. This gives a).

If $p = 1$, $\alpha = w(0) = -2\ln\left(\frac{8\pi - \delta}{8\pi}\right)$ and we have b). If $p < 1$, $\delta(0) = 0$, $\delta(1-) = 0$, and δ has a unique maximum $\delta^* = \frac{8\pi^p}{2-p}\left(\frac{1-p}{2-p}\right)^{1-p}$ at $a^* = \frac{1}{2-p}$.

Theorem 3.3 Let $n = 1$, $f(u) = e^{-u}$.

a) If $p > 2$, then there exists $\delta^* > 0$ such that BVP(2) has:

  i) two solutions for $\delta < \delta^*$,
  ii) a unique solution for $\delta = \delta^*$,
  iii) no solution for $\delta > \delta^*$.

b) If $p = 2$, BVP(2) has:

  i) a unique solution for all $\delta < \delta^* = 8$,
  ii) no solution for all $\delta \geq 8$.

c) If $p < 2$, BVP(2) has a unique solution for all $\delta > 0$.

Proof. For $n = 1$, (2) becomes

$$\left\{\begin{array}{l} u'' + \frac{\delta}{K^p} e^{-u} = 0 \\ u'(0) = 0 \quad , \quad u(1) = 0 \end{array}\right. \tag{17}$$

where $K = 2\int_0^1 e^{-u}dx$.

The solutions of (17) take the form

$$w(r) = 2\ln\frac{\cos ax}{\cos a} \tag{18}$$

provided that

$$\left\{\begin{array}{l} \delta = \delta(a) = 2^{1+p}a^{2-p}\sin^p a\cos^{p-2} a \quad , \quad 0 \leq a < \frac{\pi}{2} \\ \alpha = w(0) = -2\ln(\cos a). \end{array}\right. \tag{19}$$

If $p < 2$, $\delta(a)$ is increasing on $[0, \frac{\pi}{2})$ with $\delta(a) \to \infty$ as $a \to \frac{\pi}{2}-$ and $\alpha(a)$ is increasing on $[0, \frac{\pi}{2})$ with $\alpha(a) \to \infty$ as $a \to \frac{\pi}{2}-$. This gives c). If $p = 2$, $\alpha = \ln\frac{8}{8-\delta}$ and b) follows. If $p > 2$, $\delta(0) = 0$, $\delta(\frac{\pi}{2}-) = 0$, and δ has unique maximum at $\delta^* > 0$. This gives a).

Remark. If $n = 2$ and $f(u) = e^{-u}$, then the conclusion of Theorem 3.3 follows with $\delta^* = 8\pi^2$ in case b). One only needs to observe that $w(r) = 2\ln(1+a-ar^2)$ is the solution of (2) provided $\delta = \delta(a) = 8\pi^p a(1+a)^{1-p}$ and $\alpha = \alpha(a) = 2\ln(1+a)$, $0 \leq a < 1$ (see [23]).

For Robin boundary conditions of the form $\frac{\partial u}{\partial N} + \alpha u = 0$ with $0 < \alpha < \infty$, $\Omega = B_1(0) \subset I\!R^n$, and $n = 1$ or 2, one can easily verify the following. For $f(u) = e^u$, one has exactly the same statements as in Theorem 3.1 and Theorem 3.2. For $f(u) = e^{-u}$, the conclusion of Theorem 3.3 and above remark for $n = 2$ change. The critical exponent p^* changes to $p^* = 1$ and one has that for $p > 1$, there exists $\delta^* > 0$ such that the Robin version of BVP(2) has at least two solutions for $\delta < \delta^*$, a unique solution for $\delta = \delta^*$, and no solution for $\delta > \delta^*$. If $p \leq 1$, then the problem has exactly one solution for all $\delta > 0$.

Theorem 3.4 Let $n = 1$.

1) Let $\int_0^\infty f(s)\,ds < \infty$.

   a) If $p > 2$, then there exists δ^* such that BVP(2) has
      i) at least two solutions for $\delta < \delta^*$,
      ii) a solution for $\delta = \delta^*$,
      iii) no solution for $\delta > \delta^*$.

   b) If $p = 2$, BVP(2) has
      i) a unique solution for all $\delta < 8\int_0^\infty f(s)\,ds$,
      ii) no solution for $\delta \geq 8\int_0^\infty f(s)\,ds$.

   c) If $p < 2$, BVP(2) has a unique solution for all $\delta > 0$.

2) Let $f(u) \sim Au^q$ for $u \to \infty$ with $q \geq -1$.

   a) If $-1 \leq q < 0$ and $p > (1-q)/(-q)$, or $q > 0$ and $p < (q-1)/q$, then there exists δ^* such that BVP(2) has
      i) at least two solutions for $\delta < \delta^*$,
      ii) at least one solution for $\delta = \delta^*$,
      iii) no solution for $\delta > \delta^*$.

   b) If $q > -1$ and $p = (q-1)/q$, then there exist $0 < \delta_* \leq \delta^*$ such that BVP(2) has
      i) a unique solution for $\delta < \delta_*$,
      ii) at least one solution for $\delta_* \leq \delta < \delta^*$,
      iii) no solution for $\delta > \delta^*$.
      (If $\delta_* < \delta^*$ then there are multiple steady states in some right neighbourhood of δ_* whereas if $\delta_* = \delta^*$ there are never multiple steady states and $\|u\|_\infty \to \infty$ as $\delta \to \delta_*$.)

   c) If $q = -1$ and $p \leq 2$, $-1 < q < 0$ and $p < (q-1)/q$, $q = 0$, **or** $q > 0$ and $p > (q-1)/q$, then BVP(2) has a solution for all $\delta > 0$.

3) Let $f(u)$ be increasing and grow much faster than any power of u, in the sense that $uf'(u)/f(u) \to \infty$ as $u \to \infty$.

   a) If $p \geq 1$ then BVP(2) has a solution for all $\delta > 0$.

   b) If $0 < p < 1$ then there exists δ^* such that BVP(2) has

      i) at least two solutions for $\delta < \delta^*$,

      ii) at least one solution for $\delta = \delta^*$,

      iii) no solution for $\delta > \delta^*$.

Note that cases of $f(u) \sim Au^q$ with $q < -1$ are covered by the first part of the theorem.

The results of the second part apply equally well to cases where there are some $0 < B < D$ such that $Bu^q \leq f(u) \leq Du^q$ for large u.

For $0 \leq q \leq 1$ in part 2 case c) always applies.

Proof For part 1 we proceed as in Lacey [19]:

Problem (2) is

$$u'' + \lambda f(u) = 0, \quad u'(0) = 0 = u(1),$$

$$\delta = \left(2\int_0^1 f(u)\,dx\right)^p \lambda = 2^p\lambda^{1-p}|u'(1)|^p = 2^{3p/2}\lambda^{1-p/2}\left(\int_0^M f(s)\,ds\right)^{p/2},$$

where $M(\lambda) = u(0;\lambda) = \max_x\{u\}$.

Taking $\lambda \to \infty$, $M \to \infty$, and

$$\begin{aligned}\delta &\sim 2^{3p/2}\left(\int_0^\infty f(s)\,ds\right)^{p/2}\lambda^{1-p/2} \\ &\to \begin{cases} 0 & \text{for } p > 2 \\ 8\int_0^\infty f(s)\,ds & \text{for } p = 2 \\ \infty & \text{for } p < 2 \end{cases}.\end{aligned}$$

Since δ is a continuous function of λ (or equivalently of M) with $\delta \to 0$ as $\lambda,\ M \to 0$, a) and c) follow immediately. Case b) comes from the observation that $\delta = 8\int_0^M f(s)\,ds$ increases towards its limiting value as $M \to \infty$.

Turning now to part 2, let $F(u) = \int_0^u f(s)\,ds \sim \dfrac{A}{q+1}u^{q+1}$ for $u \to \infty$, provided that $q > -1$. Solving the auxiliary problem,

$$w'' + M^{-q}f(Mw) = 0, \quad w'(0) = 0, \quad w(0) = 1,$$

gives the solution of

$$u'' + \lambda f(u) = 0, \quad u'(0) = 0, \quad u(0) = M, \quad u(1) = 0,$$

through the relations

$$M = M(\lambda) = u(0;\lambda), \quad u(x) = Mw(\sqrt{\lambda M^{q-1}}\,x), \quad w(\sqrt{\lambda M^{q-1}}) = 0.$$

However, $w'^2 = 2M^{-q-1}[F(M) - F(Mw)]$, and $w(y)$ satisfies

$$
\begin{aligned}
y &= \frac{M^{(1+q)/2}}{\sqrt{2}} \int_w^1 [F(M) - F(Ms)]^{-1/2} ds \\
&\to \frac{1}{\sqrt{2}} \int_w^1 \left[\frac{a}{1+q}(1 - s^{1+q})\right]^{-1/2} ds \text{ as } M \to \infty,
\end{aligned}
$$

since $[F(M) - F(Ms)]/M^{q+1} = \int_{Ms}^{M} f(\sigma)\, d\sigma / M^{q+1} > M^{-q}(1-s) f(M\mathcal{S})$, with $\mathcal{S} \in [s, 1]$, $f(M\mathcal{S}) = \inf_{\sigma \in [s,1]} f(M\sigma)$ and the dominated convergence theorem can be applied.

Then $w = 0$ for $y \sim Q/\sqrt{2A}$ for $M \to \infty$, where

$$
Q(q) = \sqrt{q+1} \int_0^1 (1 - s^{q+1})^{-1/2} ds,
$$

i.e. $\lambda \sim Q^2 M^{1-q}/2A$.

Since $w'^2 = 2M^{-q-1}F(M)$ where $w = 0$,

$$
u'^2 \sim \frac{2A}{q+1}\left(M\sqrt{\lambda M^{q-1}}\right)^2 \sim Q^2 M^2/(q+1) = \left(\int_0^1 (1 - s^{q+1})^{-1/2} ds\right)^2 M^2
$$

$$
\text{and } \delta = 2^p \lambda^{1-p} |u'(1)|^p \sim 2^{2p-1}(q+1)^{-p/2} Q^{2-p} A^{p-1} M^{1+pq-q}
$$

for $M \to \infty$.

In particular:

a) if $1 + pq - q < 0$ ($-1 < q < 0$ and $p > (1-q)/(-q)$, **or** $q > 0$ and $p < (q-1)/q$) then $\delta \to 0$ as $M \to \infty$;

b) if $1 + pq - q = 0$ ($q > -1$ and $p = (q-1)/q$) then $\delta \to 2^{2p-1}(q+1)^{p-2} Q^{2-p} A^{p-1}$ as $M \to \infty$;

c) if $1 + pq - q > 0$ ($-1 < q < 0$ and $p < (1-q)/(-q)$, **or** $q > 0$ and $p > (q-1)/q$) then $\delta \to \infty$ as $M \to \infty$.

Part 2 follows except for the special case $q = -1$.

For this value $F(u) \sim A \ln u$ as $u \to \infty$ and the calculations proceed in a similar fashion except that

$$
u'(1)^2 = \left(M\sqrt{\lambda M^{-2}}\right)^2 \times 2F(M) \sim CM^2 \ln N \text{ for } M \to \infty.
$$

So now $\delta \sim CM^{2(1-p)}\left(M^2 \ln M\right)^{p/2} = CM^{2-p}(\ln M)^2$, and $\delta \to 0$ as $M \to \infty$ for $p > 2$ but $\delta \to \infty$ as $M \to \infty$ for $p \leq 2$.

Part 3 is approached in a somewhat different manner.

We again define $w(x) = u(x)/M$ so that it satisfies:

$$
w'' + \lambda f(Mw)/M = 0; \quad w'(0) = w(1) = 0 \text{ and } w(0) = 1.
$$

Since $f > 0$, $1-x$ is a lower solution in $[0,1]$ and: $-w'(1) > 1$; the point $X_1(\epsilon) = 1-\epsilon$ for any given $0 < \epsilon < 1$ satisfies $X_1 > \epsilon$.

Likewise, $(1-\epsilon)(1-x)/(1-X_1)$ is a lower solution in $[X_1, 1]$ and $-w'(X_1) \leq (1-\epsilon)/(1-X_1)$.

Since $w''' = -\lambda f'(Mw)w'$ with $f' \leq 0$ and $w' \leq 0$ for $0 < x < 1$, $g(x) \equiv w'''/2 \geq 0$ and

$$w = 1 - x^2 - \int_0^x [x^2(1-y)^2 - (x-y)^2] g(y)dy - x^2 \int_x^1 (1-y)^2 g(y)dy \leq 1 - x^2$$

$(x(1-y) > x - y$ for $0 < y < x < 1)$.

So $X_1 \leq \sqrt{\epsilon}$ and $-w'(X_1) \leq (1-\epsilon)/(1-\sqrt{\epsilon})$.

Of course the point X_2 at which $w = 1 - 2\epsilon$ satisfies $2\epsilon \leq X_2 \leq \sqrt{2\epsilon}$ and $w \geq (1-2\epsilon)(1-x)/(1-X_2)$ for $X_2 \leq x \leq 1$.

Taking $c = f((1-\epsilon)M) = \min_{s \in [1-\epsilon, 1]} f(Ms)$, $-w'' = \lambda f(Mw)/M \geq \lambda c/M$ for $0 < x < X_1$ (where $1 - \epsilon < w < 1$), so $-w'(X_1) \geq \lambda c X_1/M$ and $\lambda c X_1/M \leq (1-\epsilon)/(1-\sqrt{\epsilon})$. The estimate $X_1 > \epsilon$ yields

$$\lambda c/M \leq (1-\epsilon)/\epsilon(1-\sqrt{\epsilon}) < 1/\epsilon(1-\sqrt{\epsilon}).$$

Now, for $X_2 < x < 1$ $(0 < w < 1 - 2\epsilon)$,

$$0 < -w'' = \lambda f(Mw)/M \leq D(M,\epsilon)\lambda c/M < D(M,\epsilon)/\epsilon(1-\sqrt{\epsilon}),$$

where $D(M,\epsilon) = \dfrac{1}{c} \sup_{(0,1-2\epsilon)} f(Ms) = \dfrac{f((1-2\epsilon)M)}{f((1-\epsilon)M)} \to 0$ as $M \to \infty$.

Then taking M to be sufficiently large,

$$1 - x \leq w \leq \frac{(1-x)(1-\epsilon)}{1 - X_2}.$$

Since the right-hand side of this inequality can be made arbitrarily close to $1-x$ by making ϵ small, we see that

$$w \to 1 - x \text{ and } w'(1) \to -1 \text{ as } M \to \infty,$$

and, because $D \to 0$ (for fixed ϵ), $w'(x) \to -1$ as $M \to \infty$ for any fixed $x > 0$.

Then for $p \geq 1$, $\delta = 2^p \lambda^{1-p} M^p |w'(1)|^p \to \infty$ (because $\lambda \to 0$) as $M \to \infty$.

Now defining X by $w(X) = \frac{1}{2}$, so $X \to \frac{1}{2}$ as $M \to \infty$,

$$-w'(X) = \frac{\lambda}{M} \int_0^X f(Mx)dx > \lambda f(M/2)/2M.$$

But, by the above, $-w'(X) \to 1$ so $\lambda < 3M/f(M/2)$ for large enough M and, for $p < 1$,

$$\delta < 2^p 3^{1-p} M |w'(1)|^p / f(M/2)^{p/2} \to 0,$$

since $f(s) \to \infty$ faster than any power of s.

This concludes the proof.

Most of the results of Theorem 3.4 are expected to carry over to higher dimensions. Certainly for part 1, Theorem 2.2 shows the existence of the critical δ^* for $p \geq 2$ (with $\delta^* \geq 2|\partial\Omega|^2$) and the existence of at least one steady state for all δ if $p < 2$.

With $f(u) \sim Au^q$ as $u \to \infty$ for $q > -1$ it is easy to see that there is a large solution of $\Delta u + \lambda f(u) = 0$ in Ω, $u = 0$ on $\partial\Omega$, approximately $u = \lambda^{\frac{1}{1-q}} v$ for $\lambda \to 0$, $q > 1$, and $\lambda \to \infty, -1 < q < 1$. Here v satisfies $\Delta v + Av^q = 0$ in Ω, $v = 0$ in $\partial\Omega$, and $\frac{\partial v}{\partial N}$ is bounded on $\partial\Omega$. (This asymptotic result can be obtained via upper and lower solutions. For the special case $q = 1$, u becomes large as λ approaches an appropriate eigenvalue.) Most of part 2 is then recovered by using $\delta = \lambda^{1-p}\left(\int_{\partial\Omega} |\frac{\partial u}{\partial N}| dx\right)^p$.

It can be seen from Theorem 3.2 that part 3 needs modification, at least for $f(u) = e^u$. In one dimension there is, for $p = 1$, a solution for all δ but it can be seen that, in the special case, when $n = 2$, $||u||_\infty \to \infty$ as $\delta \to 2|\partial\Omega|^2$ and some δ^* exists.

Theorem 3.4 can also be adapted to cover the case of Robin boundary data, say $\frac{du}{dx} \pm \alpha u = 0$ on $x = \pm 1$. Part 3 of the theorem comes over with little change. For cases of $f(u) \sim Au^q$ as $u \to \infty$ (part 2) u and $\frac{du}{dx}$ grow as $\lambda^{\frac{1}{1-q}}$ for $\lambda \to 0$, $q > 1$, and $\lambda \to \infty$, for all $q < 1$. There is no longer a boundary layer for $q < -1$ and $q = -1$ is no longer a special case. We then see that: a) holds for all $q < 0$ and $p > (q-1)/q$ or $q > 0$ and $p < (q-1)/q$; b) holds for all q with $p = (q-1)/q$; c) holds for all $q < 0$ and $p < (q-1)q$, $q = 0$, or $q > 0$ and $p > (q-1)/q$. The regions for a) - c) for both Dirichlet and Robin data are indicated in the figure.

Figure The behavior of steady state for the one-dimensional problem

A: Case a), $\delta \to 0$ as $||u||_\infty \to \infty$, for both Dirichlet and Robin conditions.

B: Case a), $\delta \to 0$ as $||u||_\infty \to \infty$ for the Robin condition but case c) $\delta \to \infty$ as $||u||_\infty \to \infty$ for the Dirichlet condition.

C: case c), $\delta \to \infty$ as $||u||_\infty \to \infty$ for both Dirichlet and Robin conditions.

——— : case b), $\delta \to \delta_\infty$ as $||u||_\infty \to \infty$ for both Dirichlet and Robin conditions.

- - - - - : case b), $\delta \to \delta_\infty$ as $||u||_\infty$ for the Dirichlet condition, but case a), $\delta \to 0$ as $||u||_\infty \to \infty$ for the Robin condition.

.... : case b) $\delta \to \delta_\infty$ as $||u||_\infty \to \infty$ for the Robin condition, but case c), $\delta \to \infty$ as $||u||_\infty \to \infty$ for the Dirichlet condition.

For this one-dimensional Robin problem, it can be observed that as $q \to \pm\infty$ the critical $p \to 1$, which is the value for part 3 (very rapidly growing f) and e^{-u} (very rapidly decaying f); these exponential cases are easily approached by, again, explicitly solving the equations. A brief formal consideration of a boundary layer suggests that for f which decays faster than a power (such as e^{-u}) the critical value is $p = 1$. Part 1 then has for such rapidly shrinking f a) holds for $p > 1$ and c) (with at least one solution instead of a unique solution) for $p \leq 1$.

We finally note that for three or more dimensions with rapidly growing f, such as e^u, the situation is rather different in that we do not expect there to be a critical value of p. This is indicated by the fact that the steady state for the local problem with $f(u) = e^u$ makes $\int_\Omega f(u)dx$ bounded as $u(0) \to \infty$.

4. Global Existence and Asymptotic Stability

We now wish to determine when global solutions exist for IBVP(1).

Theorem 4.1 Assume $f(s) \geq c > 0$ for $s \geq 0$. Let $n = 1$, $\Omega = (-1, 1)$. For any nonnegative smooth initial data $u_0(x)$, IBVP(1) has a unique bounded solution on $[-1, 1] \times [0, \infty)$ provided $p \geq 1$.

Proof. Let $u(x,t)$ be the solution of IBVP(1) which exists on $B_1(0) \times [0, T)$. Note that $u(x,t)$ satisfies

$$u_t - u_{xx} = g(x,t), \quad x \in B_1(0), \quad t \in [0, T)$$

where $\int_{-1}^{1} g(x,t)dx = \dfrac{\delta}{K^{p-1}}$ where $K = \int_{-1}^{1} f(u)dx$.

Write $u(x,t) = u_1(x,t) + u_2(x,t)$ where u_1 satisfies

$$\text{(20)} \qquad \begin{cases} u_t - u_{xx} = 0 \\ u(-1,t) = 0 = u(1,t), \quad u(x,0) = u_0(x) \end{cases}$$

and u_2 satisfies

$$\text{(21)} \qquad \begin{cases} u_t - u_{xx} = g(x,t) \\ u(-1,t) = 0 = u(1,t), \quad u(x,0) = 0. \end{cases}$$

Then $||u_1(\cdot,t)||_\infty \leq ||u_0||_\infty$.

The solution u_2 of (21) can be expressed by

$$\text{(22)} \qquad u_2(x,t) = \int_0^t \int_{-1}^{1} G(x,y,s)g(y,s)dyds$$

where $\int_{-1}^{1} g(y,s)dy = \delta K^{1-p}$. Thus, $u_2(x,t) \leq \dfrac{\delta}{K^{p-1}} \int_0^t \sup_y G(x,y,s)ds$. Since $G(x,y,s) \sim \dfrac{1}{2(\pi s)^{\frac{1}{2}}} e^{-(x-y)^2/4s}$ for s small and $G(x,y,s) \sim e^{-\frac{\pi^2 s}{4}} \sin\dfrac{(x+1)\pi}{2} \sin\dfrac{(y+1)\pi}{2}$ for s large, $\sup_y G(x,y,s) \sim \dfrac{1}{2\sqrt{\pi s}}$ for $s << 1$, and $\sup_y G(x,y,s) \sim \sin\dfrac{(x+1)\pi}{2} e^{-\frac{\pi^2 s}{4}}$ for s large. Thus, $\sup_y G(x,y,s)$ is integrable on $[0, \infty)$ for each $x \in [-1, 1]$.

We then have

$$||u(\cdot,t)||_\infty \leq ||u_0||_\infty + \frac{\delta}{K^{p-1}} \sup_x \int_0^\infty \sup_y G(x,y,s)ds$$

and this implies $u(x,t)$ exists globally and is bounded.

Theorem 4.2 Let $f(s) = e^s$, $n = 2$, $\Omega = B_1(0) \subset I\!R^2$. Assume u_0 is radially symmetric.

1) If $p > 1$, $\delta > 0$, then IBVP(1) has a unique globally bounded solution $u(x,t)$ on $B_1(0) \times [0, \infty)$.

2) If $p = 1$, $0 < \delta < 8\pi$, then IBVP(1) has a unique bounded global solution.

Proof.
1) Let $p > 1$ and let $u(x,t)$ be the solution of IBVP(1) with $f(u) = e^u$ which exists on $B_1(0) \times [0,T)$.

Consider the Lyapunov function

$$V[u](t) \equiv \int_\Omega |\bigtriangledown u|^2 dx + \frac{2\delta}{p-1}\left(\int_\Omega e^u dx\right)^{1-p} \tag{23}$$

then $V'[u](t) = -2\int_\Omega u_t^2 dx \le 0$. Thus $V[u](t)$ is non-increasing and

$$\int_\Omega |\bigtriangledown u|^2 dx + \frac{2\delta}{p-1}\left(\int_\Omega e^u dx\right)^{1-p} \le E_0 \equiv V[u_0]. \tag{24}$$

¿From (24), $\int_\Omega |\bigtriangledown u|^2 dx \le E_0$ and hence $\int_\Omega u_r^2 dx \le E_0$. For $\Omega = B_1(0) \subset I\!R^2$, and assuming that u_0 is radially symmetric, this means that $\int_0^1 u_r^2 r dr \le \frac{E_0}{2\pi} = C^2$. Since $u(1) = 0$,

$$\begin{aligned} 0 \le u(r) = -\int_r^1 u'(r)dr &\le \int_r^1 |u'| r^{\frac12} r^{-\frac12} dr \\ &\le \sqrt{\int_r^1 r u'^2 dr \int_r^1 r^{-1} dr} \\ &\le C|\ln r|^{\frac12} \end{aligned}$$

. Then

$$\int_\Omega e^{\beta u} dx = 2\pi \int_0^1 r e^{\beta u} dr \le 2\pi \int_0^1 r \exp(\beta C\sqrt{|\ln r|})dr \equiv 2\pi F(\beta C), \quad \beta > 0,$$

where $F(s) = \int_0^1 r\exp(s|\ln r|^{\frac12})dr$ satisfies $F(0) = \frac12$, F is increasing, and $F(s) \sim \frac{s}{\sqrt{8\pi}} e^{\frac{s^2}{8}}$ as $s \to \infty$. This implies that $\frac{e^u}{(\int_\Omega e^u)^p} \in L_\beta$ and $\frac{e^u}{(\int_\Omega e^u)^p}$ is bounded in L_β for $1 \le \beta < \infty$. By standard L_p estimates and Sobolev imbedding, u is Hölder continuous for $0 \le t \le T$ for any $T > 0$. Thus u is a global bounded classical solution of IBVP(1) on $B_1(0) \times [0,\infty)$.
2) For $\Omega = B_1(0) \subset I\!R^2$ and $p = 1$, we take the Lyapunov functional

$$V[u](t) \equiv \int_\Omega |\bigtriangledown u|^2 dx - 2\delta \ln \int_\Omega e^u dx. \tag{25}$$

Then $V'[u] = -2\int_\Omega u_t^2 dx \le 0$ and

$$\int_\Omega |\bigtriangledown u|^2 dx \le E_0 + 2\delta \ln \int_\Omega e^u dx. \tag{26}$$

By assumption u_0 is radially symmetric so $g(t) \equiv \int_0^1 r u_r^2 dr = \frac{1}{2\pi}\int_\Omega |\nabla u|^2 dx$ satisfies

$$g(t) \le \frac{E_0}{2\pi} + \frac{\delta}{\pi}\ln\int_\Omega e^u dx. \tag{27}$$

Recall $F(s) = \int_0^1 r\exp(s|\ln r|^{\frac{1}{2}})dr$. Since

$$\begin{aligned} u(r) = -\int_r^1 u'(r)dr &\le \textstyle\int_r^1 |u'| r^{\frac{1}{2}} r^{-\frac{1}{2}} dr \\ &\le \sqrt{\int_0^1 u'^2 r dr}\sqrt{\int_r^1 r^{-1}dr} \\ &= (g(t))^{\frac{1}{2}}(|\ln r|)^{\frac{1}{2}}, \end{aligned}$$

we have from (27):

$$\begin{aligned} g(t) &\le \frac{E_0}{2\pi} + \frac{\delta}{\pi}\ln\left(2\pi\int_0^1 re^u dr\right) \\ &\le \frac{E_0}{2\pi} + \frac{\delta}{\pi}\ln\left(2\pi\int_0^1 r\exp(\sqrt{g(t)}\cdot\sqrt{|\ln r|})\right)dr \\ &= \frac{E_0}{2\pi} + \frac{\delta}{\pi}\ln\left(2\pi F(\sqrt{g(t)})\right). \end{aligned} \tag{28}$$

But

$$\ln(2\pi F(\sqrt{g})) = \ln 2\pi + \ln F(\sqrt{g}) \sim \frac{g}{8} \quad \text{for } g \text{ large} \tag{29}$$

Thus, if $\delta < 8\pi$, $g(t)$ is bounded.

As in 1), this implies u is a global, bounded, classical solution.

Remark. If the Dirichlet boundary condition in IBVP(1) is replaced by a Robin condition of the type

$$1'_a) \qquad \frac{\partial u}{\partial N} + \alpha u = 0, \quad x \in \partial\Omega, \quad t > 0, \quad 0 < \alpha < \infty$$

where $\Omega = B_1(0) \subset I\!R^2$, $f(u) = e^u$, and $p \ge 1$, then similar results hold for radially symmetric initial data $u(x)$.

For $p > 1$, the Lyapunov functional

$$V[u](t) = \int_\Omega |\nabla u|^2 + \frac{2\delta}{p-1}\left(\int_\Omega e^u\right)^{1-p} + \int_{\partial\Omega}\alpha u^2$$

satisfies $V'[u](t) = -2\int_\Omega u_t^2 \le 0$ and hence $V[u](t) \le E_0 \equiv V[u_0]$. Thus, $2\pi\alpha u^2(1,t) \le \int_{\partial\Omega}\alpha u^2 \le E_0$ and $u^2(1,t) \le E_0/2\pi\alpha = C^2$. This implies that $u(1,t)$ is bounded. As in the proof of Theorem 4-2, $u(r,t) -$

$u(1,t) \leq C|\ln r|^{1/2}$ and hence for $1 \leq \beta < \infty$, $\int_\Omega e^{\beta u} = 2\pi \int_0^1 r e^{\beta u} dr \leq 2\pi \int_0^1 r e^{\beta(u(1,t)+C|\ln r|^{1/2}} dr = 2\pi \cdot K \cdot \int_0^1 r e^{\beta C|\ln r|^{1/2}} dr = 2\pi K F(\beta C)$ where $F(s) = \int_0^1 r \exp(s|\ln r|^{1/2}) dr$. This implies that u is a bounded global classical solution of IBVP$(1')$.

If $\alpha = 0$ so that a Neumann boundary condition is imposed, then $u(1,t)$ is bounded from below and $u(r,t) \geq u(1,t) - C|\ln r|^{1/2}$. This gives

$$\frac{e^u}{(\int_\Omega e^u)^p} \leq \frac{e^u (2\pi)^{-p} e^{-pu(1,t)}}{F(-C)^p} \leq \frac{(2\pi)^{-p} e^{-(p-1)u(1,t)} e^{C|\ln r|^{1/2}}}{F(-C)^p}.$$

This in turn implies for $p > 1$, $\beta \in [1,\infty)$, that

$$\int_\Omega \left(\frac{e^u}{(\int_\Omega e^u)^p} \right)^\beta dx \leq (2\pi)^{-(\beta p - 1)} \frac{e^{-\beta(p-1)u(1,t)}}{F(-C)^{\beta p}} F(\beta C)$$

Thus, $\dfrac{e^u}{(\int_\Omega e^u)^p} \in L^\beta$ for $1 \leq \beta < \infty$ and hence IBVP$(1'')$ with Neumann boundary conditions has a global classical solution u.

For Robin and Neumann boundary conditions $0 \leq \alpha < \infty$ in the case $p = 1$ with $\Omega = B_1(0) \subset I\!\!R^2$, the Lyapunov functional

$$V[u](t) = \int_\Omega |\bigtriangledown u|^2 - 2\delta \ln \int_\Omega e^u + \int_{\partial\Omega} \alpha u^2$$

gives

$$\int_\Omega |\bigtriangledown u|^2 + \int_{\partial\Omega} \alpha u^2 \leq E_0 + 2\delta \ln \left[\int_\Omega e^u \right].$$

Again assuming radial symmetry, as in Theorem 4.2, part 2, set $g(t) \equiv \int_0^1 r u_r^2 dr = \frac{1}{2\pi} \int_\Omega |\bigtriangledown u|^2 dx$. Then

$$\begin{aligned} g(t) + \alpha u^2(1,t) &\leq \frac{E_0}{2\pi} + \frac{\delta}{\pi} \ln \left[\int_\Omega e^u \right] \\ &\leq \frac{E_0}{2\pi} + \frac{\delta}{\pi} \ln \left(2\pi e^{u(1,t)} F \sqrt{g(t)} \right) \\ &= \frac{E_0}{2\pi} + \frac{\delta}{\pi} \ln 2\pi + \frac{\delta}{\pi} u(1,t) + \frac{\delta}{\pi} \ln F(\sqrt{g}). \end{aligned}$$

For $\delta < 8\pi$, this implies $g(t)$ and $u(1,t)$ are bounded. As before, this implies u is a bounded, global, classical solution for $\delta < \delta^* = 8\pi$ and $\alpha > 0$.

Theorem 4.3. Let $n = 2$, $f(s) = e^s$, and let $\Omega \subset I\!\!R^2$ be a smooth bounded domain. If $p > 1$, $\delta > 0$, then IBVP(1) has a unique globally bounded solution on $\Omega \times [0,\infty)$.

Proof. Let $u(x,t)$ be the solution of IBVP(1) on $\Omega \times [0,T)$, and again consider the Lyapunov functional (23):

$$V[u](t) \equiv \int_\Omega | \bigtriangledown u|^2 dx + \frac{2\delta}{p-1} \left(\int_\Omega e^u dx \right)^{1-p}.$$

Then $V[u](t)$ is non-increasing with $V[u](t) \leq V[u_0] = E_0$ which implies $\int_\Omega | \bigtriangledown u|^2 dx \leq E_0$.

By the Gilbarg-Trudinger sharpening of the Sobolev inequality (see Fila [10], p. 237),

$$\int_\Omega \exp \left[C_1 \frac{|u|}{||u||} \right]^2 \leq C_2 |\Omega|$$

for any $u \in H_0^1(\Omega)$. This gives via Young's inequality

$$\int_\Omega e^{\beta u} dx \leq C_2 |\Omega| e^{\bar{\beta} ||u||} \quad , \quad \beta > 0.$$

This implies $e^u \in L_\beta$ and e^u is bounded in L_β for $1 \leq \beta < \infty$. By standard L_p estimates and Sobolev imbedding, u is Hölder continuous for $0 \leq t \leq T$ for any $T > 0$. Thus, u is a bounded global classical solution of IBVP(1) on $\Omega \times [0, \infty)$.

Theorem 4.4 Let $f(u) = e^u$.

1) For any nonnegative $u_0 \in H_0^1(\Omega)$ for $n = 1$ and $p \geq 1$, the unique global, bounded solution $u(x,t)$ of IBVP(1) converges in $H_0^1(\Omega)$ to the unique solution $\phi(x)$ of BVP(2) for any $\delta > 0$.

2) Let $n = 2$, $p > 1$, $\Omega = B_1(0)$. Assume $u_0(x) \in H_0^1(\Omega)$ is radially symmetric and nonnegative, then the unique global, bounded solution $u(x,t)$ of IBVP(1) converges in $H_0^1(\Omega)$ to the unique solution $\phi(x)$ of BVP(2) for any $\delta > 0$.

3) Let $n = 2$, $p = 1$, $\Omega = B_1(0)$. Assume $u_0(x) \in H_0^1(\Omega)$ is radially symmetric and nonnegative, then the unique global, bounded solution $u(x,t)$ of IBVP(1) converges to the unique solution $\phi(x)$ of BVP(2) for any $\delta < 8\pi$.

Proof. The result is immediate since in each case a Lyapunov functional exists and $u(x,t)$ is the unique globally bounded solution (see Bebernes and Talaga [3]).

Stability of steady states in the one-dimensional problems with $f(u) = e^u$ or linear was shown by Freitas [11]. For problems where f is decreasing stability of minimal steady states and global stability of unique steady states can be proved by using comparison methods (c.f. Lacey [18, 19]).

5. Finite-Time Blow-up for the Exponential

In this section, we consider IBVP(1) and the associated steady-state problem BVP(2) when $f(u) = e^u$, $p < 1$, and $n = 1$ or 2.

In spatial dimensions $n = 1$ or 2, IBVP(1) defines a local semiflow in $X^\alpha = H_0^1(\Omega)$, $\alpha = \frac{1}{2}$, where $X = L^2(\Omega)$ (Henry [13]).

For $0 < p < 1$, this local semiflow has a Lyapunov functional given by

$$\begin{aligned} V[u](t) &= \frac{1}{2}\int_\Omega | \bigtriangledown u|^2 dx + \frac{\delta}{p-1}\left(\int_\Omega e^u dx\right)^{1-p} \\ &= \frac{1}{2}||u||^2 + \frac{\delta}{p-1}\left(\int_\Omega e^u dx\right)^{1-p}. \end{aligned} \tag{30}$$

This semiflow is gradient-like in the sense that for any $t \in [0, \sigma)$

$$\int_o^t ||u_t||_2^2 + V[u](t) = V[u_0] \tag{31}$$

We now proceed as in Fila [10].

Lemma 5.1. If u is a global solution of IBVP(1), then there exists $\kappa = \kappa(u_0)$ such that

$$||u(t,u_0)||_2 = \left(\int_\Omega |u|^2\right)^{\frac{1}{2}} \leq \kappa \ \text{ for all } t \geq 0. \tag{32}$$

Proof. Multiplying (1a) by u and integrating over Ω, we have

$$\frac{1}{2}\frac{d}{dt}\int_\Omega |u|^2 = -\int_\Omega | \bigtriangledown u|^2 + \int_\Omega \frac{\delta u e^u}{(\int_\Omega e^u)^p}. \tag{33}$$

For $\varepsilon \geq 0$, using (30) and (31), from (33) we have

$$\frac{1}{2}\frac{d}{dt}\int_\Omega |u|^2 \geq -2V[u_0] + \frac{\delta}{(\int e^u)^p}\left[\int_\Omega \left\{ue^u - \left(\frac{2}{1-p} + \varepsilon\right)e^u\right\} + \varepsilon\int_\Omega e^u\right]. \tag{34}$$

It is straightforward to verify that there exists $C_1(\epsilon) > 0$ such that

$$ue^u - \left(\frac{2}{1-p} + \varepsilon\right)e^u \geq -C_1. \tag{35}$$

From (34), we have using (35):

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\int_\Omega |u|^2 &\geq -2V[u_0] + \frac{\delta}{(\int_\Omega e^u)^p}\left[-C_1|\Omega| + \varepsilon\int_\Omega e^u\right] \\ &\geq -2V[u_0] - \delta C_1|\Omega|^{1-p} + \delta\varepsilon\left(\int_\Omega e^u\right)^{1-p}. \end{aligned} \tag{36}$$

Hence,

$$\frac{d}{dt}\int_{\Omega}|u|^2 \geq 2\delta\varepsilon\left(\int_{\Omega}e^u dx\right)^{1-p} - C_2 \tag{37}$$

where $C_2 = 2(2V[u_0] + \delta C_1|\Omega|^{1-p})$.

Setting $\alpha = 1-p$, $\beta = 4/\alpha$ and applying Hölder's inequality, there are constants C'_3, $C_3 > 0$ such that

$$\left(\int_{\Omega}e^u\right)^{\alpha} \geq C'_3\left(\int_{\Omega}u^{\beta}dx\right)^{\alpha} \geq C_3\left(\int_{\Omega}u^2\right)^2. \tag{38}$$

From (37) and (38), we have

$$\frac{d}{dt}\left(\int_{\Omega}|u|^2\right) \geq 2\delta\varepsilon C_3\left(\int_{\Omega}|u|^2\right)^2 - C_2, \tag{39}$$

and $w(t) = \displaystyle\int_{\Omega}|u|^2$ is an upper solution of

$$y' = 2\delta\varepsilon C_3 y^2 - C_2. \tag{40}$$

If there is a $t_0 \geq 0$ for which $w(t_0) > \kappa \equiv \left(\dfrac{C_2}{2\delta\varepsilon C_3}\right)^{\frac{1}{2}}$, then $w(t)$ blows up in finite time. This is a contradiction since u is global. Hence $|u(t,u_0)|_2 \leq \kappa$.

Lemma 5.2. If $||u(t,u_0)|| = \left(\displaystyle\int_{\Omega}|\bigtriangledown u|^2dx\right)^{\frac{1}{2}} \to \infty$ as $t \to t_m$, then $t_m < \infty$.

Proof. Assume $t_m = \infty$ so that $u(t,u_0)$ exists globally.
Set

$$M(t) \equiv \int\int_{Q_t}u^2 = \int\int_{Q_t}|u|^2.$$

Then $M'(t) = \displaystyle\int_{\Omega}u^2 = \int_{\Omega}u_0^2 + \int\int_{Q_t}(u^2)_t = \int_{\Omega}u_0^2 + 2\int_0^t\left(-||u||^2 + \int_{\Omega}\frac{\delta u e^u}{(\int_{\Omega}e^u)^p}\right)$

and, using (30) and (31),

$$
\begin{aligned}
M''(t) &= 2\left(-||u||^2 + \int_\Omega \frac{\delta u e^u}{(\int_\Omega e^u)^p}\right) \\
&= 2\left(-(2+\varepsilon)V[u](t) + \frac{\varepsilon}{2}||u||^2 \right. \\
&\quad \left. + \int_\Omega \frac{\delta u e^u}{(\int_\Omega e^u)^p} + (2+\varepsilon)\frac{\delta}{p-1}\left(\int_\Omega e^u\right)^{1-p}\right) \\
&= 2\left(-(2+\varepsilon)V[u_0] + (2+\varepsilon)\int_0^t |u_t|_2^2 + \frac{\varepsilon||u||^2}{2} \right. \\
&\quad \left. + \int \frac{\delta u e^u}{(\int_\Omega e^u)^p} + (2+\varepsilon)\frac{\delta}{p-1}\left(\int_\Omega e^u\right)^{1-p}\right) \\
&= 2(2+\varepsilon)\int_0^t |u_t|_2^2 + 2\left(-(2+\varepsilon)V[u_0] + \frac{\varepsilon}{2}||u||^2 \right. \\
&\quad \left. + \int_\Omega \frac{\delta u e^u}{(\int_\Omega e^u)^p} + (2+\varepsilon)\frac{\delta}{p-1}\left(\int_\Omega e^u\right)^{1-p}\right) .
\end{aligned}
$$

By (35),

$$
\int \frac{\delta u e^u}{(\int_\Omega e^u)^p} + \frac{(2+\varepsilon)\delta}{p-1}\left(\int e^u\right)^{1-p} \geq -\delta C_1(\epsilon/(1-p))|\Omega|^{-p} \equiv -C.
$$

Thus,

$$
M''(t) \geq 2(2+\varepsilon)\int_0^t |u_t|_2^2 + 2\left(-(2+\varepsilon)V[u_0] + \frac{\varepsilon}{2}||u||^2 - C\right).
$$

Since $\varepsilon > 0$ and $||u|| \to \infty$ as $t \to \infty$, $M''(t) \to \infty$ as $t \to \infty$. But by Lemma 5.1, $M'(t) \leq \kappa^2$ for $t \geq 0$. This is a contradiction and we must conclude $t_m < \infty$.

Lemma 5.3. Let $u(t, u_0)$ be a global solution with $\omega(u_0) \neq \emptyset$. If $w \in \omega(u_0)$ is an equilibrium solution, then $||w|| \leq K = K(u_0)$.
Proof. By (31), we have $V[u_0] \geq V[u](t)$. Thus, for an equilibrium $w \in \omega(u_0)$

$$
(2+\varepsilon)V[u_0] \geq (2+\varepsilon)V[w].
$$

But $||w||^2 = \delta \int_\Omega \frac{we^w}{(\int_\Omega e^w)^p}$ gives

$$
(2+\varepsilon)V[w] = \frac{\varepsilon}{2}||w||^2 + \frac{\delta}{(\int_\Omega e^w)^p}\left[\int_\Omega we^w + \frac{(2+\varepsilon)}{p-1}\left(\int_\Omega e^w\right)\right].
$$

There exists $C > 0$ such that

$$
\int_\Omega we^w - \frac{(2+\varepsilon)}{1-p}\int_\Omega e^w > -C.
$$

Hence

$$\begin{aligned}(2+\varepsilon)V[u_0] &> \frac{\varepsilon}{2}||w||^2 - \frac{\delta}{(\int_\Omega e^w)^p}C \\ &> \frac{\varepsilon}{2}||w||^2 - \frac{\delta C}{|\Omega|^p}\end{aligned}$$

and we have

$$||w|| \le K(u_0)$$

for all equilibria $w \in \omega(u_0)$.

Lemma 5.4. Assume $n = 1$ or 2 and $u_0 \in H_0^1(\Omega)$. Let $u(t, u_0)$ be a global solution with

$$\liminf_{t\to\infty} ||u(t,u_0)|| = k < \infty \qquad \limsup_{t\to\infty} ||u(t,u_0)|| = +\infty \tag{41}$$

then for every B sufficiently large there is an equilibrium $w \in \omega(u_0)$ with $||w|| = B$.

Proof. By (41), there exist sequences $\{t_n\}, \{s_n\},\ s_n,\ t_n \to \infty$ satisfying

a) $||u(t_n, u_0)|| = B$
b) $||u(t, u_0)|| \le B \quad , \quad t \in [t_{2n},\ t_{2n+1}]$
c) $s_n \in (t_{2n}, t_{2n+1}),$ with $||u(s_n, u_0)|| = k + 1.$

By the variation of constants formula (Henry [13]),

$$u(t,u_0) = e^{(t-t_0)A}u(t_0) + \int_{t_0}^{t} e^{(t-s)A}\frac{\delta e^{u(s)}}{[\int_\Omega e^u dx]^p}ds. \tag{42}$$

There exists $M > 0$ such that for $t_0 = s_n\ ,\ t = t_{2n+1}$

$$||u(t_{2n+1})|| \le M||u(s_n)|| + M\int_{s_n}^{t_{2n+1}} (t_{2n+1}-s)^{-\frac{1}{2}}|e^{u(s)}|_2 ds. \tag{43}$$

On $[t_{2n}, t_{2n+1}]$, $|e^{u(s)}|_2$ is bounded by a constant $L(B) = L$. Let $B > M(k+1)$, then, since $||u(t_{2n+1})|| = B$ and $||u(s_n)|| = k+1$, we see that there exists $\delta > 0$ such that $t_{2n+1} - t_{2n} \ge \delta > 0$ for all n. For any $\beta \in (\frac{1}{2}, 1)$, from (42) we have

(44)

$$||u(t_{2n+1})||_\beta \le M\delta^{-(\beta-\frac{1}{2})}||u(t_{2n+1}-\delta)|| + M\int_{t_{2n+1}-\delta}^{t_{2n+1}}(t_{2n+1}-s)^{-\beta}|e^{u(s)}|_2 ds$$

where $||\cdot||_\beta$ is the norm in X^β. Since the right-hand side of (44) is bounded and X^β is compactly embedded into X^α if $\beta > \alpha$, there exists a convergent subsequence of $\{u(t_{2n+1})\}$, converging in $X^\alpha = H_0^1(\Omega)$. Because the local flow admits a Lyapunov functional V given by (30), the limit w is a steady state solution of

$$\begin{aligned}-\Delta u &= \delta\frac{e^u}{(\int_\Omega e^u dx)^p},\ x \in \Omega, \quad p < 1, \\ u(x) &= 0 \qquad ,\quad x \in \delta\Omega\end{aligned}$$

with $||w|| = B$.
Lemma 5.5. If $n = 1$ or 2, $u_0 \in H_0^1(\Omega)$, and if $u(t, u_0)$ is a global solution of IBVP(1), then

i) $\sup_{t \geq 0} ||u(t, u_0)|| < \infty$

and

ii) $\sup_{t \geq \tau} |u(t, u_0)|_\infty < \infty$ for any $\tau > 0$.

Proof. By Lemma 5.2, $||u(t, u_0)|| \not\to \infty$ as $t \to \infty$. By Lemma 5.4, if $\limsup_{t \to \infty} ||u(t, u_0)|| = +\infty$ and $\liminf_{t \to \infty} ||u(t, u_0)|| < \infty$, then for B sufficiently large there exist $w \in \omega(u_0)$ where w is a steady state solution of IBVP(1) with $||w|| = B$. This contradicts Lemma 5.3. Hence, $\sup_{t \geq 0} ||u(t, u_0)|| < \infty$.

Conclusion ii) follows by (42) and the continuity of the embedding X^β into L^∞ for $\beta \in (\frac{1}{2}, 1)$.
Theorem 5.6. Let $n = 1$ or 2, $p < 1$ and $u_0 \in H_0^1(\Omega)$. If $\delta > \delta^*$, $u(t, u_0)$ blows up in finite time T.
Proof. If $u(t, u_0)$ is global, then, by Lemma 5.5, $\sup_{t \geq 0} ||u(t, u_0)|| < \infty$ and $\sup_{t \geq \tau} |u(t, u_0)|_\infty < \infty$, $\tau > 0$. This implies the existence of a steady state solution $w \in \omega(u_0)$. But $\delta > \delta^*$ implies no such steady-state solution exists. Thus, $||u(t, u_0)|| \to \infty$ as $t \to T$. By Lemma 5.2, $T < \infty$.

Remarks

1) If $p < 1$ and $f(u) = e^u$, then for every $\delta > 0$, the solution u of IBVP (1) with sufficiently large data blows up in finite time. This follows from Lemma 5.1.

2) Blow-up for radially symmetric problems with all of $\delta > \delta^*$, $p \geq 2$, f decreasing, and $\int^\infty f\, du < \infty$, can again be proved using comparison arguments (see [18, 19] or Tzanetis [23]). The same methods quickly yield unboundedness of solutions for all cases of $\delta \geq \delta^*$ with f decreasing.

Acknowledgment

The authors wish to thank Marek Fila for several useful comments particularly regarding Theorem 4.3, Lemma 5.2, and Remark 1 above.

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