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SHEAR BAND FORMATION FOR A NON-LOCAL MODEL OF THERMO-VISCOELASTIC FLOWS

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Abstract. Many materials exhibit narrow bands of plastic deformation under extreme loading conditions. These shear bands are observed in very thin zones and are regarded as a precursor to material failure. Shear band formation is caused by the heat generated in regions with high strain-rate loading. With insufficient times for diffusion of this heat, a localised thermal softening of the material occurs, enhancing plastic flow in thin zones. A one-dimensional thermoviscous model for shear bands has previously been considered. Assuming a constitutive equation for the material to be of Maxwell type, a thermo-viscoelastic version of this model leads to a highly non-local and nonlinear parabolic equation, (A). This model exhibits various possibilities for shear band formation.

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1 Introduction

Many materials exhibit narrow bands of intense plastic deformation under extreme loading conditions. These shear bands are observed in very thin zones and are generally regarded as a precursor to material failure. The formation of shear bands plays a significant role in various manufacturing processes that involve high strain rates.

Shear band formation is caused by the heat generated in regions with high strain-rate loadings. With insufficient time for diffusion of this heat, a localised thermal softening of the material occurs, enhancing plastic flow in thin zones. This thermal softening overcomes hardening effects and precipitates stress collapse, creating the possibility of shear-band formation. Wright ([16] and references therein) describes various formulations of this phenomena and presents several model variants.

Neglecting hardening effects and inertia, the one-dimensional shearing model for thermoviscoelastic materials is given by the following system of conservation laws.

Let $z \in [0, d]$ be the spatial variable across the material (see Fig. 1), $w(z,t)$ the linear displacement with $v(z,t) = w_t(z,t)$ the velocity, both in the direction perpendicular to z. Also, let $\gamma(z,t) = w_z(z,t)$ be the plastic, or viscous, shear strain and $\tau = \tau(z,t)$ the stress of an elastic material where c is the specific heat, ρ the density, k the thermal conductivity and μ the elastic modulus. With $T(z,t)$ denoting the temperature of the material, the governing system is given by:

$$\begin{array}{lll} \rho v_t & = \tau_z & \text{(momentum);} \\ \tau_t & = \mu(v_z - \gamma_t) & \text{(elasticity);} \\ \rho c T_t & = (kT_z)_z + \beta\tau\gamma_t & \text{(energy);} \\ \gamma_t & = H(\tau, T) & \text{(constitutive).} \end{array} \tag{1}$$

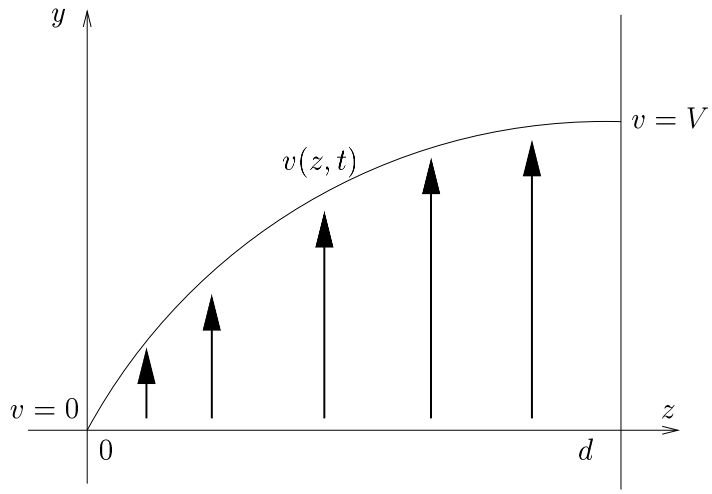

Figure 1: The velocity profile $v = w_t$ in the y direction.

A number of analytical and numerical studies using (1) have been made, usually by making simplifying assumptions such as assuming the stress to be constant and/or using specialised flow laws ([1], [2], [4], [5], [8], as well as [16] and references therein).

In this paper, we consider the non-dimensional quasi-linear system (5), presented in the next section, associated with system (1). We allow the stress to be time-dependent and assume a Maxwell-type flow law which, taking into account non-constant stress, allows the strain rate to be dependent on the rate of change of the stress.

1.1 Non-Dimensional Quasi-Static Model

Consider a material of constant density and constant specific heat with some sort of displacement X, subject some sort of forcing F. Assume it obeys a Maxwell-type constitutive law ([6], [10]),

$$\frac{\mathrm{d}X}{\mathrm{d}t} = a(T)F + b(T)\frac{\mathrm{d}F}{\mathrm{d}t}\,, \tag{2}$$

that is controlled so that, between the times t_1 and t_2, X, F and $\mathrm{d}F/\mathrm{d}t$ all go from 0 back to 0 (that is, from the reference state at rest back to the same) and T returns to its original value. Let Q be the rate of supply of energy other than mechanical work. Then the total added energy, in this case zero, is equal to the sum of the external energy supplied and the work done:

$$\begin{aligned} 0 &= \int_{t_1}^{t_2} Q\,\mathrm{d}t + \int_{t_1}^{t_2} F\frac{\mathrm{d}X}{\mathrm{d}t}\mathrm{d}t \\ &= \int_{t_1}^{t_2} Q\,\mathrm{d}t + \int_{t_1}^{t_2} \left(aF^2 - \frac{1}{2}\frac{\mathrm{d}b}{\mathrm{d}t}F^2\right)\mathrm{d}t + \left[\frac{b}{2}F^2\right]_{t_1}^{t_2} \\ &= \int_{t_1}^{t_2} Q\,\mathrm{d}t + \int_{t_1}^{t_2} \left(aF^2 - \frac{1}{2}\frac{\mathrm{d}b}{\mathrm{d}t}F^2\right)\mathrm{d}t\,. \end{aligned}$$

This indicates that the rate of mechanical dissipation is $\left(a - \frac{1}{2}\frac{\mathrm{d}b}{\mathrm{d}t}\right)F^2$.

The constitutive law (2) can be rewritten as:

$$\frac{\mathrm{d}X}{\mathrm{d}t} = \left(a - \frac{\mathrm{d}b}{\mathrm{d}t}\right)F + \frac{\mathrm{d}}{\mathrm{d}t}(bF) \equiv fF + \frac{\mathrm{d}}{\mathrm{d}t}(gF)\,, \tag{3}$$

where $b \equiv g$ and $a = f + \mathrm{d}g/\mathrm{d}t$. This gives a more obvious correspondence to a viscous liquid, when $\mathrm{d}X/\mathrm{d}t = fF$ so $f = 1/\text{viscosity}$, or to an elastic solid, when $\mathrm{d}X/\mathrm{d}t = (\mathrm{d}/\mathrm{d}t)(gF)$ so $X = gF$ and $g = 1/\text{elastic modulus}$. Thus the dissipation is then $\left(f + \frac{1}{2}\frac{\mathrm{d}g}{\mathrm{d}t}\right)F^2$.

In dimensionless form, the one-dimensional model (1) used by Wright [16] with this dissipation term can be written as:

$$\begin{aligned} \bar{\rho}v_t &= \tau_x \\ \tau_t &= \frac{1}{g}(v_x - \gamma_t) \\ u_t &= u_{xx} + \beta\tau\gamma_t \\ \gamma_t &= f(u)\tau + (g(u))_t\tau \end{aligned} \tag{4}$$

for $x \in [-1,1]$. If g is a constant, then system (4) is the same as the one-dimensional model used by Wright [16], subject to an appropriate change of variables and suitable β.

For $\bar{\rho} \ll 1$ and eliminating γ_t, (4) reduces to the quasi-static model

$$\begin{aligned} \tau_x &= 0 \\ v_x &= f(u)\tau + (g(u)\tau)_t \\ u_t &= u_{xx} + \left(f(u) + \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t} g(u) \right) \tau^2 \end{aligned} \tag{5}$$

for $x \in (-1,1)$, $t > 0$ with $u(-1,t) = 0 = u(1,t)$, $t > 0$, $u(x,0) = u_0(x) \geq 0$ for $x \in [-1,1]$, $v(-1,t) = 0$, $v(1,t) = V$ for $t \geq 0$, and $\tau(0) = 0$, if we take the temperature, as well as velocity, fixed at surfaces $x = \pm 1$.

Observe that the momentum equation in (5) implies the stress is only time dependent, $\tau = \tau(t)$.

1.2 Previous Work for a Thermoviscous Model

If $g \equiv 0$, (5) reduces to a non-local problem considered previously by the authors in ([1], [2], [11] and [12]). Since $V = \int_{-1}^{1} v_x \mathrm{d}x = \tau \int_{-1}^{1} f(u)\,\mathrm{d}x$ so that $\tau = V / \int_{-1}^{1} f(u)\,\mathrm{d}x$, we have

$$u_t - u_{xx} = \frac{V^2 f(u)}{\left(\int_{-1}^{1} f(u)\,\mathrm{d}x \right)^2}. \tag{6}$$

If $f(u) \geq c > 0$, then for any $V > 0$ the solution $u(x,t)$ of the IBVP (6) with $u(-1,t) = 0 = u(1,t)$, $u(x,0) = u_0$ exists for all $(x,t) \in [-1,1] \times [0,\infty)$ ([1], Theorem 4.1) and there is no blow-up.

For $V > 0$ fixed, blow-up is possible for $f(s) \to 0$ fast enough as $s \to \infty$. For integrable decreasing f rescaled so that $\int_0^\infty f(s)\,\mathrm{d}s = 1$, then the solution $u(x,t)$ of IBVP (6) blows up in finite time t^* for all $x \in (-1,1)$, provided $V^2 > 8$. In [12], the second author showed that $u \sim M(t)$ except in boundary layers near $x = \pm 1$ and obtained the formal estimate $\int_{-1}^{1} f(u)\,\mathrm{d}x \sim C f(M)$ for $C > 2$ as $t \to t^*$.

Then the displacement $W = w(1,t) = \int_0^t v(1,t)\,\mathrm{d}t = Vt \to Vt^*$ as $t \to t^*$.

Formally, if $f(s) \sim s^{-1-b}$ as $s \to \infty$, then $M(t) \sim C_1 (t^* - t)^{-\frac{1}{b}}$ and $\int_{-1}^{1} f(u)\,\mathrm{d}x \sim C_2 (t^* - t)^{1+\frac{1}{b}}$ as $t \to t^*$. This in turn implies

$$\left.\frac{\mathrm{d}w}{\mathrm{d}x}\right|_{x=1} = V f(0) \int_0^t \left(\int_{-1}^{1} f(u)\,\mathrm{d}x \right)^{-1} \mathrm{d}t \sim C_3 (t^* - t)^{-\frac{1}{b}} \quad \text{as} \quad t \to t^*.$$

This indicates two shear bands — one at each boundary. This case of blow-up requires the "viscosity" $= 1/f$ to get large as the temperature becomes large. We are not aware if such phenomena have been observed experimentally.

If the stress $\tau(t) = \tau$ is fixed instead of imposing V constant, then "conventional" blow-up is possible for the energy equations $u_t - u_{xx} = \tau^2 f(u)$ if $f(s) \to \infty$ as $s \to \infty$. Disregarding the logarithmic terms that should be included near blow-up points (see [3],

[7]) for $f(s) \sim s^{1+b}$ as $s \to \infty$, $u \sim C_1(t^* - t - C_2 x^2)^{-\frac{1}{b}}$, roughly speaking, as $t \to$ the blow-up time t^* and $x \to 0$ (assuming symmetry, *e.g.* $u(x,0) = 0$). Then

$$W = \int_0^t V(t)\,\mathrm{d}t = \int_0^t v(1,t)\,\mathrm{d}t = \tau \int_{-1}^1 \int_0^t f(u)\,\mathrm{d}t\,\mathrm{d}x$$

$$\sim \begin{cases} C_3\ , & b > 2 \\ C_4(t^* - t)^{1/2-1/b}, & b < 2 \end{cases}$$

as $t \to t^*$.

With $b > 2$, there are no difficulties with distances becoming infinite and $\mathrm{d}w/\mathrm{d}x$ is bounded for $x \neq 0$. But $\mathrm{d}w/\mathrm{d}x|_{x=0} \sim C_5 \int^t (t^* - t)^{-1-1/b}\,\mathrm{d}t \sim C_6(t^* - t)^{-1/b} \to \infty$ as $t \to t^*$. Thus, a shear band forms at the centre.

1.3 Present Work

The above results for a thermo-viscous model, although they do indicate that shear bands can form in the body of the flow, only do so for models of fixed stress τ. With fixed velocity V shear bands are not exhibited in the expected locations. The present paper considers the more general thermo-viscoelastic model (5) to examine whether this might or might not admit shear-band formation. Most attention is given to two special cases, $f \equiv g$ in Section 3 and $g \equiv 1$ in Section 4.

2 Models for Thermo-Viscoelastic Materials with $g \not\equiv 0$

For temperature dependent materials with constant specific heat and density, consider the initial-boundary value problem (5) for the quasi-linear system of conservation laws.

Integrating the second equation in (5) gives the first-order ODE:

$$\tau_t + \frac{\left[\int_\Omega f(u)\,\mathrm{d}x + \int_\Omega g'(u)u_t\,\mathrm{d}x\right]}{\int_\Omega g(u)\,\mathrm{d}x}\,\tau = \frac{V}{\int_\Omega g(u)\,\mathrm{d}x}$$

where $\Omega = (-1, 1)$.

For the initial value $\tau(0) = 0$, the solution is

$$\tau(t) = \frac{V}{\displaystyle\int_\Omega g(u)\,\mathrm{d}x} \int_0^t \exp\left[-\int_s^t R(z)\,\mathrm{d}z\right]\mathrm{d}s$$

with

$$R(t) = \frac{\int_\Omega f(u)\,\mathrm{d}x}{\int_\Omega g(u)\,\mathrm{d}x}\,.$$

Thus, the energy equation becomes

$$u_t - u_{xx} = \left(f(u) + \frac{1}{2}g'(u)u_t\right)\frac{V^2}{\left(\displaystyle\int_\Omega g(u)\,\mathrm{d}x\right)^2}\left[\int_0^t \exp\left[-\int_s^t R(z)\,\mathrm{d}z\right]\,\mathrm{d}s\right]^2$$

or

(A)
$$\left(1-\frac{V^2 g'(u)}{2\left(\displaystyle\int_\Omega g(u)\,\mathrm{d}x\right)^2}\left(\int_0^t \exp\left[-\int_s^t R(z)\,\mathrm{d}z\right]\right)^2\right)u_t-u_{xx}$$
$$=\frac{V^2 f(u)}{\left(\displaystyle\int_\Omega g(u)\,\mathrm{d}x\right)^2}\left[\int_0^t \exp\left[-\int_s^t R(z)\,\mathrm{d}z\right]\mathrm{d}s\right]^2.$$

The associated steady-state equation is:

(A′)
$$u_{xx}+\frac{V^2 f(u)}{\left(\int_\Omega f(u)\,\mathrm{d}x\right)^2}=0$$

because
$$R(t)\equiv R\,.$$

Theorem 1 *For $f(u)$ positive with $f(u)>0$ for $u\geq 0$, the BVP (A′) with zero boundary conditions has a unique solution provided that $V^2<8\int f(s)\,ds$.*

The proof of this theorem can be found in Bebernes & Lacey ([1], pp. 939–943). □

3 Special Case $f\equiv g$

If $f(u)\equiv g(u)$, then $R(t)=1$ and (A) becomes

(B)
$$\left(1-\frac{V^2 f'(u)(1-e^{-t})^2}{2\left(\int_\Omega f(u)\,\mathrm{d}x\right)^2}\right)u_t-u_{xx}=\frac{V^2 f(u)}{\left(\int_\Omega f(u)\,\mathrm{d}x\right)^2}(1-e^{-t})^2.$$

For $\Omega=(-1,1)$, consider the Dirichlet initial-boundary value problem (B) with

$$\begin{aligned} u(-1,t) &= u(1,t)=0, \quad t>0, \quad \text{and}\\ u(x,0) &= u_0(x), \quad x\in\Omega, \end{aligned}$$

where $u_0(x)\geq 0$ is smooth, $u_0(x)=0$ on $\partial\Omega$, and f is positive and C^1-smooth.

Write (B) as

(7)
$$F(x,t)u_t=u_{xx}+G(x,t)f(u)$$

where
$$F(x,t)=\left(1-\frac{V^2 f'(u)(1-e^{-t})^2}{2(\int_\Omega f(u)\,\mathrm{d}x)^2}\right)$$

and
$$G(x,t)=\frac{V^2(1-e^{-t})^2}{(\int_\Omega f(u)\,\mathrm{d}x)^2}\,.$$

(i) **Assume f increasing.**

Theorem 2 *Assume $f(u) \geq c > 0$ with $f' \geq 0$ sufficiently small so that $F(x,t) > 0$ for $x \in [-1,1]$, $t > 0$.*

If

$$w_{xx} + \frac{V^2}{4c^2} w = 0, \quad w(\pm 1) = 0 \tag{8}$$

has a solution w, then for any $0 \leq u_0(x) \leq w(x)$ on $[-1,1]$, then IBVP (7) has a unique bounded global solution $u(x,t)$.

Proof: The solution $w(x)$ of BVP (8) is an upper solution for IBVP (7). □

Remarks.

1) A similar argument establishes local existence of a bounded solution u of IBVP (B).

2) If $f(u) \to \infty$ as $u \to \infty$ with $F(x,t) > 0$, we conjecture that IBVP (B) has a unique global bounded solution. The "normal" non-local problem

$$u_t = u_{xx} + \frac{V^2 f(u)}{(\int_\Omega f(u)\,\mathrm{d}x)^2} \tag{9}$$

by Theorem 4.1 of [1], has a unique global bounded solution. For $F > 0$, we expect (B) to behave like (9).

On the other hand, if f is increasing but bounded, we have

Theorem 3 *Assume $f' > 0$, $f(0) = 1$ with $f \leq C_0$. Then some form of blow-up for IBVP (B) must take place in finite time for V sufficiently large.*

Proof: Let $u(x,t)$ be the solution of IBVP (B) and assume u experiences no form of blow-up. (Thus u is a classical global solution.) This implies $F(x,t) > 0$. Thus $u(x,t)$ can be bounded below by a lower solution $v(x,t)$ that satisfies an equation of the form

$$v_t - v_{xx} = K(1 - e^{-t})^2 f(v)\,.$$

Then $v(x,t) \to v^*(x)$ as $t \to \infty =$ where $v^*(x)$ is the solution of $-v_{xx} = Kf(v)$, $v(\pm 1) = 0$.

For $C_1 \in (0, \max_\Omega v^*)$, $u(x,t)$ must take this value at some points x for all $t \geq t_1 > 0$ for some t_1. For $V^2 > 8C_0^2/f'(C_1)$,

$$\frac{V^2 f'(C_1)}{8C_0^2} > C_2 > 1 \quad \text{for some} \quad C_2\,.$$

Then for $t_1 > 0$,

$$\begin{aligned} F(x,t) &= 1 - \frac{V^2}{2} f'(u) \left(\frac{1 - e^{-t}}{\displaystyle\int_\Omega f(u)\,\mathrm{d}x} \right)^2 < 1 - \frac{V^2}{2} f'(u) \frac{(1 - e^{-t})^2}{4C_0^2} \\ &< 1 - \frac{V^2 f'(C_1)}{8C_0^2}(1 - e^{-t})^2 < 1 - C_2(1 - e^{-t})^2 < 0 \end{aligned}$$

for some points $x \in (-1,1)$ for all $t \geq t_2$ for some $t_2 \geq t_1$. Thus, equation (B) has turned into a backward heat equation and some form of blow-up of some derivatives for u must take place at some $t^* < t_2$ with u bounded. □

(ii) **Assume f is decreasing.**

In this case there is no problem with possible ill-posedness arising from backward diffusion since

$$F(x,t) = 1 - \frac{V^2}{2} f'(u) \left(\frac{1-e^{-t}}{\int_\Omega f(u)\,\mathrm{d}x} \right)^2 \geq 1 > 0\,.$$

Note that if the "standard" non-local problem IBVP (9) has a bounded, increasing solution v, then v is an upper solution for IBVP (10) where

$$F(x,t)u_t - u_{xx} = \frac{V^2 f(u)}{(\int_\Omega f(u)\,\mathrm{d}x)^2} \tag{10}$$

and hence v bounds the solution for IBVP (B) assuming u_0 is small enough. Thus, the solution $u(x,t)$ of IBVP (B) is a global bounded solution. More particularly, we have

Theorem 4 *If $8\int_0^\infty f(s)\,ds > V^2$ and if $u_0(x) \leq z(x)$ on $[-1,1]$ where $z(x)$ is the unique solution of*

$$-z_{xx} = \frac{V^2 f(z)}{(\int_\Omega f(z)\,dx)^2}\ ,$$

then IBVP (B) has a unique global bounded solution.

Proof: This follows by Thm. 3.4 of [1] and above remarks. □

For f decreasing, since $F(x,t) > 1$, the only reason for $u(x,t)$, the solution of IBVP (B), to have a finite time of existence is that

$$\sup_{\bar\Omega} u(x,t) \to +\infty \quad \text{as} \quad t \to t^*- < \infty$$

but this can happen only if

$$h(t) = \left(\int_\Omega f(u)\,\mathrm{d}x \right)^{-2} \to \infty \quad \text{or equivalently} \quad \int_\Omega f(u)\,\mathrm{d}x \to 0 \ \text{as}\ t \to t^* -\ .$$

Because $u_0 \geq 0$ on $(-1,1)$, the solution of IBVP (B) satisfies $u \geq 0$ and $u_x(1,0) \leq 0 \leq u_x(-1,0)$ for $(x,t) \in \Omega \times (0,t^*)$. Set

$$H(t) = \left(\int_\Omega f(u)\,\mathrm{d}x \right)^{-1}$$

and integrate (B) over $\Omega = (-1,1)$ to get

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_\Omega u\,\mathrm{d}x + \frac{(1-e^{-t})^2 V^2}{2} H' = u_x(1,t) - u_x(-1,t) + V^2(1-e^{-t})^2 H$$

which gives the differential inequality

$$H' - 2H \leq -\frac{2}{(1-e^{-t})^2 V^2} \int_\Omega u_t \, \mathrm{d}x \, .$$

Solving this linear inequality and using integration by parts on $t > t_1 > 0$ gives

$$H(t) \leq e^{2(t-t_1)} H(t_1) + \left(\frac{-2e^{2t}}{V^2} \right) \left[\frac{e^{-2s}}{(1-e^{-s})^2} \int_\Omega u(x,s)\,\mathrm{d}x \Big|_{t_1}^{t} + \int_{t_1}^{t} \left(\int_\Omega u(x,s)\,\mathrm{d}x \right) \frac{2e^{-2s}\,\mathrm{d}s}{(1-e^{-s})^3} \right]$$

and hence

$$H(t) \leq e^{2(t-t_1)} H(t_1) + \frac{2e^{2t}}{V^2} \frac{1}{(e^{t_1}-1)^2} \int_\Omega u(x,t_1)\,\mathrm{d}x \, .$$

Thus $H(t)$ is bounded for bounded t and finite-time blow-up can not occur. This implies u is a global solution.

Thus, we have proven

Theorem 5 *If f is decreasing, then IBVP (B) has a unique global solution.*

If such a solution u is bounded, it would ultimately act as an upper solution to an increasing solution to a problem of the form

$$c\hat{u}_t - \hat{u}_{xx} = \widehat{V}^2 \frac{f(\hat{u})}{(\int_\Omega f(\hat{u})\,\mathrm{d}x)^2}$$

for a $\widehat{V}$ which can be arbitrarily close to V. For $8\int_0^\infty f(s)\,\mathrm{d}s < V^2$, $\widehat{V}$ can be chosen larger that $(8\int_0^\infty f(s)\,\mathrm{d}s)^{\frac{1}{2}}$ and $\hat{u}$ is unbounded (*e.g.*, by the results of [12]). It follows that

Corollary 1 *If f is decreasing with $\int_0^\infty f(s)\,ds < 8V^2$, then IBVP (B) has a unique, global, unbounded solution.*

4 Special Case $g = 1$

For $g = 1$, $R(t) = \int_\Omega f(u)\,\mathrm{d}x$ and the energy balance equation (A) becomes

(C) $$u_t - u_{xx} = V^2 f(u) \left[\int_0^t \exp\left(-\int_s^t \left[\int_\Omega f(u(y,z))\,\mathrm{d}y \right] \mathrm{d}z \right) \mathrm{d}s \right]^2 .$$

By setting

$$I(t) \equiv \int_0^t \exp\left(-\int_s^t \int_\Omega f(u(y,z))\,\mathrm{d}y\,\mathrm{d}z \right) \mathrm{d}x \, ,$$

and writing $\lambda = V^2$, then (C) is equivalent to

(11) $$\begin{cases} u_t - u_{xx} &= \lambda I^2 f(u) \\ I_t &= 1 - I \int_\Omega f(u)\,\mathrm{d}y \end{cases} .$$

Theorem 6 *The solution* $(u(t), I(t))$ *of the IVP (12) where*

$$\left\{ \begin{array}{rcl} u_t & = & \lambda I^2 f(u) \\ I_t & = & 1 - 2If \\ u(0) & = & u_0, \quad I(0) = 0, \end{array} \right. \tag{12}$$

satisfies $u(t) \leq u_0 + \lambda t^2/2$, $I(t) > 0$ *for* $t > 0$ *and hence* $u(t)$ *is a global solution for the homogeneous IVP (C).*

Proof: Since $I(0) = 0$, $I'(t) \leq 1$, and $0 \leq I(t) \leq t$. This implies

$$u' \leq \lambda t I f = \frac{\lambda t}{2}(1 - I'(t))$$

and hence

$$\begin{aligned} u(t) & \leq u_0 + \frac{\lambda t^2}{4} - \frac{\lambda}{2} \int_0^t s I'(s) \, \mathrm{d}s \\ & = u_0 + \frac{\lambda t^2}{4} - \frac{\lambda}{2} \left[t I(t) - \int_0^t I(s) \, \mathrm{d}s \right] \\ & \leq u_0 + \frac{\lambda t^2}{4} - \frac{\lambda t}{2} I + \frac{\lambda}{2} \int_0^t s \, \mathrm{d}s \\ & = u_0 + \frac{\lambda t^2 - \lambda t I}{2} \\ & \leq u_0 + \frac{\lambda t^2}{2}. \end{aligned}$$

□

Corollary 2 *If* f *is decreasing, then the solution* $u(x,t)$ *of*

$$\left\{ \begin{array}{rcl} u_t - u_{xx} & = & V^2 f(u) \left[\int_0^t \exp\left(- \int_s^t \left[\int_\Omega f(u) \, dy \right] dz \right) ds \right]^2 \\ u(x,0) & = & u_0(x) \geq 0 \\ u_x(-1,t) & = & 0 = u_x(1,t) \end{array} \right. \tag{C$'$}$$

satisfies

$$0 \leq u(x,t) \leq F^{-1} \left(\frac{V^2 t^3}{3} + F \left(\max_{[-1,1]} u_0(x) \right) \right)$$

where $F(z) = \displaystyle\int \frac{dz}{f(z)}$. *Thus,* $u(x,t)$ *is a global solution of (C′) and of (C).*

Proof: Since f is decreasing, the non-local integral term is monotone increasing in u. This means comparison techniques may be used. Then note that $z(t)$, the solution of

$$z' = V^2 t^2 f(z), \qquad z(0) = \max_{[-1,1]} u_0(x)$$

is an upper solution of (C′) and of (C). □

We now impose the additional condition that $f(u) = \nu e^u$, then

$$\text{(C)} \qquad u_t - u_{xx} = V^2 \nu e^u \left[\int_0^t \exp\left(-\nu \int_s^t \int_\Omega e^u \,\mathrm{d}y \,\mathrm{d}z \right) \mathrm{d}s \right]^2$$

is equivalent to

$$\text{(C}_\nu\text{)} \qquad \begin{cases} u_t - u_{xx} &= \lambda I^2 e^u \\ I_t &= 1 - \nu I \int_\Omega e^u \,\mathrm{d}z \end{cases}$$

with

$$\begin{cases} u(x,t) = 0, \quad x \in \partial\Omega, \quad t > 0 \\ u(x,0) = 0, \quad I(0) = 0, \quad x \in \Omega \end{cases}$$

when $\lambda = V^2 \nu$, by setting

$$I(t) = \int_0^t \exp\left(-\int_s^t \left\{ \int_\Omega \nu e^u \,\mathrm{d}y \right\} \mathrm{d}z \right) \mathrm{d}s \,.$$

Note that I is essentially the stress τ.

Lemma 1 *For $\lambda > 0$ and $\nu = 0$, the solution of IBVP (C_0) is given by $(u(x,t), I(t)) = (u_0(x,t), t)$ on $(-1,1) \times [0, t_0)$ where $t_0 < \infty$ and $u_0(x,t)$ is monotonic increasing with single-point blow-up at $x = 0$ as $t \to t_0-$.*

Proof: Observe that (C_0) becomes $u_t - u_{xx} = \lambda t^2 e^u$. □

Lemma 2 *For $\nu > 0$, the solution $(u(x,t), I(t))$ of IBVP (C_ν) satisfies:*

i) *$I(t) \le t$,*

ii) *$u(x,t)$ is radially symmetric and radially decreasing with $u(x,t) < u_0(x,t)$, $|x| < 1$, $0 \le t < t_0$,*

iii) *$u(x,t)$ depends continuously on x, t, ν for $\nu > 0$ and (x,t) in compact subsets of $(-1,1) \times [0, t_0)$,*

iv) *$u(x,t) \to u_0(x,t)$ uniformly on compact sets of $(-1,1) \times [0,t_0)$ as $\nu \to 0+$.*

Lemma 3 *Given I_1, I_2 such that*

$$0 < I_2 < I_1 < t_0 \,,$$

there exist $\nu_1 > 0$, $t_1 \in (0, t_0)$ such that $I(t_1) \ge I_1$ for all $\nu \in (0, \nu_1]$.

Proof: $I_\nu(t) \to I_0(t) = t$ as $\nu \to 0+$. □

Lemma 4 *For $\nu \in (0, \nu_1)$, there exist $q \in (0,1)$, $\varepsilon > 0$ such that*

$$(13) \qquad u(x,t) \le -\frac{1}{q} \ln \left[\frac{\varepsilon q x^2}{2} + e^{-qM(t)} \right] \le -\frac{1}{q} \ln \left[\frac{\varepsilon q x^2}{2} \right]$$

for $t \ge t_1$, $|x| \le 1$ as long as the solution $u(x,t)$ of IBVP (C_ν) exists.

Proof: Let $M(t) \equiv u(0,t)$. Following a Friedman–McLeod type of approach (see [9]), note that there exists some $c > 0$ such that $-u_x \geq cx$ for $0 \leq x \leq 1$, $t \geq t_1$, for all $0 < \nu \leq \nu_1$ as long as $I \geq I_2$. Define

$$J(x,t) = u_x + \varepsilon x e^{qu}$$

for some chosen $q \in (0,1)$ and $\varepsilon > 0$ to be determined. For ε taken so that $\varepsilon e^{qu_0(0,t_1)} \leq c$, then $J \leq 0$ on $x \in [0,1]$, $t = t_1$ and for $x = 0$ or $x = 1$ with $t \geq t_1$.

A straightforward computation yields

$$J_t - J_{xx} = \lambda I^2 e^u \left(1 - \frac{2\varepsilon q e^{(q-1)u}}{\lambda I^2}\right) J + \varepsilon x e^{qu}(\lambda I^2 e^u (q-1) + 2\varepsilon q e^{qu}) - \varepsilon x q^2 e^{qu} u_x^2.$$

For $0 < q < 1$ fixed, we can choose $\varepsilon > 0$ sufficiently small to have:

$$J_t - J_{xx} \leq \lambda I^2 e^u \left(1 - \frac{2\varepsilon q e^{(q-1)u}}{\lambda I^2}\right) J \equiv \lambda I^2 e^u \beta J\,.$$

Thus, as long as $I \geq I_2$ and solution $u(x,t)$ exists for $0 \leq \nu \leq \nu_1$, $J_t - J_{xx} - \beta\lambda I^2 e^u J \leq 0$ for $x \in (0,1)$, $t \geq t_1$. Thus, $J(x,t) \leq 0$ on $0 \leq x \leq 1$, $t \geq t_1$. A similar argument works on $[-1,0]$. This implies

$$u(x,t) \leq -\frac{1}{q} \ln \left[\frac{\varepsilon q}{2} x^2 + e^{-qu(0,t)}\right] \leq -\frac{1}{q} \ln \left[\frac{\varepsilon q}{2} x^2\right]$$

for $|x| \leq 1$, $t > t_1$, as long as $I \geq I_2$ and the solution exists. □

Lemma 5 *Given $x_1 \in (0,1)$,*

$$\frac{\partial u_0}{\partial t} \geq 3\eta e^{u_0(x,t)}, \quad |x| \leq x_1, \quad t \in [t_1, t_0)\,,$$

where $\eta > 0$ is chosen so that

$$\eta \leq \frac{e^{-u_0}}{3} \frac{\partial u_0}{\partial t} \tag{14}$$

for $|x| \leq x_1$, $t = t_1$, and for $|x| = x_1$, $t \in [t_1, t_0)$.

Proof: Let $x_1 \in (0,1)$ be given. By the Maximum Principle, $\partial u_0/\partial t$ is bounded away from zero on $[-x_1, x_1] \times [t_1, t_0)$. Thus we can choose $\eta > 0$ such that $\eta < e^{-u_0(x,t)} \dfrac{\partial u_0}{\partial t}(x,t)$ for $|x| \leq x_1$, $t = t_1$ and for $|x| = x_1$, $t \in [t_1, t_0)$. Next we define $J(x,t) \equiv \partial u_0/\partial t - 3\eta e^{u_0}$. It is straightforward to show that

$$J_t - J_{xx} - \lambda t^2 e^{u_0} J = 2\lambda e^{u_0} t + 3\eta e^{u_0} u_{0x}^2 \geq 0$$

for $x \in (-x_1, x_1)$, $t \in (t_1, t_0)$. Thus, $\partial u_0/\partial t \geq 3\eta e^{u_0}$ for $|x| \leq x_1$, $t \in [t_1, t_0)$ provided (14) holds. □

Lemma 6 *For t_2 such that $t_1 \le t \le t_2 < t_0$ and given $x_2 \in (0, x_1)$, there exists $\nu_2 = \nu(t_2)$ such that for $0 < \nu \le \nu_2 < \nu_1$*

$$\frac{\partial u}{\partial t} \ge 2\eta e^{u_0} \ge 2\eta > 0$$

for $|x| = x_2$, $t_1 \le t \le t_2$ and for $|x| \le x_2$, $t = t_1$ where η is given by (14).

Proof: This follows by continuous dependence of u and u_t on ν and t. □

Lemma 7 *For some $x_3 \in (0, x_2)$ and some $\Delta t > 0$ (independent of t_2),*

$$\frac{\partial u}{\partial t} \ge \eta > 0 \quad for \quad |x| = x_3, \quad t \in [t_2, t_2 + \Delta t]\,,$$

and

$$\frac{\partial u}{\partial t} \ge \eta e^{u_0} \ge \eta e^{u} \quad for \quad t = t_2, \quad |x| < x_3\,,$$

provided that

$$I \ge I_4 \ge I_2 \quad with \quad I_3 - C_1\eta \le I_4 \le I_3 \equiv I(t_2) \tag{15}$$

for some $C_1 > 0$.

Proof: By Lemma 4, u is bounded, away from $x = 0$, and $I(t)$ is bounded above by t. Thus, by standard continuity results, u_x, u_{xx}, u_{xxx} are bounded on compact subsets of

$$R \equiv \{(x,t) : 0 < x_3 < x < x_2,\ 0 < t_1 < t < t_0\}.$$

From $u_t = u_{xx} + \lambda I^2 e^u$ with I, u, and u_x bounded as long as u exists, $|\Delta u| \le C_0(\Delta t)^{1/2}$. Again as long as u exists, from

$$u_{xxt} = u_{xxxx} + \lambda I^2 e^u (u_x^2 + u_{xx})$$

with $I, u, u_x, u_{xx}, u_{xxx}$ bounded, we have

$$|\Delta u_{xx}| \le C_2 |\Delta t|^{1/2}.$$

Thus, since $I \le t$,

$$\begin{aligned}
\Delta u_t &= \Delta u_{xx} + \lambda \Delta(I^2 e^u) \\
&\ge -C_2(\Delta t)^{1/2} - \lambda t^2 e^m |\Delta u| + 2\lambda t e^m (\Delta I)^- \\
&\ge -(C_2 + \lambda t^2 e^m C_0)(\Delta t)^{1/2} + 2\lambda t e^m (\Delta I)^- \\
&\ge -(C_2 + \lambda(t_0 + \Delta t)^2 e^m C_0)(\Delta t)^{1/2} + 2\lambda(t + \Delta t)e^m(\Delta I)^-\,,
\end{aligned}$$

where

$$m = \sup_R \left(-\frac{1}{q} \ln\left(\frac{\varepsilon q x^2}{2}\right)\right).$$

We take Δt small enough so that $(C_2 + \lambda(t_0 + \Delta t)^2 e^m C_0)(\Delta t)^{1/2} < \eta/2$ and choose $C_1 = 1/(4\lambda(t_0 + \Delta t_0)e^m)$. Note that $(\Delta I)^- \ge I_4 - I_3 \ge -C_1\eta$ provided that (15) holds. Then

$$u_t = u_t|_{t=t_2} + \Delta u_t > 2\eta - \eta = \eta\,.$$

This means that $u_t \ge \eta$ for $|x| = x_3$, $t_2 \le t \le t_2 + \Delta t$ as well as $u_t \ge \eta e^u$ for $t = t_2$, $|x| < x_3$. □

Lemma 8 *For t_2 sufficiently close to t_0, ν sufficiently close to 0 such that $I_3 > I_4$, then for some $b > 0$*

$$u_t \geq be^u$$

for t near t_2 and as long as solution $u(x,t)$ exists with $I \geq I_4$.

Proof: Define $J(x,t) \equiv u_t - (\lambda I^2 - a)e^u$ on $\{(x,t) : |x| \leq x_3, t_2 \leq t\}$ with $a > 0$ to be determined.

On $D \equiv \{(x,t) : |x| < x_3,\ t \in [t_2, t_2 + \Delta t]\}$, J satisfies $J_t - J_{xx} \geq \lambda I^2 e^u J$ provided that

$$a < \lambda I^2 \tag{16}$$

which is guaranteed by $a < \lambda I_4^2$ as long as $I \geq I_4 \geq I_2$. By Lemma 7, $J \geq \eta - (\lambda I^2 - a)e^u$ on $|x| = x_3$ provided, by (13),

$$a > \lambda(t_0 + \Delta t)^2 - \eta \left(\frac{\varepsilon q x_3^2}{2}\right)^{1/q} \tag{17}$$

and

$$J \geq \eta e^u - (\lambda I_3^2 - a)e^u > 0$$

at $t = t_2$ provided $a \geq \lambda I_3^2 - \eta$ which holds if

$$a > \lambda t_0^2 - \eta\,. \tag{18}$$

Take

$$I_4^2 \in (\max\{t_0^2 - C_3\eta, t_0^2 - \eta/\lambda\}, t_0^2)$$

for some $C_3 < (\varepsilon q x_3^2/2)^{1/q}$ with $\lambda \max\{t_0^2 - C_3\eta, t_0^2 - \eta/\lambda\} < a < \lambda I_4^2$. Then (17) and (18) hold, taking if need be a smaller value of Δt. For t_2 close to t_0 and ν_0 close to 0 so that $I_3 > I_4 \geq I_3 - C_1\eta$, then (15) also holds. Thus, $J \geq 0$ and

$$u_t \geq (\lambda I^2 - a)e^u \tag{19}$$

as long as $I \geq I_4$. From (19), we have $u_t > be^u$ with $b > 0$ for t near t_2 as long as solution exists with $I \geq I_4$. □

Theorem 7 *Consider the IBVP (C_ν) where*

$$\left\{ \begin{array}{rl} u_t - u_{xx} & = \lambda I^2 e^u \\ I_t & = 1 - \nu I \int_\Omega e^u \, dx \end{array} \right. \tag{C_ν}$$

with

$$u(x,0) = 0, \quad x \in \Omega = (-1,1),\ I(0) = 0\,,$$
$$u(x,t) = 0, \quad x \in \partial\Omega,\ t > 0\,,$$

and $\lambda > 0$, $\nu \geq 0$. For any ν sufficiently small, there exists $t_\nu \in (t_0, \infty)$ such that

$$\sup_{x\in\Omega} u(x,t) \to \infty \quad as \quad t \to t_\nu -\,.$$

Proof: Let $M(t) = u(0,t)$. Then by Lemma 8

$$M' \geq be^M \tag{20}$$

provided $I \geq I_4$ for $t_2 \leq t \leq t_3$. From (20)

$$M(t) \geq -\ln[e^{-M(t_2)} - b(t - t_2)]$$

which implies $M(t)$, and therefore $u(x,t)$, must blow up by a time

$$t_3 \equiv t_2 + \frac{1}{b} e^{-M(t_2)} \geq t_2 + \Delta t$$

provided *(i)* $I \geq I_4$ and *(ii)* $e^{-M(t_2)} \leq b\Delta t$. On taking t_2 close to t_0 and η close to 0, *(ii)* holds.

By Lemma 4,

$$e^u \leq \left(\frac{\varepsilon q}{2} x^2 + e^{-qM(t)}\right)^{-1/q} = e^M \left(1 + \frac{\varepsilon q x^2}{2} e^{qM(t)}\right)^{-1/q}$$

so

$$\begin{aligned} \int_{-1}^{1} e^{u(x,t)}\, \mathrm{d}x &\leq e^{M(t)} \int_{-1}^{1} \left(1 + \frac{\varepsilon q x^2}{2} e^{qM(t)}\right)^{-1/q} \mathrm{d}x \\ &\leq C_3 e^{(1-q/2)M} \end{aligned}$$

where

$$C_3 = \varepsilon^{-1/2} \int_{-\infty}^{\infty} \left(1 + \frac{q}{2} x^2\right)^{-1/q} \mathrm{d}x\,.$$

This implies

$$\frac{\mathrm{d}I}{\mathrm{d}t} \geq 1 - C_3 \nu e^{(1-q/2)M} I \geq -\nu C_3 e^{(1-q/2)M} I$$

as long as $I > I_4$ and the solution exists.

Thus,

$$\frac{\mathrm{d}I}{\mathrm{d}M} \geq -\frac{\nu C_3}{b} e^{-\frac{qM}{2}} I$$

so

$$I \geq I_3 \exp\left[\frac{2\nu C_3}{qb}\left(e^{-\frac{qM}{2}} - e^{\frac{-qM(t_2)}{2}}\right)\right] \geq I_4$$

provided

$$\nu \leq \frac{qb}{2C_3} \ln\left(\frac{I_3}{I_4}\right) e^{\frac{qM(t_2)}{2}}$$

which implies *(i)*. Thus $u(x,t)$ blows up as $t \to t_\nu -$. □

Corollary 3 *For the IBVP (C_ν), the blow-up set T for $u(x,t)$ consists of the single point $x = 0$.*

Proof: By Lemma 4,

$$u(x,t) \le -\frac{1}{q}\ln\frac{\varepsilon q x^2}{2} \quad on \quad [0,t_\nu)$$

where

$$\limsup_{t\to t_\nu} u(x,t) = \infty\,.$$

□

Theorem 8 *For ν sufficiently large, the solution $u(x,t)$ of IBVP (C_ν) is bounded for all $t \ge 0$.*

Proof: Since $I' \le 1-2\nu I$, $I(t) \le (1-e^{-2\nu t})/2\nu \le 1/2\nu$ which gives $u_t - u_{xx} \le \lambda e^u/4\nu^2$. Thus IBVP ($\mathrm{C}_\nu$) has no blow-up if $\lambda/4\nu^2 \le \delta^*$ where

$$\delta^* = \sup\{\delta > 0 : w'' + \delta e^w = 0\,,\ w(-1) = 0 = w(1), \quad \text{has a solution}\}.$$

Thus, for

$$\nu \ge \frac{1}{2}\left(\frac{\lambda}{\delta^*}\right)^{1/2},$$

IBVP (C_ν) has no blow-up. □

5 Conclusions

We have investigated two fairly special thermo-viscoelastic versions of the earlier thermoviscous model for shear bands. Both models assume a constitutive equation for the material to be of Maxwell type. The velocities at the surface of the material are fixed and constant, leading to non-local models for the temperature u.

The first version of the model has v_x = strain rate = $f(u)\tau + (f(u)\tau)_t$ for a shear stress $\tau(t)$. With f (which might be thought of as a measure of the weakness of the material) bounded away from zero, then for small enough V no blow-up of temperature occurs, and no shear bands form. However, even for f both bounded and bounded away from 0, if f is increasing, a sufficiently large given velocity V leads to some form of "blow-up" taking place with some derivative of u, but not u itself, becoming unbounded in finite time. It is not clear what the implications of such behaviour might be regarding shear bands. Conversely, if f is decreasing then no blow-up of any type takes place, although if $\int_0^\infty f(s)\,\mathrm{d}s$ is bounded and V is large enough the temperature turns out to be unbounded. Again, no shear bands can form in a finite time.

The second version of the model is $v_x = f(u)\tau + \tau_t$. The homogeneous version of the problem, which requires uniform initial temperature and insulating boundary conditions, is seen to have a global solution, albeit an unbounded one; no shear band forms. Similarly, for f decreasing (the material hardens with increasing temperature), a bound for temperature as a function of time can be obtained.

Most attention is focused on the very special case of $f(u) = e^u$, although the results are expected to carry over to other sufficiently rapidly increasing functions. For a parameter ν, roughly equivalent to V^{-2}, sufficiently small (V large), single-point blow-up of u is seen to occur, with stress τ both bounded and bounded away from zero. Integration of the constitutive relations gives

$$w_x = \tau + \int_0^t e^u \tau \, \mathrm{d}t \, .$$

As t approaches the blow-up time t^*, this remains bounded for $x \neq 0$, but must become infinite at $x = 0$. (Because $u_x < 0$ for $x > 0$ and $u_x > 0$ for $x < 0$, $u_{xx} \leq 0$ at $x = 0$. Then $u_t(0,t) \leq \lambda I^2 e^u$ so $\int_0^t e^u \mathrm{d}t \to \infty$ as $t \to t^*$.) A shear band forms along the centre of the material. With ν large, u remains bounded (as can be expected) and there is then no shear band.

References

[1] J. Bebernes and A. Lacey, *Global existence and finite-time blowup for a class of nonlocal parabolic problems*, Advances in Differential Equations **2** (1997), 927–953.

[2] J. Bebernes, C. Li and P. Talaga, *Single-point blowup for nonlocal parabolic problems*, Physica D **134** (1999), 48–60.

[3] S. Bricher, *Blow-up behaviour for nonlinearly perturbed similinear parabolic equations*, Proc. Royal Soc. Edinburgh, **124A** (1994), 947–969.

[4] T. J. Burns, *A mechanism for shear band formation in the high strain-rate torsion test*, J. Appl. Mech. **57** (1990), 836–844.

[5] T. J. Burns, *Does a shear band result from a thermal explosion?*, Mechanics of Materials **17** (1994), 261–27.

[6] R. J. Crawford, *Plastics Engineering*, Butterworth–Heinemann, 1998.

[7] S. Filippas and R. V. Kohn, *Refined asymptotics for the blowup of $u_t - \Delta u = u^p$*, Comm. Pure Appl. Math **45** (1992), 821–869.

[8] R. Flemming, W. Olmstead and S. Davis, *Shear localization with Arrhenius flow law*, SIAM J. Appl. Math. **60** (2000), 1867–1886.

[9] A. Friedman and B. McLeod, *Blowup of positive solutions of semilinear heat equations*, Indiana Univ. Math. J. **34** (1985), 425–447.

[10] S. C. Hunter, *Mechanics of Continuous Media*, Ellis Horwood, 1983.

[11] A. Lacey, *Thermal runaway in a non-local problem modelling Ohmic heating: Part I*, Euro. Jnl. of Applied Mathematics **6** (1995), 127–144.

[12] A. Lacey, *Thermal runaway in a non-local problem modelling Ohmic heating: Part II* Euro. Jnl. of Applied Mathematics **6** (1995), 201–224.

[13] A. Marchand and J. Duffy, *An experimental study of the formation of adiabatic shear bands in a structural steel*, J. Mech. Phys. Solids **36** (1988), 251–283.

[14] W. Olmstead, S. Nemat-Nasser and L. Ni, *Shear bands as surfaces of discontinuity*, J. Mech. Phys. Solids **42** (1994), 697–709.

[15] J. Walter, *Numerical experiments on adiabatic shear band formation in one dimension*, Int. J. Plasticity **8** (1992), 657–693.

[16] T. Wright, *The physics and mathematics of adiabatic shear bands*, Cambridge University Press, Cambridge, 2000.

[17] T. Wright and R. Batra, *The initiation and growth of adiabatic shear bands*, Int. J. Plasticity **1** (1985), 205–212.