MOCK EXAM, November 2015.

(1) (a) (i) Define the Lie algebras $\mathfrak{gl}_n$ and $\mathfrak{sl}_n$, and prove that $\mathfrak{sl}_n$ is an ideal of $\mathfrak{gl}_n$.

(ii) Suppose that S is an $n \times n$ matrix, and define

$$\mathfrak{gl}_S := \{x \in \mathfrak{gl}_n \mid x^T S = -Sx\}.$$

Show that $\mathfrak{gl}_S$ is a subalgebra of $\mathfrak{gl}_n$. **[8 marks]**

(b) Suppose that L is a Lie algebra.

(i) Define what is meant by an L-module.

(ii) If M and N are both L-modules, define what is meant by a L-*module homomorphism*

$$\theta \colon M \to N.$$

(iii) If M is an L-module, and K is a submodule of M, briefly explain how the quotient vector space M/K acquires the structure of an L-module, and show that this is well-defined. **[8 marks]**

(c) Consider the Lie algebra of upper triangular 2×2 matrices $L = \mathfrak{b}_2$. State with justification whether the following claims are TRUE or FALSE. If they are true give a short proof, whereas if they are false give a counterexample.

(i) L has only two ideals.

(ii) L is semisimple.

(iii) All indecomposable L-modules are one-dimensional. **[9 marks]**

(2) Let L be a finite dimensional Lie algebra.

(a) Define what it means for a representation of L to be *faithful*. Give a precise criterion for when the adjoint homomorphism $L \to \mathfrak{gl}(L)$ is faithful. **[4 marks]**

(b) Define the *lower central series* of L, and define what it means for L to be *nilpotent*. **[4 marks]**

(c) Give an explicit example of:

(i) A 3-dimensional abelian Lie algebra.

(ii) A 3-dimensional solvable Lie algebra.

(iii) A 3-dimensional semisimple Lie algebra.

You do not need to justify your answer. **[4 marks]**

(d) State both forms of Engel's Theorem. You do not need to give the proof. **[4 marks]**

(e) Prove directly that the Lie algebra $\mathfrak{n}_3$ (strictly upper triangular 3×3 matrices) is nilpotent. **[9 marks]**

(3) Let L be a finite dimensional Lie algebra.

(a) Define the *Killing form* $\kappa(-,-)$, and show that it satisfies

$$\kappa([a,b],c) = \kappa(a,[b,c])$$

for all $a,b,c \in L$. **[5 marks]**

(b) Define the *radical* $\operatorname{rad} L$ of L. Prove that $L/\operatorname{rad}L$ is semisimple. You may use standard results from the lectures, provided that they are clearly stated. **[4 marks]**

(c) State both of Cartan's criteria. You do not need to give a proof. **[4 marks]**

(d) The Lie algebra $\mathfrak{sl}_2$ has basis

$$\left\{ e = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, f = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \right\}.$$

(i) With respect to the basis $\{e,h,f\}$, show that

$$\operatorname{ad}_h = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -2 \end{pmatrix}, \quad \operatorname{ad}_e = \begin{pmatrix} 0 & -2 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \quad \operatorname{ad}_f = \begin{pmatrix} 0 & 0 & 0 \\ -1 & 0 & 0 \\ 0 & 2 & 0 \end{pmatrix}.$$

(ii) Using this, or otherwise, compute $\kappa_{\mathfrak{sl}_2}(a,b)$ for all $a,b \in \{e,h,f\}$.

(iii) Using the above, deduce that $\mathfrak{sl}_2$ is semisimple. **[12 marks]**

(4) Let L be a finite dimensional semisimple Lie algebra.

(a) Define what is meant by a *Cartan subalgebra* of L. State the Cartan subalgebra of $\mathfrak{sl}_3$, and confirm that it is two-dimensional. **[5 marks]**

For the remainder of this question, you may assume that L has a non-zero Cartan subalgebra H that satisfies $H = C_L(H)$.

(b) Let Φ denote the set of roots.

(i) Briefly justify why the formula

$$\dim L = \dim H + |\Phi|$$

holds. **[5 marks]**

(ii) Deduce that there cannot be a semisimple Lie algebra of dimension 5. **[5 marks]**

(c) State with justification whether the following claims are TRUE or FALSE. If they are true give a brief justification, whereas if they are false give a justification or counterexample.

(i) There exists a semisimple Lie algebra with precisely four roots.

(ii) There exists a semisimple Lie algebra with precisely five roots.

(iii) There exists a semisimple Lie algebra with precisely six roots. **[10 marks]**

[End of Paper]